8. In a given circle to inscribe a symmetrically irregular hexagon which shall be pericyclic.

9. In a given circle to inscribe a symmetrically irregular heptagon which shall be pericyclic.

10. In a given circle to inscribe a symmetrically irregular octagon which shall be pericyclic.

Geometrical Notes.

By R. E. ALLARDICE, M.A.

1. Some points of difference between polygons with an even number and polygons with an odd number of sides.

A polygon with an odd number of sides is determined when its angles are given and it is such that a circle of given radius may be circumscribed about it; while a polygon with an even number of sides is not *determined by these conditions.*

To prove this it is sufficient to show that in general one polygon and only one with given angles can be inscribed in a given circle if the number of sides be odd; and that if the number of sides be even either it is impossible to inscribe any such polygon or it is possible to inscribe an infinite number.

Let ABCDE (fig. 76) be a polygon with an odd number of sides. In a circle take any point A′; make an arc A′C′ to contain an angle equal to B; an arc C′E′ to contain an angle equal to D; an arc E′B′ to contain an angle equal to A; and so on. Thus the problem is in general a possible one; but it is evident that, though no definite relation among the angles is required (except that connecting the angles of any n-gon), there are limits in which the angles must lie in order that a solution be possible. Thus when the above construction is made, the point E′ must fall between C′ and A′, the point B′, between A′ and C′, and so on.

If the angles are $A_1, A_2, A_3 \ldots\ldots A_{2n+1}$, the necessary and sufficient conditions are of the form

$$A_2 + A_4 + \ldots\ldots + A_{2n} > (n-1)\pi$$
$$A_2 + A \ + \ldots\ldots + A_{2n} + A_1 < n\pi.$$

https://doi.org/10.1017/S0013091500001437 Published online by Cambridge University Press

Consider next a polygon ABCDEF (Fig. 77) with an even number of sides. Take any point A′ in a circle; make the arc A′C′ to contain an angle equal to B; an arc C′E′ to contain an angle equal to D; then if the arc E′A′ contains an angle equal to F, but not otherwise, the problem is possible. Suppose this condition to be satisfied. Take any point B′ in the arc A′C′; make the arc B′D′ to contain an angle equal to C; and the arc D′F′ to contain an angle equal to E. Then A′B′C′D′E′F′ and ABCDEF are equiangular. Hence either there is no solution or there are an infinite number.

The above condition is equivalent to $B + D + F = 2\pi$, or, in the case of a polygon with $2n$ sides,

$$A_2 + A_4 + \ldots\ldots + A_{2n} = (n-1)\pi.$$

It may be shown that the above data involve, in both cases, the number of conditions necessary to determine an n-gon. For $n-3$ conditions must be satisfied in order that a circle may be drawn to circumscribe a given n-gon; and one more, in all $n-2$ conditions, in order that a *given* circle may be so drawn. These with the $n-1$ conditions involved in the giving of the angles, give the $2n-3$ necessary conditions. In the case of a polygon with an even number of sides, one of the conditions that a circle may be drawn to circumscribe it, is expressible in terms of the angles of the polygon alone; while for a polygon with an odd number of sides, this is not the case.

A particular case of the above is that an equi-angular polygon inscribed in a circle is necessarily regular if it have an odd number of sides, but not if it have an even number of sides.

This particular case is given in Charles Hutton's *Miscellanea Mathematica*, London, 1775, p. 271, in the form :—

"An equilateral figure inscribed in a circle is always equiangular.

"But an equiangular figure inscribed in a circle is not always equilateral, except when the number of sides is odd. If the sides be of an even number, then they may be either all equal, or else half of them will always be equal to each other, the equals being placed alternately."

[I am indebted for this reference to Dr Mackay.]

The dual of this theorem, modified slightly, as sides instead of angles are now concerned, is also true, namely—

It is, in general, possible to describe one and only one n-gon with n given lines for sides such that a circle may be inscribed in it, if n

https://doi.org/10.1017/S0013091500001437 Published online by Cambridge University Press

be an odd number; but if n be an even number, either there is no such n-gon or there are an infinite number.

The case of a pentagon may be selected as the method of proof is general.

Let AB, BC, &c. (fig. 78), be the given sides; then points G, H, K, L, M, may be found in one way and in one way only, such that AG = AM, BH = BG, &c.

Let $$AG = x_1,\ BH = x_2,\ \&c.$$
$$AB = a,\ BC = b,\ \&c.$$

Then $$x_1 + x_2 = a \qquad x_2 + x_3 = b \qquad x_3 + x_4 = c$$
$$x_4 + x_5 = d \qquad x_5 + x_1 = d;$$

which equations give unique values for x_1, x_2, &c., namely,

$$2x_1 = a - b + c - d + e,\ \&c.$$

Suppose, now, that a pentagon could be found with these sides such that a circle could be inscribed in it (fig. 78).

Let AOG = α, BOH = β, &c.

It may be shown that α, β, &c., and the radius of the circle may be determined uniquely, as follows:—

It is evident that $\alpha + \beta + \gamma + \delta + \epsilon = \pi$.

Take any line (fig. 79), and from a point P in it cut off parts PQ_1, PQ_2, &c., equal to x_1, x_2, &c.

Draw a perpendicular through P to this line; then we have to find a point R in this line, such that

$$PRQ_1 + PRQ_2 + \ldots\ldots = \pi.$$

It is evident that one such position and only one exists, since when R is at P each of these angles is a right angle, and when R is at an infinite distance each of them is zero, and each of the angles diminishes continuously as R moves away from P.

It is obvious that PR is the radius of the inscribed circle and that PRQ_1, PRQ_2, &c., are the angles subtended at the centre by x_1, x_2, &c.

There is thus one and only one solution of the problem.

If, however, there is an even number of sides (say six) there are the equations

$$x_1 + x_2 = a \qquad x_2 + x_3 = b \qquad x_3 + x_4 = c$$
$$x_4 + x_5 = d \qquad x_5 + x_6 = e \qquad x_6 + x_1 = f$$

which give $$a - b + c - d + e - f = o;$$

so that, unless this relation is satisfied, it is impossible to find any polygon of the kind required; and if it is satisfied, there are an infinite number.

For in AB (fig. 80) take any point G; make BH = BG, CK = CH, ...FN = FM; then, on account of the above relation, AG will be equal to AN.

There are thus an infinite number of ways of dividing the sides as required; and, corresponding to each method of division, it is possible to construct a hexagon in which a circle may be inscribed, by finding the radius of the inscribed circle and the angles subtended at the centre by the segments of the sides, as in the case of a polygon of an odd number of sides.

Corollary. An equilateral polygon circumscribing a circle is necessarily regular, if it have an odd number of sides; but if it have an even number, this is not necessarily the case. In the latter case, however, alternate angles are equal.

It is obvious that an equiangular polygon circumscribing a circle is necessarily regular.

Some other cases of differences between polygons with an even and those with an odd number of sides are well known. Thus if all the sides of a polygon of the first kind are produced, a crossed polygon is formed; while if the sides of one of the second kind are produced, two polygons are formed.

There are also differences in the formation of the different crossed n-gons with the sides of a given n-gon, according as n is a prime number or not.

Again, when the middle points of the sides of a polygon are given, one and only one polygon can in general be formed, if the number of sides is odd; but either none or an infinite number, if the number of sides is even. [Dr Mackay gives me a reference for this theorem to Catalan's *Théorèmes et Problèmes de Géométrie Elémentaire* (6th ed., pp. 10, 11); and Catalan gives a reference to Prouhet, *Nouvelles Annales*, tome III. The latter reference I have verified.]

The treatment of this latter question by means of co-ordinates is perhaps of some interest.

Let (x_1, y_1), (x_2, y_2), &c, be the co-ordinates of the vertices required; (a_1, b_1), (a_2, b_2), &c., those of the middle points.
Then

$$x_1 + x_2 = 2a_1$$
$$x_2 + x_3 = 2a_2, \text{ \&c.}$$

https://doi.org/10.1017/S0013091500001437 Published online by Cambridge University Press

These equations have a determinate solution if n is odd, namely,

$$x_1 = a_1 - a_2 + a_3 - a_4 + \dots\dots, \text{ \&c.};$$

but if n is even there is no solution unless

$$a_1 - a_2 + a_3 - a_4 + \dots\dots = 0$$

or

$$a_1 + a_3 + \dots\dots = a_2 + a_4 + \dots\dots,$$

which means that the centres of gravity of the two polygons, got by taking the given points alternately, coincide. If this condition is satisfied, there are an infinite number of solutions. It may be noted that the given points may be taken in any order, with the exception that when n is even, the points taken once in odd order must always be taken in odd order.

II. If a straight line (fig. 81) meets the three diagonals AC, BD, EF, of a complete quadrilateral in the points L, M, N, and if L′, M′, N′, are the harmonic conjugates of L, M, N, with respect to A and C, B and D, E and F respectively, then L′, M′, N′, are collinear.

Since P and Q, and also M and M′, are harmonically conjugate with respect to B and D, DM′PBQM is an involution of which B and D are the double points.

Hence $$(\text{BMDQ}) = (\text{BM}'\text{DP}),$$

that is, $$\frac{\text{BD.MQ}}{\text{BQ.MD}} = \frac{\text{BD.M}'\text{P}}{\text{BP.M}'\text{D}};$$

$$\therefore \quad \frac{\text{M}'\text{P}}{\text{MQ}} = \frac{\text{BP}}{\text{BQ}} \cdot \frac{\text{M}'\text{D}}{\text{MD}}.$$

Similarly from the relation

$$(\text{BM}'\text{DQ}) = (\text{BMDP}),$$

$$\frac{\text{MP}}{\text{M}'\text{Q}} = \frac{\text{BP}}{\text{BQ}} \cdot \frac{\text{MD}}{\text{M}'\text{D}};$$

$$\therefore \quad \frac{\text{MP}}{\text{MQ}} = \frac{\text{M}'\text{Q}}{\text{M}'\text{P}} \cdot \frac{\text{BP}^2}{\text{BQ}^2}.$$

$$\therefore \quad \frac{\text{MP}}{\text{MQ}} \cdot \frac{\text{LR}}{\text{LP}} \cdot \frac{\text{NQ}}{\text{NR}} = \frac{\text{M}'\text{Q}}{\text{M}'\text{P}} \cdot \frac{\text{L}'\text{P}}{\text{L}'\text{R}} \cdot \frac{\text{N}'\text{R}}{\text{N}'\text{Q}} \cdot \left(\frac{\text{BP}}{\text{BQ}} \cdot \frac{\text{CR}}{\text{CP}} \cdot \frac{\text{EQ}}{\text{ER}}\right)^2$$

Now $$\frac{\text{MP}}{\text{MQ}} \cdot \frac{\text{LR}}{\text{LP}} \cdot \frac{\text{NQ}}{\text{NR}} = 1, \text{ and } \frac{\text{BP}}{\text{BQ}} \cdot \frac{\text{CR}}{\text{CP}} \cdot \frac{\text{EQ}}{\text{ER}} = 1;$$

$$\therefore \quad \frac{\text{M}'\text{Q}}{\text{M}'\text{P}} \cdot \frac{\text{L}'\text{P}}{\text{L}'\text{R}} \cdot \frac{\text{N}'\text{R}}{\text{N}'\text{Q}} = 1;$$

and hence, L′, M′, N′, are collinear.

Cor. 1. The middle points of the diagonals are collinear.

https://doi.org/10.1017/S0013091500001437 Published online by Cambridge University Press

Cor. 2. If the first corollary be proved independently, the theorem may be deduced from it by projection.

Cor. 3. The dual of this theorem is also true. It may be stated as follows:—If lines l, m, n, be drawn through any point to the three diagonal points of a complete quadrangle, and the harmonic conjugates l', m', n', of l, m, n, be taken with respect to the sides of the quadrangle that pass through these points, then l', m', n', will be concurrent.

Sixth Meeting, April 8th, 1887.

J. S. MACKAY, Esq., LL.D., in the Chair.

On the value of $\Delta^n 0^m / n^m$, when m and n are very large.

By Professor TAIT.

I had occasion, lately, to consider the following question connected with the Kinetic Theory of Gases:—

Given that there are 3.10^{20} particles in a cubic inch of air, and that each has on the average 10^{10} collisions per second; after what period of time is it even betting that any specified particle shall have collided, once at least, with each of the others?

The question obviously reduces to this:—Find m so that the terms in

$$X^m, = (x_1 + x_2 + x_3 + \dots\dots + x_n)^m$$

which contain each of the n quantities, once at least, as a factor, shall be numerically equal to half the whole value of the expression when $x_1 = x_2 = \dots\dots = x_n = 1$. Thus we have

$$X^m - \Sigma(X - x_r)^m + \Sigma(X - x_r - x_s)^m - \dots\dots = \tfrac{1}{2}X^m$$

or

$$\Delta^n 0^m / n^m = \tfrac{1}{2}.$$

It is strange that neither Herschel, De Morgan, nor Boole, while treating differences of zero, has thought fit to state that Laplace had, long ago, given all that is necessary for the solution of such questions. The numbers $\Delta^n 0^m$ are of such importance that one would naturally expect to find in any treatise which refers to them at least a state-