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BARRELLED SPACES WITH(OUT) SEPARABLE QUOTIENTS

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Abstract

While the separable quotient problem is famously open for Banach spaces, in the broader context of barrelled spaces we give negative solutions. Obversely, the study of pseudocompact X and Warner bounded X allows us to expand Rosenthal's positive solution for Banach spaces of the form $C_c(X)$ to barrelled spaces of the same form, and see that strong duals of arbitrary $C_c(X)$ spaces admit separable quotients.

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1. Introduction

In this paper locally convex spaces (lcs's) and their quotients are assumed to be infinite-dimensional and Hausdorff. The classic separable quotient problem asks if all Banach spaces have separable quotients. Do all barrelled spaces? We find some that do not (Section 2), and add to the long list of those that do (Section 3).

Schaefer [27, para. 7.8(i), page 161] and others observe that:

(∗) If F is a subspace of a normed space E, then $E'_\beta/F^\perp \approx F'_\beta$, where E'_β and F'_β denote the respective strong duals of E and F.

Consequently, reflexive Banach spaces have separable quotients [30, Example 15-3-2]. When X is compact, the Banach space $C_c(X)$ contains c_0, and ℓ^1 is a separable quotient of $C_c(X)'_\beta$ by (∗). An over-arching recent result [1] of Argyros, Dodos and Kanellopoulos (ADK) is that strong duals of arbitrary Banach spaces admit separable quotients.

Rosenthal [17] proved that for any compact space X, either ℓ^2 or c is a [separable] quotient of $C_c(X)$. Venturing beyond Banach spaces, Eidelheit [5, Satz 2] showed that

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every Fréchet space (complete metrisable lcs) F which is *not* Banach must admit a quotient isomorphic to ω, the separable Fréchet space of all scalar sequences. Pérez Carreras and Bonet [2] obtained the result for finite-codimensional subspaces of F, and asked about countable-codimensional subspaces [3, Problem 13.2.2]. Saxon answered positively [19, 20], and with Narayanaswami [22, page 77] proved every proper (LF)-space (inductive limit of a properly increasing sequence of Fréchet spaces) has a separable quotient (cf. [16]).

All the quotients above are of *barrelled* spaces [21]; that is, lcs's in which every barrel (= absorbing balanced convex closed set) is a neighbourhood of zero. An lcs E is barrelled if and only if it is the strong dual of its weak dual E'_σ [12, Section 21.2(2)]. In Section 3 we extend Rosenthal: if $C_c(X)$ is barrelled, then ℓ^2, c or ω is a (separable) quotient. But echoes of ADK reach well beyond Banach, Fréchet, and even barrelled spaces: strong duals of arbitrary $C_c(X)$ spaces admit separable quotients. In fact, we prove that $C_c(X)'_\beta$ always contains a complemented copy either of ℓ^1, or of φ, the strong dual of ω.

These positive results and Popov's nonlocally convex counterexamples [15] beg the separable quotient question for (i) lcs's, (ii) strong duals of lcs's and (iii) barrelled spaces. Section 2 shows that those GM-spaces which disallow separable quotients constitute the nonempty class of quasi-Baire GM-spaces, settling all three basic questions (i)–(iii) in the negative.

Section 3 also enhances the general structure theory of $C_c(X)$ spaces and adds a large collection of Banach spaces perhaps not previously known to admit separable quotients. We recall for the reader's convenience that

$$\begin{aligned} X \text{ is compact } &\Rightarrow\ C_c(X) \text{ is a } DF\text{-space } \Rightarrow\ C_c(X) \text{ is a } df\text{-space} \\ &\Rightarrow\ X \text{ is Warner bounded } \Rightarrow\ X \text{ is pseudocompact.} \end{aligned}$$

Precisely when X is Warner bounded, $C_c(X)'_\beta$ is a normed space which (we show) contains ℓ^1 complemented; $C_c(X)'_\beta$ is a Banach space precisely when $C_c(X)$ is a df-space [8]. Therefore precisely when X is Warner bounded and not compact, $C_c(X)$ is not normable (so $(*)$ and ADK do not apply) and yet $C_c(X)'_\beta$ is a normed space which, by Section 3, has a separable quotient ℓ^1, as does the completion. Consider, for example, Morris and Wulbert's nonnormable DF-space $C_c[0,\Omega)$ [14]: the strong dual is necessarily normable (and complete, in this case) with a complemented copy of ℓ^1. In fact, examples in [7–10] show that none of the four displayed arrows can be reversed. Therefore in these examples the spaces X are nonhomeomorphic, and most are Warner bounded and not compact.

2. Barrelled spaces without separable quotients

We note that an lcs H is (isomorphic to) a dense subspace of ω if and only if $\dim H' = \aleph_0$, for in this case the weak topology $\sigma(H,H')$, being metrisable, coincides with the Mackey topology $\tau(H,H')$. An lcs E is an S_σ *space* if it is the union of an increasing sequence of proper closed subspaces.

Theorem 2.1. *Let E be an lcs. The following three statements are equivalent:*

(1) *E is an S_σ space;*
(2) *E contains a closed $\aleph_0$-codimensional subspace M;*
(3) *E'_σ contains a subspace H isomorphic to a dense subspace of ω.*

Proof. (1) $\Leftrightarrow$ (2): Suppose that $(E_n)_n$ is a properly increasing sequence of closed subspaces which covers E. Let $(f_n)_n$ be a sequence in E' where each f_n vanishes on E_n but not on all of E_{n+1}, and take $M = \cap_n f_n^\perp$. Conversely, given M, let $(x_n)_n$ be a cobasis for M in E, so that E is covered by the properly increasing sequence $(E_n)_n$ of closed subspaces, where $E_n = M + \mathrm{sp}\{x_1, \ldots, x_n\}$.

(2) $\Leftrightarrow$ (3): Given M, set $H = M^\perp$ and note that $\dim H' = \dim E/M = \aleph_0$; given H, set $M = H^\perp$. $\square$

It is easy to generalise [26, $P_1 \Leftrightarrow P_3$] from Banach spaces to lcs's:

Theorem 2.2. *An lcs E admits a separable quotient if and only if it contains a dense S_σ subspace.*

See [11] for a topological vector space analog. The main result of [26] says that *a Banach space has a separable quotient if and only if it has a dense nonbarrelled subspace*, and via the next theorem, 'nonprimitive' can replace 'nonbarrelled', but neither version is valid for general lcs's; for example consider φ.

An lcs E is *primitive* (in [25], has *property* $f|_{L_n}$) if a linear form on E is continuous when its restriction to each member of some increasing, covering sequence of subspaces is continuous. All barrelled spaces and all non-S_σ spaces are primitive; in fact, primitivity is the weakest of the usual weak barrelledness conditions.

Eberhardt and Roelcke's GM-spaces satisfy the hypothesis of our next theorem. Indeed, every dense subspace of a GM-space is barrelled [4].

Theorem 2.3. *Let E be an lcs in which every dense subspace is primitive (a GM-space, for example). E admits a separable quotient if and only if E is an S_σ space.*

Proof. Every S_σ space E has a closed subspace M with $\dim E/M = \aleph_0$. Necessarily, E/M is separable.

Conversely, if E/M is separable, then M is a (closed) $\aleph_0$-codimensional subspace of a dense subspace F. The proof is complete if we show that $F = E$. Suppose that there is some $x \in E \backslash F$. Then the dense $G := F + \mathrm{sp}\{x\}$ is primitive with closed $\aleph_0$-codimensional subspace M, and is the union of a sequence $M = M_1 \subset M_2 \subset \cdots$ of subspaces with each $\dim M_{n+1}/M_n = 1$. Thus every linear form on G which vanishes on M has continuous restrictions to M_n and is continuous. In particular, the linear form which vanishes on F and is 1 at x must be continuous, a contradiction of density. $\square$

From the proof it is clear that any separable quotient of a GM-space E must be $\aleph_0$-dimensional without proper dense subspaces; in particular, E admits no *proper* separable quotients in the sense of Robertson [16]. In [18] we call non-S_σ barrelled spaces *quasi-Baire* (QB). Every Baire lcs (for example each Fréchet space) is QB, and a barrelled space E is QB if and only if $E \not\approx E \times \varphi$ [23, Theorem 1 d)].

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Corollary 2.4. *An lcs in which every dense subspace is barrelled (for example a GM-space) admits separable quotients if and only if it is not QB.*

The (negative) answer to questions (i)–(iii) follows:

Corollary 2.5. *Some barrelled spaces do not admit separable quotients.*

Proof. Some GM-spaces are Baire [4, 3.5 and 3.6], hence QB. □

3. Barrelled spaces with separable quotients

The infinite Tichonov (completely regular Hausdorff) space X is pseudocompact if and only if the set $B := \{f \in C(X) : |f(x)| \leq 1 \text{ for all } x \in X\}$ is a barrel in $C_c(X)$, where $C(X)$ is the linear space of all real-valued continuous functions on X, and $C_c(X)$ denotes $C(X)$ endowed with the compact-open topology. If X is pseudocompact and $C_c(X)$ is barrelled, then B is a neighbourhood of zero; equivalently, X is compact. We proved [8, Theorem 3.1] that X is not pseudocompact if and only if $C_c(X)$ contains a copy of ω. In any lcs, copies of ω are complemented [3, Corollary 2.6.5(iii)]. Since X is either pseudocompact or not, we have the following theorem.

Theorem 3.1. *If $C_c(X)$ is barrelled, then it admits a separable quotient. When X is compact, this is Rosenthal's result; some quotient is an isomorph of either the Hilbert space ℓ^2 or the Banach space c. When X is not compact, the barrelled space $C_c(X)$ must contain a complemented copy of ω.*

Observation (∗) produces, independent of ADK, a separable quotient $E'_\beta/F^\perp$ for every choice of normable E having subspace F with F'_β separable. The observation fails when E is a general lcs [12, Section 31.7]; nevertheless, we shall prove that the quotient $E'_\beta/F^\perp$ is still separable and isomorphic to F'_β provided F'_β is isomorphic to either φ or the Banach space ℓ^1.

Lemma 3.2. *If M is a closed $\aleph_0$-codimensional subspace of an lcs E then, under the strong topology $\beta(E, E')$, every algebraic complement N is a topological complement isomorphic to φ.*

Proof. Let V be an absolutely convex absorbing set in N. The proof is complete if we show that $M + V$ is a $\beta(E, E')$-neighbourhood of 0. One routinely finds a biorthogonal sequence $\{x_n, f_n\}_n \subset E \times E'$ with $\{x_n\}_n$ a Hamel basis for N and each $f_n \in M^\perp$. Choose $\varepsilon_n > 0$ with each $\varepsilon_n x_n \in V$, and note that $\{2^n \varepsilon_n^{-1} f_n(x)\}_n$ is eventually 0 for each $x \in E$. Hence $\{2^n \varepsilon^{-1} f_n\}_n$ is $\sigma(E', E)$-bounded, and its polar, a $\beta(E, E')$-neighbourhood of 0, is contained in $M + V$ by convexity. □

The desired dual version follows.

Theorem 3.3. *Let E be a locally convex space with subspace H isomorphic to a dense subspace of ω. If E' is endowed with any locally convex topology finer than $\sigma(E', E)$, then $H^\perp$ is closed in E' and the quotient $E'/H^\perp$ is separable, indeed is $\aleph_0$-dimensional.*

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For the strong dual in particular, every algebraic complement of $H^\perp$ is a topological complement in E'_β isomorphic to φ.

Certainly, the choice of H is limited; for example, we cannot interchange the roles of ω and φ. Indeed, let E be any $(LF)_2$-space; that is, an (LF)-space which is non-S_σ and contains a copy H of φ [23]. We claim that E'_β does *not* contain a copy G of ω, the strong dual of φ. Otherwise, by minimality, G would be a copy of ω in E'_σ, and so, by Theorem 2.1, the space E would be S_σ, a contradiction.

We generalise a choice of H known for Banach spaces.

Theorem 3.4. *Let E be a locally convex space with subspace H isomorphic to a dense subspace of the Banach space c_0. If E' is endowed with any locally convex topology between $\sigma(E',E)$ and $\beta(E',E)$, then $H^\perp$ is closed in E' and the quotient $E'/H^\perp$ is separable. For the strong dual in particular, $H^\perp$ is complemented in E'_β by an isomorph N of the Banach space ℓ^1 (the strong dual of c_0).*

Proof. We prove the last sentence first. Let $T : \ell^1 \to H'_\beta$ be an isomorphism from ℓ^1 onto H'_β, and let $(g_n)_n$ be the Schauder basis for H'_β which is the image of the natural Schauder basis for ℓ^1, so that

$$T((a_n)_n) = \sum_n a_n g_n$$

for each $(a_n)_n \in \ell^1$. Let U denote the unit ball $\{(a_n)_n \in \ell^1 : \sum_n |a_n| \le 1\}$. Now $T(U)$ is a bounded neighbourhood of zero in H'_β, which implies that

$T(U)$ is equicontinuous on the normed space H, and
$B^\bullet \subset T(U)$, where $B^\bullet$ is the polar in H' of some bounded set B in H.

Thus $(g_n)_n$, a subset of $T(U)$, is equicontinuous on H, and the Hahn–Banach theorem provides linear extensions f_n of the g_n such that $(f_n)_n$ is equicontinuous on E. Let W be a closed absolutely convex neighbourhood of zero in E such that

$$(f_n)_n \subset W^\circ.$$

For each $\varepsilon > 0$ and $(a_n)_n \in \varepsilon \cdot U$ the series $\sum_n a_n f_n$ converges pointwise to a sum $S((a_n)_n) \in \varepsilon \cdot W^\circ$, and the mapping S thus defined is linear from ℓ^1 onto its image $S(\ell^1) =: N$. If A is any bounded set in E, then there is some $\varepsilon > 0$ with $\varepsilon \cdot A \subset W$, so that $A^\circ \supset \varepsilon \cdot W^\circ \supset S(\varepsilon \cdot U)$, which proves that S is continuous. Since each $S((a_n)_n)|H = T((a_n)_n)$, the mapping is one-to-one.

Now suppose that $f \in B^\circ \cap N$. Then for some $(a_n)_n \in \ell^1$ we have $f = S((a_n)_n) \in B^\circ$. Therefore $T((a_n)_n) = f|H \in B^\bullet \subset T(U)$, and T one-to-one implies that $(a_n)_n \in U$. Hence $f \in S(U)$, and we have proved that

$$B^\circ \cap N \subset S(U), \tag{†}$$

so that S is open. Therefore S is an isomorphism from the Banach space ℓ^1 onto N with the topology induced by E'_β.

One easily checks that $E' = H^\perp + N$ and $H^\perp \cap N = \{0\}$. Now suppose that we are given $h \in H^\perp$, $g \in N$ with $h + g \in B^\circ$. Note that h vanishes on B since $B \subset H$, which implies that $g \in B^\circ$. From (†) we conclude that $g \in S(U)$, thus $h + g \in H^\perp + S(U)$, so that $B^\circ \subset H^\perp + S(U)$, and the latter set is a neighbourhood of zero in E'_β. Hence the projection of E'_β onto N along $H^\perp$ is continuous, and the theorem's last sentence is established.

Finally, if E' is given any topology between $\sigma(E', E)$ and $\beta(E', E)$, the quotient $E'/H^\perp$ is still separable, since the Banach space ℓ^1 with any coarser topology is still separable. □

Now previous work [9] and a short argument will show that these two spaces, c_0 and ω, provide separable quotients in the strong dual of every $C_c(X)$ space. Whether every $C_c(X)$ itself admits separable quotients remains open.

$C_c(X)$ is always non-S_σ [10, 13]. Not so for the weak dual $C_c(X)'_\sigma$; we prove that it is non-S_σ precisely when X is *Warner bounded*, that is when, for every disjoint sequence $(U_n)_n$ of nonempty open sets in X, there is a compact set in X that meets infinitely many of the U_n. We also prove that, while $(LF)_2$-spaces are strong duals that characteristically contain φ and not φ complemented [3, 23], the strong dual $C_c(X)'_\beta$ must contain φ complemented if it contains φ at all.

Theorem 3.5. *Let X be an infinite Tichonov space. The following eight statements are equivalent:*

(1) *X is Warner bounded;*
(2) *$C_c(X)$ does not contain a dense subspace of ω;*
(3) *$C_c(X)$ contains c_0 and no dense subspace of ω;*
(4) *$C_c(X)'_\beta$ does not contain φ;*
(5) *$C_c(X)'_\beta$ does not contain φ complemented;*
(6) *$C_c(X)'_\beta$ is normed* (Warner)*;*
(7) *$C_c(X)'_\beta$ is normed and contains ℓ^1 complemented;*
(8) *$C_c(X)'_\sigma$ is non-S_σ.*

Proof. In [9, Corollary 2.6] we proved that (1), (2) and (4) are equivalent, and that (4) ⇒ (5) is trivial. Theorem 3.3 says that (5) ⇒ (2). Duality invariance and Theorem 2.1 yield (2) ⇔ (8). Warner [29] (cf. [9]) proved (1) ⇔ (6). Thus we have the equivalence of (1), (2), (4), (5), (6) and (8).

Trivially, (3) ⇒ (2) and (7) ⇒ (6).

(1) ⇒ (3): Half the work is already done, since (1) ⇒ (2). Assuming that X is Warner bounded, we must show that $C_c(X)$ contains a copy H of c_0. There always exists a disjoint sequence $(U_n)_n$ of nonempty open sets in X. By definition, there is a compact set K in X which meets infinitely many of the U_n, so there are a subsequence $(V_n)_n$ of $(U_n)_n$ and a sequence $(x_n)_n$ with $x_n \in K \cap V_n$ for each $n \in \mathbb{N}$. If Q is any compact set in X, the seminorm ρ_Q on $C(X)$ defined by the equation $\rho_Q(f) = \sup\{|f(x)| : x \in Q\}$ is continuous on $C_c(X)$, whose topology is generated by

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the totality of such seminorms. For each n, a well-known extension theorem yields $f_n \in C(X)$ such that $f_n(x_n) = 1$, $f_n(X \backslash V_n) = \{0\}$, and $|f_n(x)| \le 1$ for all $x \in X$. For each $(a_n)_n \in c_0$ it is readily seen that the series $\sum_n a_n f_n$ converges pointwise to a continuous function on X, so that

$$H := \Big\{ \sum_n a_n f_n : (a_n)_n \in c_0 \Big\}$$

is a well-defined subspace of $C_c(X)$. Furthermore, for each $(a_n)_n \in c_0$ and compact set Q in X, have

$$\rho_Q\Big(\sum_n a_n f_n\Big) \le \sup\{|a_n| : n \in \mathbb{N}\} = \rho_K\Big(\sum_n a_n f_n\Big).$$

Therefore ρ_K is a norm on H which generates the topology induced by $C_c(X)$, and H is norm-isomorphic to c_0.

Finally, the equivalent statements (1)–(6) and (8) and Theorem 3.4 imply (7). □

Non-S_σ, *primitive* and *dual locally complete* (dlc) [24] are duality invariant properties, generally distinct. Our paraphrase [9, Theorem 2.4(6)] of Warner says that (1) is equivalent to the following statement:

(9) *X is pseudocompact and $C_c(X)'_\sigma$ is dlc.*

Moreover, (1) is also equivalent to:

(10) *X is pseudocompact and $C_c(X)'_\sigma$ is primitive.*

Indeed, (9) ⇒ (10) is immediate. We easily argue that ¬(8) ⇒ ¬(10): if M is a closed $\aleph_0$-codimensional subspace of a primitive $C_c(X)'_\sigma$, then $M^\perp$ is a copy of ω in $C_c(X)$, so X is not pseudocompact [8]. □

Hence (1)–(10) are equivalent.

In [6] we prove that $C_p(X)'_\sigma$, while never non-S_σ, is dlc if and only if it is primitive, if and only if X is a P-space.

Corollary 3.6. *Let X be an arbitrary (infinite) Tichonov space. Both the strong and weak duals of $C_c(X)$ admit separable quotients. In fact:*

(A) *If X is Warner bounded, $C_c(X)$ contains the Banach space c_0; otherwise it contains a dense subspace of ω.*

(B) *If $C_c(X)'_\beta$ is normed, it contains a complemented copy of the Banach space ℓ^1; otherwise it contains a complemented copy of φ.*

Every separable Banach space is a quotient of ℓ^1 (cf. [20]), thus also of each normed $C_c(X)'_\beta$, by (B). Separable quotients exist for $C_p(X)'_\beta$ and $C_p(X)'_\sigma$ [6].

Tweddle and Yeomans [28] used $\mathfrak{c}$-dimensional bounded sets to see that every barrelled $C_c(X)$ space has a barrelled countable enlargement. Since each barrelled $C_c(X)$ contains either ω or c_0, enlargements for just those two spaces readily yield the general result. Also potentially useful is the fact that a quasibarrelled $C_c(X)$ is either a Banach space or contains a dense subspace of ω.

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