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BRAESS'S PARADOX AND POWER-LAW NONLINEARITIES IN NETWORKS

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Abstract

We study flows in physical networks with a potential function defined over the nodes and a flow defined over the arcs. The networks have the further property that the flow on an arc a is a given increasing function of the difference in potential between its initial and terminal node. An example is the equilibrium flow in water-supply pipe networks where the potential is the head and the Hazen-Williams rule gives the flow as a numerical factor k_a times the head difference to a power $s > 0$ (and $s \cong 0.54$). In the pipe-network problem with Hazen-Williams nonlinearities, having the same $s > 0$ on each arc, given the consumptions and supplies, the power usage is a decreasing function of the conductivity factors k_a. There is also a converse to this. Approximately stated, it is: if every relationship between flow and head difference is not a power law, with the same s on each arc, given at least 6 pipes, one can arrange (lengths of) them so that Braess's paradox occurs; i.e. one can increase the conductivity of an individual pipe yet require more power to maintain the same consumptions.

1. Introduction

Braess in [2] gave an example of a network, in the setting of traffic flow, in which an extra arc (i.e. road) was added and the total travel time for any road user was increased. Braess's example has just one origin-destination pair, so that it is, like the physical network flows in this paper, a single-commodity

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flow. (See Rockafellar [21, Section 1K], for definitions.) The phenomenon was reconsidered in several works on traffic flow (such as Murchland [18], LeBlanc [16], Fisk [14]) and in popularisations (such as Toint [23]).

Nonlinear networks occur in many contexts, see Dembo *et al.* [10], Rockafellar [21]. In the context of flows in physical networks, Braess's paradox can be restated informally as follows.

BRAESS'S PARADOX. *The power consumed in a nonlinear network can increase if an arc's conductivity is increased with consumptions held constant.*

Theorem 11 of the present paper states that Braess's paradox cannot occur in a two-terminal series-parallel network. See Keady [15, Part I, Appendix A], for a discussion.

One of the examples of physical networks is the flow of water in a pipe network. When the commonly-used Hazen-Williams rule (with the same exponent on each arc, e.g. Brebbia and Ferrante [3]) is assumed, our Theorem 1 shows that Braess's paradox cannot occur. This was known already in two-terminal resistive networks where Ohm's law holds for each resistor, that is, $s = 1$ for all arcs a, as shown in Shannon and Hagelbarger [22], Melvin [17].

Define a network to be of *type* (H) if, for each arc A, the flow is a numerical factor k_a times the head difference to a power $s > 0$ independent of a. (The definition of type (H) has been introduced as an aid to readers because there are two separate meanings of the word "power" in this paper.) We provide various characterisations, both of when a network is type (H), and also of when a network can exhibit Braess's paradox. The two subjects are related: see our main theorems, Theorems 1 and 2 of Section 3. In a recent paper on Braess's paradox, Cohen and Horowitz [8] write: "The task remains of specifying the general conditions under which such paradoxes can occur, for general network topologies and broad classes of components ...". Our theorems summarise some progress with this task.

Our first exposition, in [15], treated these matters in the setting of two-terminal networks. Rockafellar [21, Section 8N], calls these "black boxes". Here the exposition is in terms of the power-loss in a general network, with known consumptions at each node. Two-terminal networks are just a special case.

2. Notation

Our notation follows that of Bertsekas *et al.* [1] and of Rockafellar [21]. A *network* $G = (N, A)$, or the directed graph associated with the network, consists of two finite sets A and N and a function that assigns to each $a \in A$ a pair $(i, k) \in N \times N$ such that $i \neq k$. The elements of A are called *arcs* (or edges); the elements of N are called *nodes* (or vertices). We may label the arcs with the first $|A|$ positive integers, i.e. starting from 1, and identify A with this set. We label the nodes with the $|N|$ nonnegative integers from 0 to $|N| - 1$. The correspondence of arc j with its nodes is written $j \sim (i, k)$. We say that i is the *initial* or *start* node of arc j and k is the *terminal* or *end* node of arc j.

Let E be a *node-arc incidence matrix* for G; that is, assign a column vector to each arc, with 0 everywhere except for a 1 for one node of the arc and -1 for the other. More precisely, the entries e_{ij} of E are given by

$$e_{ij} = \begin{cases} +1 & \text{if } i \text{ is the initial node of arc } j, \\ -1 & \text{if } i \text{ is the terminal node of arc } j, \\ 0 & \text{otherwise.} \end{cases}$$

We always assume that G is connected, so that $|A| \geq |N| - 1$, and the rank of E is $|N| - 1$.

For $i \in N$, we let $p(i) \in \mathbb{R}$ be the head (or potential or voltage or time) at node i. For $a \in A$, let $q_a \in \mathbb{R}$ denote the flow on arc a, from start to end. For $i \in N$, let $b(i)$ be the deficit (or input, or consumption, or current supplied) at node i.

For each arc $a \sim (i, k) \in A$, suppose there is given a conductivity function $\sigma_a\ \mathbb{R} \to \mathbb{R}$ and a nonnegative real number, the conductivity factor k_a. The flow function $k_a \sigma_a$ will relate the head differences and flows by (2.2).

DEFINITION. We say the network conductivity functions satisfy *Assumption* $A(\sigma)$ if, for each $a \in A$, σ_a is continuous, $\sigma_a(0) = 0$, $\sigma_a(-t) = -\sigma_a(t)$ for all t, $\sigma_a(t) \to \infty$ as $t \to \infty$, σ_a is C^1 on $(0, \infty)$, and $\sigma_a'(t) > 0$ for all $t > 0$.

DEFINITION. We say the network conductivity factors satisfy *Assumption* $A(k)$ if, with $A(k) = \{a \in A | k_a > 0\}$, $(N, A(k))$ is a connected graph.

As all of our results depend on G being connected and assumption $A(k)$ being satisfied, these are presumed to be satisfied throughout the paper.

Rockafellar [21, Section 8H], defines the *network equilibrium problem* as follows. Given b with

$$\sum_{i \in N} b(i) = 0, \tag{2.1}$$

given conductivity functions σ_a satisfying Assumption $A(\sigma)$, and given conductivity factors k_a satisfying Assumption $A(k)$, find a head vector $p \in \mathbb{R}^{|N|}$ such that

$$q_a = k_a \sigma_a(p(i) - p(k)), \qquad \forall a \sim (i, k) \in A, \tag{2.2}$$

and

$$Eq = b. \tag{2.3}$$

We remark that the above only determines p up to a $p + ce$, $c \in \mathbb{R}$, where e is the vector all of whose entries are one. When, as from Section 4 onwards, we want to make p unique, we suppose its 0th entry is 0.

Informally, the σ_a can be thought of as giving the form of the conductivity law. In the context of the pipe-network problem with Hazen-Williams flow functions, varying a conductivity factor k_a amounts to varying some aspect of pipe a's geometry, its length or diameter or roughness. Outside the setting of Hazen-Williams laws, varying a conductivity factor seems to be physically unnatural. See the discussion at the end of Section 7.

The "power-law nonlinearity" in our title refers to the situation when the flow q_a on an arc $a \sim (i, k)$ satisfies

$$q_a = k_a \sigma(p(i) - p(k), s_a) \qquad \text{where } \sigma(t, s) = t|t|^{s-1}, \ s > 0. \tag{2.4}$$

Type (H) means that there exists $s > 0$ such that for all $a \in A$, $\sigma_a(t) = \sigma(t, s)$.

The *power-loss* P in the networks is defined by

$$P = \sum_{(i,k) \sim a \in A} (p(i) - p(k)) q_a. \tag{2.5}$$

Where there are arcs in parallel joining i to k, the summation is over all of them. We always have $i < k$. Both of these conventions will be used elsewhere in this paper. Provided all the σ_a are odd functions, the power-loss P is nonnegative and, by Rockafellar [21, Section 1I],

$$P = b^{\mathsf{T}} p. \tag{2.6}$$

In a two-terminal network, where $b(i) = 0$ except for the two nodes, one of which we always label $i = 0$ and the other $n_t \leq |N| - 1$, we then have

$P = b(0)(p(0) - p(n_t))$. From this, in the language of DC electric circuits, with $b(0) > 0$ the applied voltage difference $p(0) - p(n_t)$ is a nonincreasing function of any conductivity factor k_a precisely when P is.

Rockafellar [21, Section 8B] gives the following definition.

DEFINITION. The set of pairs $(b(0), p(0) - p(n_t))$ when p solves the equilibrium problem is called the *characteristic curve* of the two-terminal network with terminals 0 and n_t.

Lemma 3, a variant of Theorem 2, stated in Section 3, implies that, unless the network is type (H), we can increase resistance, or characteristic curve, on an arc, yet lower the overall characteristic curve of the network. This is a restatement, for two-terminal networks, of Braess's paradox.

3. The main theorems

THEOREM 1. *Suppose (N, A) is a type* (H) *network and that b satisfies* (2.1). *Then, for any a, the power-loss P is a nonincreasing function of k_a.*

The proof will be given in Section 5 together with a much stronger result with the same hypotheses. Before we state a converse in Theorem 2, we give an additional example of Braess's paradox, this time involving only power-law nonlinearities on the arcs, which shows that it is necessary in Theorem 1 that the powers be the same on all the arcs.

DEFINITION. The *Wheatstone bridge graph* is the graph $G = (N, A)$ with

$$N = \{0, 1, 2, 3\}, \qquad A = \{(0, 1), (0, 2), (1, 2), (1, 3), (2, 3)\}. \tag{3.1}$$

Four of the arcs form a quadrilateral, the other forms a diagonal.

EXAMPLE OF BRAESS'S PARADOX. Consider the Wheatstone bridge graph, with the diagonal arc $(1, 2)$ having variable conductivity. The flow functions $k_{i,j}\sigma_{i,j}$ on the arcs (i, j) are

$$k_{01}\sigma_{01}(x) = k_{23}\sigma_{23}(x) = x, \qquad k_{02}\sigma_{02}(x) = k_{13}\sigma_{13}(x) = x|x|^{-1/2},$$

and the variable conductivity arc has

$$k_{12}\sigma_{12}(x) = kx.$$

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Choose b such that $b(1) = 0 = b(2)$, so that the network is a two-terminal one.

The result is that the power-loss $P = b(3)(p(3) - p(0))$ can either decrease or increase as k increases, depending on the value of $b(3) = -b(0)$ and the value of k at the start.

There is a lot of symmetry in the network and the flows, heads and power-loss can be calculated explicitly. The elementary calculations are in [15].

For networks with $|A| \leq 5$, the Wheatstone network is the only two-terminal instance where Braess's paradox can occur. For $|A| > 5$, we have the following.

THEOREM 2. *Suppose* $\{\sigma_a | a \in \mathbb{N}_\nu\}$ *is a set of* ν *functions,* $\nu \geq 6$*, satisfying Assumption* $A(\sigma)$*. Let* $\mathcal{N}$ *denote the set of all networks* (N, A) *with flow functions* $k_a \sigma_{\varphi(a)}$ *on arcs* $a \in A$*, where* φ *ranges over all the one-to-one mappings from* A *to* $\mathbb{N}_\nu$*. Suppose that for any network* (N, A) *in* $\mathcal{N}$*, and any* b *satisfying* (2.1)*, the power-loss* P *is a nonincreasing function of each* k_a*. Then there is an* $s > 0$ *such that for all* $a \in \mathbb{N}_\nu$ *and* $t \geq 0$*,* $\sigma_a(t) = \sigma_a(1)t^s$*.*

Theorem 2 follows immediately from the following result.

LEMMA 3. *Let* $\mathcal{N}_2$ *denote the set of all two-terminal networks* (N, A)*, with the terminal nodes being* 0 *and* $n_t \leq |N| - 1$*. Let* $\nu \geq 6$ *and let Assumption* $A(\sigma)$ *be satisfied as in Theorem* 2*. With, in Theorem* 2*,* $\mathcal{N}$ *replaced by* $\mathcal{N}_2$ *and* b *satisfying* $b(i) = 0$ *for* $i \neq 0$*,* n_t *and* $b(n_t) = -b(0)$*, the conclusion of Theorem* 2 *holds. That is, if for any network* (N, A) *in* $\mathcal{N}_2$ *and any* b *satisfying the preceding restriction the power-loss* P *is a nonincreasing function of each* k_a*, then there is an* $s > 0$ *such that for all* $a \in \mathbb{N}_\nu$ *and* $t \geq 0$*,* $\sigma_a(t) = \sigma_a(1)t^s$*.*

We remark that Theorem 2 and Lemma 3 can be shown to follow from their $\nu = 6$ versions.

The proof of Lemma 3, given in Section 7, depends in part on a detailed analysis of a general nonlinear Wheatstone bridge network.

(If readers find the requirement $\nu \geq 6$ to be unpleasant, we remark that Theorem 2 can be modified to allow its removal to $\nu \geq 1$. The modification is to allow repetition of the conductivity functions in the elements in the test networks, i.e. to drop the requirement that φ be one-to-one.)

4. The convex optimisation problem: primal form

The pipe-network problem has been treated as an optimisation problem in Duffin [11, 12], Collins *et al.* [9], Rockafellar [20, 21], Dembo *et al.* [10].

The proof of Theorem 1 depends on a variational, or optimisation, argument. In Section 2 we defined the *network equilibrium problem.* Rockafellar [21, Section 8H] is the standard reference that this problem is equivalent to two others, namely the *optimal differential problem*, Problem (P) given in the next paragraph, and the optimal distribution problem, Problem (D) given in Section 5. This section contains more material than is needed for the narrow aim of Theorem 1. Some of this, such as existence and uniqueness, is, however, reassuring in that it shows that the counterintuitive flows of Theorem 2 actually exist. The results in this section are not new and, for reasons of space the proofs are suppressed. For detailed proofs and references see [15, 21].

The *optimal differential problem* defined in Rockafellar [21, Section 8G] is similar to the following. Define

$$S_a(t) = \int_0^t \sigma_a(\hat{t})\, d\hat{t}.$$

Define V_b by

$$V_0 = \sum_{(i,k)\sim a\in A} k_a S_a(p(i) - p(k)), \tag{4.1}$$

$$V_b = V_0 - \sum_{i=1}^{|N|-1} b(i)p(i). \tag{4.2}$$

Define X to be the set of vectors with coordinate indexing starting at 1 in $\mathbb{R}^{|N|-1}$, augmented with a zeroth component which is zero. Problem (P) is to find p_* satisfying

$$V_b(p_*) = \min_{p\in X} V_b(p). \tag{P}$$

THEOREM 4. *With Assumptions $A(\sigma)$ on the σ_a and $A(k)$ on the k_a,*
(i) *V_0 is strictly convex on X;*
(ii) *$V_0(p)/\|p\| \to \infty$ as $\|p\| \to \infty$ in any norm on X;*
(iii) *solutions to Problem (P) exist and are unique in X;*
(iv) *p_* solves the network equilibrium problem if and only if it solves Problem* (P).

We shall call items (iii)–(iv) Duffin's Existence Theorem.

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An additional property of V_0, whose consequences are explained in [15] but which is not used in this paper, is the following. For σ_a satisfying Assumption $A(\sigma)$ in the pipe-network problem, we have the inequality

$$V_0(p_0 \wedge p_1) + V_0(p_0 \vee p_1) \leq V_0(p_0) + V_0(p_1) \qquad \forall p_0, p_1.$$

Here $p_0 \wedge p_1$ denotes the minimum of the two vectors p_0, p_1 and $p_0 \vee p_1$ denotes their maximum.

In the case of the Hazen-Williams σ we have the following additional item.

THEOREM 5. *For type* (H) *networks,* $(V_0(p))^{1/(s+1)}$ *is a strictly convex norm on* X.

5. Duality

For σ_a satisfying Assumption $A(\sigma)$, σ_a has an inverse. Call the inverse function ρ_a,

$$\rho_a(\sigma_a(t)) = t, \qquad \forall t \in \mathbb{R}.$$

Define R to be the integral of ρ from 0, $R(0) = 0$. For $q \in \mathbb{R}^{|A|}$ define

$$U(q) = \sum_{a \in A} k_a R_a \left(\frac{q_a}{k_a} \right).$$

The problem of minimising $U(q)$ over q satisfying

$$Eq = b,$$

is Problem (D). Recall that $e^{\mathsf{T}} b = 0$, where e denotes the vector all of whose entries are 1. Problem (D) is a convex separable programming problem, which Rockafellar [21, Section 8D] calls the *optimal distribution problem.*

(The functions R_a and S_a are Fenchel conjugate convex functions on $\mathbb{R}$. For σ_a a power law, both R_a and S_a are also power laws.)

Collins *et al.* [9] establish the following theorem. See also Rockafellar [21, Section 8L]. The notation involving the σ $\mathbb{R}^{|A|} \to \mathbb{R}^{|A|}$, i.e. acting on vectors, is that $\sigma(q)$ is the vector with components $\sigma_a(q_a)$. Similarly $k\sigma(q)$ is the vector with components $k_a \sigma_a(q_a)$.

THEOREM 6. *Let $G = (N, A)$ be a connected network.*

If p_* solves the primal problem (P), *defined above, then $q = k\sigma(E^{\mathsf{T}} p_*)$ solves the dual problem* (D).

If q_* solves the dual problem (D), *and $q_* = k\sigma(E^{\mathsf{T}} p)$ for some p with $p(0) = 0$, then p solves the primal problem* (P).

At the solutions, $U(q_*) = -V_b(p_*)$.

The constraints $Eq = b$ restrict q to lie on a certain hyperplane. For type (H) networks, the minimisation problem, Problem (D), is to find the point q_* on this hyperplane which is closest to the origin in a certain l_p norm. For networks not of type (H), Problem (D) is to find the point q_* on this hyperplane at which $U(q)$ is minimised.

Theorem 2 can be rephrased to give a characterisation of l_p norms amongst certain classes of convex functions U. Other recent characterisations of l_p norms occur in [6, 7].

6. Proof of Theorem 1

The theorems in this subsection do not depend on the "network" aspects of the optimisation problem. We begin with a very easy general monotonicity result.

LEMMA 7. *Consider any continuous function V $\mathbb{R}^{|N|} \times \mathbb{R}^{|A|} \to \mathbb{R}$, and denote the variables in $\mathbb{R}^{|N|}$ by p, and the parameters in $\mathbb{R}^{|A|}$ by k. Suppose*

$$k \leq \hat{k} \qquad \Rightarrow \qquad V(p, k) \leq V(p, \hat{k}) \qquad \forall p. \tag{6.1}$$

Suppose that, for all k, $p_*(k)$ exists satisfying

$$V(p_*(k), k) = \min\{V(p, k) | p \in \mathbb{R}^{|N|}\}. \tag{6.2}$$

Then

$$k \leq \hat{k} \qquad \Rightarrow \qquad V(p_*(k), k) \leq V(p_*(\hat{k}), \hat{k}).$$

PROOF. Inequality (6.1) gives, with $k \leq \hat{k}$,

$$V(p_*(\hat{k}), k) \leq V(p_*(\hat{k}), \hat{k}).$$

Inequality (6.2) gives

$$V(p_*(k), k) \leq V(p_*(\hat{k}), k).$$

The result follows from these two inequalities.

REMARK. If, as in problem *(P)*, $V_b(p,k)$ at a fixed p is concave in k, then $V_b(p_*(k),k)$ is concave in k.

The proofs of both theorems in this section depend on the homogeneity of V_0. Calculus arguments give that the homogeneity is equivalent to

$$p^{\mathsf{T}} D V_0(p) = (s+1)V_0(p) \qquad \forall p,$$

and the calculus condition for a minimum of V_b is $DV_0(p_*) = b$. In the context of the network problem, the essential consequences of homogeneity are contained in the following lemma, whose proof does not need calculus.

LEMMA 8. *For a type* (H) *network, at the solutions p_* and q_* of Problems* (P) *and* (D),

$$P = p_*^{\mathsf{T}} b = V_0(p_*) - V_b(p_*) = (s+1)V_0(p_*) = \frac{s+1}{s}U(q_*). \tag{6.3}$$

PROOF. For a type (H) network, substituting the power-law expressions for R_a and S_a in the formulae for U and V_0 respectively, and using the fact that for each arc $a \sim (i,j)$,

$$|q_{*a}| = k_a |p_*(i) - p_*(j)|^s,$$

it follows that $U(q_*) = sV_0(p_*)$. (See (6.10) and (4.1).) This gives the rightmost equality of (6.1). The duality Theorem 6 stated that $U(q_*) = -V_b(p_*) = -V_0(p_*) + p_*^{\mathsf{T}} b$, or equivalently $p_*^{\mathsf{T}} b = V_0(p_*) - V_b(p_*) = V_0(p_*) + U(q_*)$. From this and the equality of the first sentence, (6.3) follows.

PROOF OF THEOREM 1. Lemma 7 applies when $V = V_b$ and ensures that

$$\hat{k} \geq k \qquad \Rightarrow \qquad V_b(p_*(k),k) \leq V_b(p_*(\hat{k}),\hat{k}). \tag{6.4}$$

Then (6.4) combined with duality $U(q_*) = -V_b(p_*)$, and with (6.3) yields Theorem 1.

With the same hypotheses as in Theorem 1, it is possible to prove more about how the power changes as conductivity factors are changed. This additional result has been separated from Theorem 1 because it is less closely related to the Braess phenomena than is Theorem 1.

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THEOREM 9. *Consider flows in a type* (H) *network with the same network topology, the same b, but varying conductivity factors k. The power-loss P* (i) *is a convex function of the k, and* (ii) *is a concave function of the r where* $r_a = k_a^{-1/s}$.

PROOF. (i) We suppose that the conductivity factors k vary as follows:

$$k_\tau = (1-\tau)k_0 + \tau k_1, \qquad \tau \in [0, 1].$$

Let p_τ be the minimiser of $V_{b,\tau}(p) = V_{0,\tau}(p) - b^{\mathrm{T}} p$. The power-loss $P_\tau = b^{\mathrm{T}} p_\tau$. In particular

$$V_{0,0}(p_0) - P_0 \le V_{0,0}(p_\tau) - P_\tau, \qquad V_{0,1}(p_1) - P_1 \le V_{0,1}(p_\tau) - P_\tau. \quad (6.5)$$

Equation (6.3) can be applied to the minimisers p_0 and p_1, so that multiplying the two preceding inequalities by $s+1$ gives:

$$-sP_0 \le (s+1)(V_{0,0}(p_\tau) - P_\tau), \qquad -sP_1 \le (s+1)(V_{0,1}(p_\tau) - P_\tau). \quad (6.6)$$

Taking $1-\tau$ times the first of these plus τ times the second gives

$$-s((1-\tau)P_0 + \tau P_1) \le (s+1)(V_{0,\tau}(p_\tau) - P_\tau). \quad (6.7)$$

Equation (6.3) can also be applied to the minimiser p_τ, so that

$$P_\tau = (s+1)V_{0,\tau}(p_\tau). \quad (6.8)$$

Finally, we eliminate $V_{0,\tau}(p_\tau)$ and then (6.7) and (6.8) give

$$-s((1-\tau)P_0 + \tau P_1) \le -sP_\tau. \quad (6.9)$$

This establishes the convexity result which we were required to prove.

(ii) Recall that r is the vector of all the r_a and $r_a = k_a^{-1/s}$. The function U defined at Problem (D) can be written

$$U(q) = \frac{s}{s+1} \sum_{a \in A} r_a |q_a|^{(s+1)/s}. \quad (6.10)$$

We now follow steps similar to part (i). Suppose

$$r_\tau = (1-\tau)r_0 + \tau r_1, \qquad \tau \in [0, 1].$$

Let q_τ be the minimisers of U_τ over the set of q such that $Eq = b$. Let P_τ be the corresponding powers. Then

$$U_0(q_\tau) \ge U_0(q_0) = \frac{s}{s+1} P_0, \qquad U_1(q_r) \ge U_1(q_1) = \frac{s}{s+1} P_1. \quad (6.11)$$

Multiplying the first by $(1-\tau)$ and the second by τ and adding gives

$$P_\tau \geq (1-\tau)P_0 + \tau P_0.$$

This establishes the required concavity of the power in r.

An alternative argument using both parts (i) and (ii) to give the result of Theorem 1 follows. Let a be fixed. Let the power as a function of k_a be denoted $Pk(k_a)$. Let the power as a function of $r_a = k_a^{-1/s}$ be denoted $Pr(r_a)$. Omit the subscript a in the rest of this proof. Consider $k_0 < k_1$. Since Pk is convex in k and Pr is concave in $k^{-1/s}$,

$$\frac{dPk}{dk}(k_1) - \frac{dPk}{dk}(k_0) \geq 0, \qquad \frac{dPr}{dr}(k_1^{-1/s}) - \frac{dPr}{dr}(k_0^{-1/s}) \geq 0.$$

Now

$$\frac{dPk}{dk}(k) = -\frac{1}{s}k^{-(s+1)/s}\frac{dPr}{dr}(k^{-1/s}).$$

Combining the immediately preceding displayed items gives

$$(k_0^{(s+1)/s} - k_1^{(s+1)/s})\frac{dPk}{dk}(k_0) \geq 0, \qquad \text{hence } \frac{dPk}{dk}(k_0) \leq 0,$$

as required.

7. Proof of Theorem 2

In contrast to Section 6, all the proofs in Sections 7 and 8 depend on the network structure. In this section, we complete the proof of Theorem 2 by proving Lemma 3.

The next lemma says that if two arcs with the same conductivity function σ (up to a multiplicative constant) placed in series give the same function σ (up to a multiplicative constant—which, we remark, is a very strong requirement), then σ is a power law, $\sigma(t) = \sigma(1)t^s$ for $t \geq 0$. The lemma is effectively a uniqueness statement for a certain functional equation.

LEMMA 10. *Let σ $(0,\infty) \to (0,\infty)$ be nondecreasing and absolutely continuous on compact intervals. Suppose for all $h > 0$ there exists $\theta(h) > 0$ such that for all $v > 0$, there is a $t \in (0, v)$ such that*

$$\theta(h)\sigma(v) = h\sigma(t) = \sigma(v-t).$$

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Then, there exists $s > 0$ *such that* $\sigma(t) = \sigma(1)t^s$.

PROOF. Taking $h = 1$, there is a $\theta(1)$ such that, for all v, there is a t with

$$\theta(1)\sigma(v) = \sigma(t) = \sigma(v - t).$$

Since σ is nondecreasing for all $v > 0$,

$$\sigma(v/2) = \theta(1)\sigma(v). \tag{7.1}$$

Hence for $n \in \mathbb{N}$,

$$\sigma(v/2^n) = \theta(1)^n\sigma(v). \tag{7.2}$$

Take $h = 1/\theta(1)$. For all v, there is a t with

$$\theta(\theta(1)^{-1})\sigma(v) = \theta(1)^{-1}\sigma(t) = \sigma(v - t).$$

The central term is $\sigma(2t)$ by (7.1). If $t \le v/3$, we have $2t \le 2v/3 \le v - t$, giving $\theta(h)\sigma(v) = \sigma(2v/3)$, since σ is nondecreasing. If $t > v/3$, we have $v - t \le 2v/3 \le 2t$, and again $\theta(h)\sigma(v) = \sigma(2v/3)$. Using (7.1), this gives

$$\sigma(v/3) = \theta(1)\theta(\theta(1)^{-1})\sigma(v). \tag{7.3}$$

Continuing, for all $m, n \in \mathbb{N}$, there is a $\lambda(m, n) > 0$ such that, for all $v > 0$,

$$\sigma(mv/2^n) = \lambda(m, n)\sigma(v). \tag{7.4}$$

Hence, unless σ is identically zero, $\sigma(v) > 0$ for all $v > 0$. Letting $v = \exp\xi$, for $\xi \in \mathbb{R}$, and assuming the left-hand side is positive so we can take logarithms,

$$\log(\sigma(\exp(\xi + \log(m/2^n)))) = \log(\lambda(m, n)) + \log(\sigma(\exp\xi)). \tag{7.5}$$

Dividing by $\log(m/2^n)$ and letting $(m/2^n)$ tend to one, we have, for ξ where $(\log\circ\sigma\circ\exp)'(\xi)$ exists, that it is a positive constant independent of ξ (as the m, n are independent of ξ).

Thus there are $s \ge 0$ and $c \in \mathbb{R}$, with

$$\log\circ\sigma\circ\exp(\xi) = s\xi + c \quad \forall\xi, \qquad \text{or} \qquad \sigma(v) = \exp(c)v^s \quad \forall v.$$

The next two theorems from earlier papers, and our discussion in Section 3, give some indications towards our proof of Theorem 2. We begin with definitions from Riordan and Shannon [19].

DEFINITION. Let n_0 and n_f be given nodes of a network G. The network G is said to be *series-parallel* with respect to n_0 and n_f if through each arc of G there is at least one path from n_0 to n_f not touching any node twice, and no two of these paths pass through any arc in opposite directions.

An equivalent inductive definition is as follows. The one-arc graph G_0 with $A_0 = (n_0, n_f)$ is defined to be series-parallel. A network is series-parallel with respect to n_0 and n_f if it is either (i) a connection of a series-parallel network with respect to n_0 and n_i in series with a second series-parallel network with respect to n_i and n_f, or (ii) a parallel connection of two networks both series-parallel with respect to n_0 and n_f.

(The Wheatstone bridge graph of the example in Section 3 is not series-parallel with respect to $\{0, 3\}$, but is series-parallel with respect to $\{1, 2\}$.) A two-terminal network is said to be series-parallel if it is series-parallel with respect to the two terminal nodes.

THEOREM 11. *For series-parallel two-terminal networks, the power-loss P decreases when any conductivity factor k increases.*

Characterisations of series-parallel networks are also known. See Duffin [13].

THEOREM 12. *A two terminal network is series-parallel if and only if there is no embedded network having the Wheatstone bridge configuration.*

We now return to the proof of Lemma 3, knowing that Wheatstone bridges must be used as test networks.

LEMMA 13. *Consider the Wheatstone bridge graph with nodes numbered as in (3.1). The arcs in A are indexed by the numbers 1 to 5 in the order listed in (3.1). Suppose the conductivity functions satisfy Assumption $A(\sigma)$.*

Let $M(k, b)$ be the statement: for the given value of b and for all positive values of k_1, k_2, k_4, k_5 and nonnegative values of k_3, the power-loss is nonincreasing as the conductivity factor k_3 of the arc $3 = (1, 2)$ increases.

Suppose there exists a nonzero b with $b(1) = 0 = b(2)$, such that $M(k, b)$ holds. Then there exists λ such that

$$\frac{\sigma_1'(x)}{\sigma_1(x)}\frac{\sigma_2(x)}{\sigma_2'(x)} = \lambda \ \forall x > 0, \qquad \frac{\sigma_4'(y)}{\sigma_4(y)}\frac{\sigma_5(y)}{\sigma_5'(y)} = \lambda \ \forall y > 0, \tag{7.6}$$

and there exist numbers $\mu_{1,2}$, $\mu_{4,5}$ such that

$$\sigma_1(x) = \exp(\mu_{1,2})(\sigma_2(x))^\lambda \ \forall x > 0, \qquad \sigma_4(y) = \exp(\mu_{4,5})(\sigma_5(y))^\lambda \ \forall y > 0.$$

REMARK. (i) If there exists a nonzero b such that $M(k, b)$ holds, then for all nonzero b, $M(k, b)$ holds. To see this, consider scaling both b and k by the same positive factor.

(ii) The lemma remains true when $M(k, b)$ is replaced by $M_0(k, b)$ involving k_3 increasing from zero, rather than any initial nonnegative value of k_3.

PROOF. The flow function on arc j is $k_j\sigma_j$, $j = 1, \ldots, 5$. Take $p(0) = 0$, let $p(i)$ denote the head at node i, let $p = (p(1), p(2), p(3))^{\mathrm{T}}$ and let $k = (k_1, k_2, k_3, k_4, k_5)$. Define $F\ \mathbb{R}^3 \oplus \mathbb{R}^5 \to \mathbb{R}^3$ by

$$\begin{aligned}
F_1(p, k) &= k_1\sigma_1(p(1)) + k_3\sigma_3(p(1) - p(2)) + k_4\sigma_4(p(1) - p(3)), \\
F_2(p, k) &= k_2\sigma_2(p(2)) + k_3\sigma_3(p(2) - p(1)) + k_5\sigma_5(p(2) - p(3)), \\
F_3(p, k) &= k_4\sigma_4(p(3) - p(1)) + k_5\sigma_5(p(3) - p(2)).
\end{aligned}$$

At fixed k, $k_j > 0$ for $j \neq 3$, $k_3 \geq 0$, $F = DV_0$ and, by Duffin's Existence Theorem (stated with Theorem 4), there is a unique p satisfying the network equilibrium problem,

$$F(p, k) = (0, 0, b(3)). \tag{7.7}$$

Consider the Jacobian $D_pF(p, k)$, where D_p denotes the derivative with respect to p with k held constant. We have

$$D_pF(p, k) = \begin{pmatrix} k_1\sigma_1' + k_3\sigma_3' + k_4\sigma_4' & -k_3\sigma_3' & -k_4\sigma_4' \\ -k_3\sigma_3' & k_2\sigma_2' + k_3\sigma_3' + k_5\sigma_5' & -k_5\sigma_5' \\ -k_4\sigma_4' & -k_5\sigma_5' & k_4\sigma_4' + k_5\sigma_5' \end{pmatrix},$$

where, for $a \sim (i, k)$, $\sigma_a' \equiv \sigma_a'(p(i) - p(k))$. (Recall that $\sigma_a'(-t) = \sigma_a'(t)$.) Either directly from this, or indirectly from the convexity of V_0, we have $\delta = \det(D_pF(p, k)) > 0$. The Cramer's-rule formula for the inverse of the matrix $D_pF(p, k)$ is also useful. It is

$$\delta(D_pF(p, k))^{-1} = \begin{pmatrix} m(2,3;2,3) & -m(1,3;2,3) & m(1,2;2,3) \\ -m(1,3;2,3) & m(1,3;1,3) & -m(1,2;1,3) \\ m(1,2;2,3) & -m(1,2;1,3) & m(1,2;1,2) \end{pmatrix},$$

where

$$m(i, j; k, l) = \frac{\partial(F_i, F_j)}{\partial(p(k), p(l))}.$$

Since the smoothness hypotheses of the Implicit Function Theorem hold (provided $p(1) \neq p(2)$, since then all the q_a will be nonzero so that σ_a is C^1 there), (7.7) gives p as a C^1 function of k with

$$\frac{\partial p(3)}{\partial k_3} = -(0,0,1)(D_pF(p,k))^{-1}\frac{\partial F}{\partial k_3}.$$

Now

$$\frac{\partial F}{\partial k_3} = \sigma_3(p(1)-p(2))(1,-1,0)^{\mathrm{T}},$$

from which

$$\frac{\partial p(3)}{\partial k_3} = -(\det(D_pF(p,k)))^{-1}\sigma_3(p(1)-p(2))M, \tag{7.8}$$

where

$$\begin{aligned} M &= -(0,0,1)\delta(D_pF(p,k))^{-1}(1,-1,0)^{\mathrm{T}}, \\ &= m(1,2;2,3) + m(1,2;2,3), \\ &= k_2\sigma_2'(p(2))k_4\sigma_4'(p(3)-p(1)) - k_1\sigma_1'(p(1))k_5\sigma_5'(p(3)-p(2)). \end{aligned}$$

Now $P = b(3)p(3)$. If $p(3)$ is to be monotonic in k_3, M must change sign at points where $\sigma_3(p(1)-p(2))$ changes sign. Hence $p(1) = p(2)$ implies

$$k_2\sigma_2'(p(2))k_4\sigma_4'(p(3)-p(1)) = k_1\sigma_2'(p(1))k_5\sigma_5'(p(3)-p(2)). \tag{7.9}$$

Since $F_1(p,k) = 0$ and $F_2(p,k) = 0$, $p(1) = p(2)$ also implies

$$k_1\sigma_1(p(1)) = k_4\sigma_4(p(3)-p(1)), \tag{7.10}$$

$$k_2\sigma_2(p(2)) = k_5\sigma_5(p(3)-p(2)). \tag{7.11}$$

With $b(3) > 0$, given $x, y > 0$, there are positive values of k_1, k_2, k_4, k_5 such that $p(1) = x = p(2)$ and $p(3) = x + y$. (We just take k_1 so that $k_1\sigma_1(x) < b(3)$ and define k_2 by

$$k_1\sigma_1(x) + k_2\sigma_2(x) = b(3),$$

and k_4 and k_5 are given by (7.10) and (7.11).) Thus, (7.9), (7.10) and (7.11) give

$$\frac{\sigma_1'(x)}{\sigma_1(x)}\frac{\sigma_5'(y)}{\sigma_5(y)} = \frac{\sigma_2'(x)}{\sigma_2(x)}\frac{\sigma_4'(y)}{\sigma_4(y)}.$$

From this, with $\Lambda_j = \log\sigma_j$, there exists $\lambda \in \mathbb{R}$ such that

$$\frac{\Lambda_1'(x)}{\Lambda_2'(x)} = \lambda \qquad \forall x > 0, \qquad \frac{\Lambda_4'(y)}{\Lambda_5'(y)} = \lambda \qquad \forall y > 0. \tag{7.12}$$

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Rewriting the second of these gives

$$\Lambda_4'(y) = \lambda \Lambda_5'(y) \qquad \forall y > 0,$$

giving

$$\Lambda_4(y) = \lambda \Lambda_5(y) + \mu \qquad \forall y > 0,$$

and

$$\sigma_4(y) = \exp(\mu)(\sigma_5(y))^{\lambda}.$$

The result with arcs 1 and 2 is proved similarly, and the proof of the lemma is then complete.

PROOF OF LEMMA 3. The proof is in two parts, the first of which applies with $\nu \geq 5$.

(i) Let(N_1, A_1) be as in (3.1) with arcs indexed as in Lemma 13, and let $\varphi A_1 \to \mathbb{N}_\nu$ satisfy φ on arc (0, 2) is 2, and φ on arc (0, 1) is $j \neq 2$. By Lemma 13, there are positive λ_j and μ_j such that for all $x \in \mathbb{R}$, $\sigma_j(x) = \mu_j \sigma_2(x)^{\lambda_j}$. By (7.6), since

$$\frac{\sigma_j'(x)}{\sigma_j(x)} = \lambda_j \frac{\sigma_2'(x)}{\sigma_2(x)},$$

if l, m, n and 2 are four elements of $\mathbb{N}_\nu$, then, with λ_2 defined to be 1,

$$\lambda_2 \lambda_m = \lambda_l \lambda_n. \tag{7.13}$$

Equation (7.13) is true for all permutations of the different integers l, m, n. From this and $\lambda_2 = 1$ it follows that $\lambda_j = 1$ for all j. Writing σ for σ_2, for all $j \in \mathbb{N}_\nu$ there is μ_j with $\sigma_j = \mu_j \sigma$.

(ii) We now move to $\nu \geq 6$. Define

$$\begin{aligned} N_2 &= \{0, 1, 2, 3, 4\}, \\ A_2 &= \{(0,4), (4,1), (0,2), (1,2), (1,3), (2,3)\}. \end{aligned}$$

(We have inserted the node 4 into the arc (0, 1). The two arcs $a_1 = (0,4)$, $a_2 = (4,1)$ are in series. The set is essentially a Wheatstone bridge network, with the pair not on the central arc $(1,2)$.) We shall regard this as a two-terminal network with the selected terminals 0 and 3, and $b(4) = 0$. (This is different than conventions elsewhere in this paper and in [15]: it is only used in the proof.) Rather than repeating all steps of an analysis like Lemma 13 on

(N_2, A_2), we apply Lemma 13 to the Wheatstone bridge (N_1, A_1) with $(0, 1)$ having conductivity function $\tilde{\sigma}$ with

$$\tilde{\sigma}^{-1}(q) = \sigma^{-1}(q/h) + \sigma^{-1}(q), \tag{7.14}$$

where $h > 0$ is given. That is, $\tilde{\sigma}^{-1}$ has as its graph the characteristic curve of the two-terminal network (N_3, A_3) where $N_3 = \{0, 4, 1\}$ and $A_3 = \{(0, 4), (4, 1)\}$, $b(4) = 0$, with flow functions $h\sigma$ on $(0, 4)$ and σ on $(4, 1)$. In both the networks (N_1, A_1) and (N_2, A_2), $b(i) = 0$ except when $i = 0$ and $i = 3$. We will have the same relationship between heads, flows, and b in (N_1, A_1), with flow functions $k_1\tilde{\sigma}$ on $(0, 1)$, $k_2\sigma$ on $(0, 2)$, $k_3\sigma$ on $(1, 2)$, $k_4\sigma$ on $(1, 3)$, $k_5\sigma$ on $(2, 3)$, as in (N_2, A_2), with flow functions $k_1 h\sigma$ on $(0, 4)$, $k_1\sigma$ on $(4, 1)$, and exactly the same as in (N_1, A_1) on the other four arcs.

We use, again, the fact that the power is nonincreasing in K_a. Applying Lemma 13 to (N_1, A_1), there is $\theta(h) > 0$ and $\lambda > 0$ such that for $x \in \mathbb{R}$,

$$\tilde{\sigma}(x) = \theta(h)\sigma(x)^{\lambda},$$

and, by (7.13), $\lambda = 1$.

We now apply Lemma 10 to (N_3, A_3) with flow functions $h\sigma$ on $(0, 4)$ and σ on $(4, 1)$. Given potentials $p(0) = 0$ and $p(1)$, the flow q is given by $q = \theta(h)\sigma(p(1))$. There exists a potential $p(4)$ such that on $(0, 4)$ we have $q = h\sigma(p(4))$ and on $(4, 1)$ we have $q = \sigma(p(1) - p(4))$. Hence there is an $s > 0$ such that, for all $t \geq 0$, $\sigma(t) = \sigma(1)t^s$.

Hence, for all $a \in \mathbb{N}_\nu$, $\sigma_a(t) = \sigma_a(1)t^s$. This completes the proof.

REMARK. Work is in progress on the following variation on Theorem 2. Rather than fixing conductivity functions σ_a and varying conductivity factors k_a, it may be appropriate to fix resistance functions ρ_a and vary resistance factors r_a. These can be defined by changing the position of the scalar factor in (2.2) to give $q_a = \sigma_a((p(i) - p(k))/r_a)$. In the context of the pipe-network problem, changes in r_a come from varying the lengths of the pipes.

The analogue of Lemma 10 is as follows. If two arcs with the same resistance function ρ (up to a multiplicative constant) placed in parallel give the same function ρ (up to a multiplicative constant) then ρ is a power law, $\rho(t) = \rho(1)t^{1/s}$ for $t \geq 0$.

An informal way to consider possible variations on Theorem 2 and Lemma 3 is as follows. Suppose that one is given ν (≥ 6) very large rolls of wire, with each roll being identifiable by its colour, say. The rolls of wire are made of rather exotic electrical conductors. For each type of wire w, the form of the resistance function ρ_w is constant along the wire in roll w. By cutting a length l of wire from roll w, one obtains an element with resistance $r(l)\rho_w$ (with larger $r(l)$ corresponding to longer l). Electrical networks can be built making their arcs from lengths l_w of wire from roll w assembled in various kinds of networks. If, no matter which electrical network is built, the power-loss P increases when$r$ is increased, we expect that the network would have to be of type (H). See the authors' 1992 University of Auckland research report for results.

8. Further results

The following theorems (i) show, for the Wheatstone graph, that type (H) is not necessary to have power-loss decreasing in k_a, and (ii) show that $\nu \geq 6$ is necessary in Lemma 3.

THEOREM 14. *Suppose, in addition to the hypotheses of Lemma* 13, *that the conductivity functions on each arc are the same, that is,* $k_a\sigma_a = k_a\sigma$ *for some* σ *satisfying Assumption* $A(\sigma)$. *If it is given that at any* $k \in \mathbb{R}^5$, $k \geq 0$ *satisfying Assumption* $A(k)$, *the power-loss is a decreasing function of any* k_a *then* $\log\sigma$ *is concave on* $(0, \infty)$.

PROOF. Given $0 < p(2) < p(1)$, we let $p(3) = p(1) + p(2)$, and choose k_1, k_2, k_4, k_5 positive so that (7.10) and (7.11) hold, and $k_3 = 0$. Let $0 \leq b(3) = F_3(p, k)$. Since $\partial p(3)/\partial k_3 \leq 0$ at these values of p and k, (7.8) gives $M \geq 0$, which gives $(\Lambda'(p(2)))^2 \geq (\Lambda'(p(1)))^2$ where $\Lambda = \log\sigma$, and hence $\log\sigma$ is concave on $(0, \infty)$.

THEOREM 15. *Consider the Wheatstone bridge graph, and two-terminal flows with* $b(1) = 0 = b(2)$. *Suppose that the conductivity functions on each arc are the same, that is* $k_a\sigma_a = k_a\sigma$ *for some* σ *satisfying Assumption* $A(\sigma)$. *The power-loss does not increase if* k_a *is increased*

(i) *with no further restrictions on* σ, *if* a *is any arc except* $(1, 2)$,

(ii) *if* a *is* $(1, 2)$ *provided* $\log\ \sigma$ *is concave.*

PROOF. Suppose $b(3) > 0$. There are two cases to consider: when a is the central arc $3 = (1, 2)$ and when it is one of the other arcs.

(i) We begin with the case that a is not the central arc. To show $p(3)$ is decreasing in k_j, $j \neq 3$, by symmetry it is enough to let $j = 1$, that is, $a = (0, 1)$. Since

$$\frac{\partial p(3)}{\partial k_1} = -(0,0,1)(D_p F(p,k))^{-1}\frac{\partial F}{\partial k_1},$$

and

$$\frac{\partial F}{\partial k_1} = \sigma(p(1))(1,0,0)^{\mathsf{T}},$$

then

$$\frac{\partial p(3)}{\partial k_1} = -\sigma(p(1))(\det(D_p F(p,k)))^{-1} M_1,$$

where

$$M_1 = m(1,2;2,3) = \det\left(\frac{\partial(F_1, F_2)}{\partial(p(2), p(3))}\right).$$

The fact that $\det(D_p F(p,k)) > 0$ shows that the proof depends on establishing the sign of M_1. Expanding the determinant defining M_1 gives

$$M_1 = k_3\sigma'(p(1) - p(2))k_5\sigma'(p(2) - p(3)) + k_4\sigma'(p(1) - p(3))\frac{\partial F_2}{\partial p(2)},$$

where

$$\frac{\partial F_2}{\partial p(2)} = k_2\sigma'(p(2)) + k_3\sigma'(p(2) - p(1)) + k_5\sigma'(p(2) - p(3)).$$

Since $\sigma' > 0$ this establishes the sign of M_1 and hence shows that $p(3)$ is a decreasing function of k_1.

(ii) Next consider the case $a = (1, 2)$, the central arc. By (7.8), to show $p(3)$ is decreasing in k_3, we need to show $M \geq 0$ for $p(1) > p(2)$, and $M \leq 0$ for $p(1) < p(2)$. Here M is the expression occurring in (7.8). Write $\Lambda = \log\sigma$ and eliminate the σ' terms from M using $\sigma' = \sigma\Lambda'$.

First consider $p(1) > p(2)$. As we have also supposed $b(3) > 0$, we have $p(3) > p(1) > p(2) > p(0) = 0$. We have

$$M = k_1 k_2 \sigma(p(1))\sigma(p(2))(L_1 K_1 + L_2 K_2)$$

where

$$L_1 = (\Lambda'(p(3) - p(1))\Lambda'(p(2)) - \Lambda'(p(1))\Lambda'(p(3) - p(2))),$$

$$L_2 = (\Lambda'(p(1))\Lambda'(p(3) - p(2))),$$
$$K_1 = \left(1 + \frac{k_3\sigma(p(1) - p(2))}{k_1\sigma(p(1))}\right),$$
$$K_2 = \left(\frac{k_3\sigma(p(1) - p(2))}{k_1\sigma(p(1))} + \frac{k_3\sigma(p(1) - p(2))}{k_2\sigma(p(2))}\right).$$

Since Λ' is decreasing, with $p(1) > p(2)$ and hence $p(3)-p(2) > p(3)-p(1)$, one can deduce that $L_1 \geq 0$. Since $p(3) > p(1) > p(2) > p(0) = 0$, both K_1 and K_2 are nonnegative. Since $\sigma' \geq 0$, $L_2 \geq 0$. This establishes that $M \geq 0$ as required.

A similar argument, involving a different reorganisation of the terms of M, can be used to show $M \leq 0$ if $p(1) < p(2)$.

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