BULL. AUSTRAL. MATH. SOC. 30C50
VOL. 46 (1992) [391–400]

ON THE SCHWARZIAN COEFFICIENTS OF UNIVALENT FUNCTIONS

STEPHEN M. ZEMYAN

Dedicated to Glenn E. Schober

For $f \in S$, we study the Schwarzian coefficients s_n defined by $\{f, z\} = \sum s_n z^n$. Sharp bounds on s_0, s_1 and s_2 are given, together with an order of growth estimate as $n \to \infty$. We use the Grunsky Inequalities to estimate combinations of coefficients.

1. DEFINITIONS AND EXAMPLES

Let S denote the class of functions $f(z) = z + a_2 z^2 + \dots$ which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$.

The *Schwarzian derivative* of a function $f(z)$ in S is defined by the relation

$$\{f, z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{1}$$

and the *Schwarzian coefficients* of the function $f(z)$ are the Taylor coefficients in the series expansion

$$\{f, z\} = \sum_{n=0}^{\infty} s_n z^n . \tag{2}$$

By way of example, we consider the Koebe function

$$k(z) = \frac{z}{(1-z)^2} \tag{3}$$

and its mth root transform

$$k_m(z) = \frac{z}{(1-z^m)^{2/m}} \cdot \quad (m \geqslant 2) \tag{4}$$

Received 13th November, 1991
The author wishes to express his appreciation to D. Saunders for his assistance in computing the formulas (7) and to Y.J. Leung for pointing out Jenkins' Inequality which was used to prove (20).

Both of these functions belong to S and their geometric and analytic properties have been described in detail in numerous references [**2**, **8**, **10**]. Elementary computations show that

$$\{k,z\} = \frac{-6}{(1-z^2)^2} = \sum_{n=0}^{\infty} -6(n+1)z^{2n} \tag{5}$$

and that

$$\begin{aligned}\{k_m,z\} &= 2z^{m-2}\left\{\frac{m^2}{(1+z^m)^2} - \frac{1}{(1-z^m)^2}\right\}\\ &= \sum_{n=0}^{\infty} 2(n+1)[(-1)^n m^2 - 1]z^{mn+m-2} \quad (m \geqslant 2)\end{aligned} \tag{6}$$

clearly displaying the Schwarzian coefficients of k and k_m.

Each Schwarzian coefficient may be written in terms of the coefficients of $f(z)$. Indeed, using MACSYMA, one quickly obtains:

$$\begin{aligned}s_0 &= 6(a_3 - a_2^2)\\ s_1 &= 24(a_4 - 2a_2a_3 + a_2^3)\\ s_2 &= 12(5a_5 - 10a_2a_4 - 6a_3^2 + 17a_2^2a_3 - 6a_2^4)\\ s_3 &= 24(5a_6 - 10a_2a_5 - 13a_3a_4 + 18a_2^2a_4 + 21a_2a_3^2 - 29a_2^3a_3 + 8a_2^5)\end{aligned} \tag{7}$$

and

$$\begin{aligned}s_4 = 6\,(&35a_7 - 70a_2a_6 - 95a_3a_5 + 130a_2^2a_5 - 52a_4^2 + 328a_2a_3a_4\\ &- 224a_2^3a_4 + 63a_3^3 - 387a_2^2a_3^2 + 352a_2^4a_3 - 80a_2^6)\end{aligned}$$

Since it is well known [**8**, p.20] that $|a_3 - a_2^2| \leqslant 1$, we easily have $|s_0| \leqslant 6$. However, obtaining precise bounds on succeeding coefficients, order of growth estimates, *et cetera*, in terms of the above representations is unnecessarily difficult, if not impossible. For this reason, we introduce alternate representations for the Schwarzian coefficients.

Many of the results of this paper will depend upon the close relationship between the Schwarzian coefficients s_n and the Grunsky coefficients c_{nk} of a function $f(z)$ belonging to S, which are generated from $f(z)$ by setting

$$\Phi(z,\zeta) = \log\left(\frac{f(z)-f(\zeta)}{z-\zeta}\right) = +\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{nk}z^n\zeta^k. \tag{8}$$

Note that Φ is analytic in $U \times U$ since $f(z)$ is univalent in U.

A tedious computation shows that

$$\{f,z\} = 6 \cdot \lim_{\zeta\to z} \frac{\partial^2\Phi(z,\zeta)}{\partial z\partial\zeta}\,, \tag{9}$$

and, indeed, (9) may be taken as an alternate definition of the Schwarzian derivative.

Substituting (8) into (9), and then comparing the result with (2), we see that

$$s_n = 6\sum_{k=1}^{n+1} k(n+2-k)c_{k,n+2-k}. \tag{10}$$

Thus, each Schwarzian coefficient is a "slanted" linear combination of Grunsky coefficients. Using the explicit formulas developed by Hummel [**5**, p.147] or Todorov [**12**, p.437] for the Grunsky coefficients c_{nk} of $f(z)$ in terms of the coefficients a_n of $f(z)$, it is possible to express *all* Schwarzian coefficients s_n in terms of the coefficients a_n, as we had begun to do in (7).

It will be very useful to define Schwarzian coefficients for another related class of functions. Let $\sum$ denote the class of functions $g(z)$ of the form

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n} \tag{11}$$

which are meromorphic and univalent in $\Delta = \{z : |z| > 1\}$. Analogously, the coefficients γ_{nk} generated by the relation

$$\Psi(z,\zeta) = \log\left(\frac{g(z)-g(\zeta)}{z-\zeta}\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \gamma_{nk} z^{-n}\zeta^{-k} \tag{12}$$

are called the Grunsky coefficients of $g(z)$. The Schwarzian derivative of a function $g \in \sum$ may be defined [**10**, p.117] alternately as either

$$\{g,z\} = \left(\frac{g''(z)}{g'(z)}\right)' - \frac{1}{2}\left(\frac{g''(z)}{g'(z)}\right)^2 \tag{13}$$

or

$$\{g,z\} = 6 \cdot \lim_{\zeta\to z} \frac{\partial^2 \Psi(z,\zeta)}{\partial z \partial \zeta}. \tag{14}$$

Also, the Schwarzian coefficients σ_n of $g(z)$ are defined to be the coefficients in the Laurent expansion

$$\{g,z\} = z^{-4}\sum_{n=0}^{\infty} \sigma_n z^{-n}. \tag{15}$$

By substituting (12) into (14), we obtain the relation

$$\sigma_n = 6\sum_{k=1}^{n+1} k(n+2-k)\gamma_{k,n+2-k}. \tag{16}$$

Since Schur [**11**, p.41], Todorov [**12**, p.435] and Harmelin [**3**, p.355] have given explicit formulas for the Grunsky coefficients γ_{nk} in terms of the coefficients b_n of $g(z)$, one may express all Schwarzian coefficients σ_n in terms of the coefficients b_n as well. Indeed, we have

$$\begin{aligned}
\sigma_0 &= -6b_1 \\
\sigma_1 &= -24b_2 \\
\sigma_2 &= -12(5b_3 + b_1^2) \qquad (17)\\
\sigma_3 &= -24(5b_4 + 3b_1b_2) \\
\text{and} \quad \sigma_4 &= -6(35b_5 + 25b_1b_3 + 17b_2^2 + 3b_1^3)
\end{aligned}$$

and so forth. These formulas will be useful in obtaining coefficient bounds.

We close this section by pointing out a relationship between the coefficients s_n and σ_n. If $f(z) \in S$, then $g(z) = 1/f(1/z) \in \sum$. Conversely, if $g(z) \in \sum$ and $g(z) + c \neq 0$ for all $z \in \Delta$ and some complex number c, then there exists $f_c(z) \in S$ such that $g(z) + c = 1/f_c(1/z)$. Consequently, we have the following

PROPOSITION 1. *Let $f \in S$ and $g \in \sum$. If $g(z) = 1/f(1/z)$ or if $g(z) + c = 1/f_c(1/z) \neq 0$ for some complex number c, then*

(a) $$c_{nk}(f) = \gamma_{nk}(g) = \gamma_{nk}(g + c) = c_{nk}(f_c) \qquad (n, k > 1)$$

and

(b) $$s_n(f) = \sigma_n(g) = \sigma_n(g + c) = s_n(f_c) \qquad (n \geqslant 0)$$

PROOF: Since

$$\log\left(\frac{g(1/z) - g(1/\zeta)}{1/z - 1/\zeta}\right) = \log\left(\frac{f(z) - f(\zeta)}{z - \zeta}\right) - \log\left(\frac{f(z)}{z}\right) - \log\left(\frac{f(\zeta)}{\zeta}\right) \quad (z, \zeta \in U)$$

the coefficients $\gamma_{nk}(g) = c_{nk}(f)$ for all $n, k \geqslant 1$, whenever $g(z) = 1/f(1/z)$. Part (b) follows from (10) and (16). ∎

2. BOUNDS ON COEFFICIENTS, ORDER OF GROWTH

By using well-known results, it is easy to prove

THEOREM 1. *Let s_n and σ_n denote the Schwarzian coefficients of $f \in S$ and $g \in \sum$, respectively. Then we have the following coefficient bounds:*

(a) $$\max_{g \in \sum} |\sigma_0(g)| = \max_{f \in S} |s_0(f)| = 6$$

https://doi.org/10.1017/S0004972700012053 Published online by Cambridge University Press

Equality holds in $\sum$ *if and only if* $g(z) = g_{c,\eta} = z + c + \eta^2/z$, *where* $|\eta| = 1$ *and* c *is an arbitrary complex number. Equality holds in* S *if and only if* $f(z) = f_{\alpha,\eta}(z) = z/(1 + 2\alpha\eta + \eta^2 z^2)$, *where* $|\eta| = 1$ *and* $\alpha \in [-1, +1]$.

(b) $$\max_{g\in\sum} |\sigma_1(g)| = \max_{f\in S} |s_1(f)| = 16$$

Equality holds in $\sum$ *if and only if* $g(z) = g_{c,\eta}(z) = z(1 - \eta^3 z^{-3})^{2/3} + c$, *where* $|\eta| = 1$ *and* c *is an arbitrary complex number. Equality holds in* S *if and only if* $f(z) = f_{c,\eta}(z) = 1/g_{c,\eta}(1/z)$ *where* c *is chosen so that* $g_{c,\eta}(z) \neq 0$. *(If* $c = 0$, *then* $f_{0,\eta}(z) = \overline{\eta} k_3(\eta z)$.*)*

PROOF: (a) If $g(z) = z + b_0 + b_1 z^{-1} + \ldots \in \sum$, then a well-known [4, p.348] consequence of Gronwall's Outer Area Theorem is that $|b_1| \leqslant 1$, and that equality holds if and only if $g(z) = g_{c,\eta}(z) = z + c + \eta^2/z$, where $|\eta| = 1$. Consequently, $|\sigma_0(g)| = 6\ |b_1(g)| \leqslant 6$, and $|\sigma_0(g_{c,\eta})| = 6$, for arbitrary choice of c. The function $g_{c,\eta}$ maps Δ onto the whole plane slit along the segment $[-2\eta + c, +2\eta + c]$. If we choose $c = 2\alpha\eta, \alpha \in [-1, +1]$, then $g_{2\alpha\eta,\eta}(z) \neq 0$ for all $z \in \Delta$. Thus, by Proposition 1, $|s_0(f_{\alpha,\eta})| = |\sigma_0(g_{2\alpha\eta,\eta})| = 6$, where $f_{\alpha,\eta}(z) = 1/g_{2\alpha\eta,\eta}(1/z) = z/(1 + 2\alpha\eta z + \eta^2 z^2)$. Also, if $f \in S$ and $f \neq f_{\alpha,\eta}$, then $|s_0(f)| < 6$, since $g(z) = 1/f(1/z) \in \sum$ and $g \neq g_{c,\eta}$ for any choice of c and η. (b) Schiffer [9] established the result that if $g(z) = z + b_1 z^{-1} + \ldots \in \sum$, then $|b_2| \leqslant 2/3$. Equality holds here if and only if $g(z) = g_\eta(z) = z(1 - \eta^3 z^{-3})^{2/3}, |\eta| = 1$. Consequently, $|\sigma_1(g)| = 24\,|b_2(g)| \leqslant 16$, and $|\sigma_1(g_{c,\eta})| = 16$, where $g_{c,\eta}(z) = g_\eta(z) + c$. The function $g_\eta(z) = \eta/k_3(\eta/z)$ maps Δ onto the complement of three equally spaced line segments each of length $2^{2/3}$ emanating from the origin. If $-c$ lies on any of these line segments, then $g_{c,\eta}(z) \neq 0$, and by Proposition 1, $|s_1(f_{c,\eta})| = |\sigma_1(g_{c,\eta})| = 16$, where $f_{c,\eta}(z) = 1/g_{c,\eta}(1/z)$. Finally, if $f \in S$ and $f \neq f_{c,\eta}$, then $|s_1(f)| < 16$, since $g(z) = 1/f(1/z) \neq g_{c,\eta}(z)$ for any choice of c and η. ∎

On the basis of this theorem and the series $\{k_{n+2}, z\} = 2(n+1)(n+3)z^n + \ldots$, we are tempted to formulate the conjecture that, for each $n \geqslant 0$,

(18) $$\max_{g\in\sum} |\sigma_n(g)| = \max_{f\in S} |s_n(f)| = 2(n+1)(n+3)\,.$$

However, it is possible to show that this conjecture is *false* for every even $n \geqslant 2$. The odd case will be dealt with in a later paper.

THEOREM 2. *For every $n \geqslant 1$, there exists $\widehat{f}_n \in S$ and $\widehat{g}_n \in \sum$ such that*

$$(19) \qquad |\sigma_{2n}(\widehat{g}_n)| = \left|s_{2n}\left(\widehat{f}_n\right)\right| = 2(2n+1)(2n+3)\left(1+2e^{-\beta_n}\right)$$

where

$$\beta_n = \frac{26}{3} + \frac{8}{n(n+2)} .$$

Furthermore,

$$(20) \qquad \max_{g \in \sum} |\sigma_2(g)| = \max_{f \in S} |s_2(f)| = 30\left(1+2e^{-\beta_1}\right) = 30.00071804\ldots .$$

PROOF: Let $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in S$, and let $m \geqslant 2$. Then, the mth root transform $f_m(z) = z\left(1 + a_2 z^m + a_3 z^{2m} + \ldots\right)^{1/m}$ also belongs to S, and in series form, we have

$$f_m(z) = z + \frac{a_2}{m} z^{m+1} + \left(\frac{a_3}{m} - \frac{(m-1)}{2m^2} a_2^2\right) z^{2m+1} + \ldots .$$

Consequently,

$$\{f_{n+1}, z\} = n(n+2) a_2 z^{n-1} + 2(2n+1)(2n+3)\left(a_3 - \alpha_n a_2^2\right) z^{2n} + \ldots$$

where

$$\alpha_n = \frac{13n^2 + 26n + 12}{4(2n+1)(2n+3)} .$$

We now invoke the Fekete–Szegö Theorem [**8**, p.165] to conclude that

$$\left|a_3 - \alpha_n a_2^2\right| \leqslant 1 + 2e^{-2\alpha_n/(1-\alpha_n)} = 1 + 2e^{-\beta_n}$$

for all $f \in S$, and that there exists a function $\widehat{f}_n \in S$ for which equality holds. This establishes (19), where we choose $\widehat{g}_n(z) = 1/\widehat{f}_n(1/z)$.

To prove (20), we use a known result of Jenkins [**6**] which states that for any function $g(z) = z + b_0 + b_1 z^{-1} + \ldots \in \sum$, the inequality $\left|2b_3 + \alpha b_1^2\right| \leqslant 1 + 2e^{-\beta}$ is valid, where $\alpha \in [0,1)$ and $\beta = (6+2\alpha)/(1-\alpha)$, and that there exists a function $g_\alpha \in \sum$ for which equality holds. If $\alpha = 2/5$, then $\beta = 34/3 = \beta_1$. Hence,

$$|\sigma_2(g)| = 30\left|2b_3(g) + \frac{2}{5} b_1^2(g)\right| \leqslant 30\left(1 + 2e^{-34/3}\right) = \left|\sigma_2\left(g_{2/5}\right)\right| .$$

If c is chosen so that $g_{2/5}(z) + c \neq 0$, then $\left|s_2\left(f_{c,2/5}\right)\right| = \left|\sigma_2\left(g_{2/5} + c\right)\right| = \left|\sigma_2\left(g_{2/5}\right)\right|$, where $g_{2/5}(z) + c = 1/f_{c,2/5}(1/z)$, by Proposition 1. ∎

The next result is concerned with the order of growth of the Schwarzian coefficients.

https://doi.org/10.1017/S0004972700012053 Published online by Cambridge University Press

Proposition 2. *Let*

$$m_n = \max_{f \in S} \{ |s_n(f)| / n^2 \}$$

Then,

(21) $$2.000688\ldots = 2 + 4e^{-26/3} \leqslant \overline{\lim_{n \to \infty}}\, m_n \leqslant 3\pi/4 = 2.356194\ldots$$

Proof: We consider first the inequality on the left. For the function $\widehat{f}_n$, defined within the proof of Theorem 2, we conclude from (19) that

$$\frac{\left| s_{2n}\left(\widehat{f}_n\right) \right|}{(2n)^2} = 2\left(1 + \frac{1}{2n}\right)\left(1 + \frac{3}{2n}\right)\left(1 + 2e^{-\beta_n}\right) \leqslant m_{2n}\,,$$

which justifies the conclusion, since $\beta_n \to -26/3$ as $n \to \infty$.

To obtain an upper estimate on $|s_n|$, we use a well-known [**10**, p.119] estimate for the Grunsky coefficients. Since $|c_{km}| \leqslant 1/\sqrt{km}$ for all $k, m \geqslant 1$, we have

(22) $$|s_n(f)| = \left| 6 \sum_{k=1}^{n+1} k(n+2-k) c_{k,n+2-k}(f) \right| \leqslant 6 \sum_{k=1}^{n+1} \sqrt{k(n+2-k)}$$

Since the points $\left(k, \sqrt{k(n+2-k)}\right)$ lie on a circle, a careful geometric estimate (using trapezoids) allows us to conclude that

(23) $$|s_n(f)| < \frac{3\pi}{4}(n+2)^2$$

for every $f \in S$, and every $n \geqslant 0$. Hence, $m_n < (3\pi/4)((n+2)/n)^2$ for every n, and the right side of inequality (21) has been verified. ∎

For small n, it is convenient to use (18), (19) and (22) to provide estimates. For example,

$$48 \leqslant \max_{f \in S} |s_3(f)| \leqslant 12\left(2 + \sqrt{6}\right) = 53.393876\ldots$$

and

$$70.008870\ldots = 70 + 140e^{-29/3} \leqslant \max_{f \in S} |s_4(f)| \leqslant 6\left(3 + 4\sqrt{2} + 2\sqrt{5}\right) = 78.773941\ldots$$

For large n, it is more convenient to use (23).

The functions with which we are concerned here, $\widehat{f}_n$ and k_n, are unbounded. For an example of a bounded function, see [**2**, p.83].

3. Linear Combinations

Several techniques are useful in deriving inequalities involving linear combinations of Schwarzian coefficients. One method involves the use of classical inequalities. We first prove

Theorem 3. *Let $f \in S$ and let $s_n (n = 0, 1, 2, \dots)$ denote the Schwarzian coefficients of f. Then,*

$$\left| \sum_{n=0}^{N} (N+1-n) s_n z^n \right| \leqslant 6 \sum_{n=0}^{N} (N+1-n)(n+1) |z|^{2n} \tag{24}$$

for every integer $N \geqslant 0$ and every complex z. In particular, for $z = 1$, we get

$$\left| \sum_{n=0}^{N} (N+1-n) s_n \right| \leqslant (N+1)(N+2)(N+3). \tag{25}$$

Proof: Using (10), we rearrange coefficients to obtain

$$\begin{aligned} \sum_{n=0}^{N} (N+1-n) s_n z^{n+2} &= 6 \sum_{n=0}^{N} (N+1-n) \left(\sum_{k=1}^{n+1} k(n+2-k) c_{k,n+2-k} z^k z^{n+2-k} \right) \\ &= 6 \sum_{r=1}^{N+1} \left(\sum_{k=1}^{N+2-r} \sum_{j=1}^{r} c_{kj} (kz^k)(jz^j) \right) . \end{aligned}$$

Applying the Generalized Weak Grunsky Inequalities [**1**, p.124], and the Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned} \left| \sum_{n=0}^{N} (N+1-n) s_n z^{n+2} \right| &\leqslant 6 \sum_{r=1}^{n+1} \left| \sum_{k=1}^{N+2-r} \sum_{j=1}^{r} c_{kj} (kz^k)(jz^j) \right| \\ &\leqslant 6 \sum_{r=1}^{n+1} \left(\sum_{k=1}^{N+2-r} k |z|^{2k} \right)^{1/2} \left(\sum_{j=1}^{r} j |z|^{2j} \right)^{1/2} \\ &\leqslant 6 \left(\sum_{r=1}^{N+1} \sum_{k=1}^{N+2-r} k |z|^{2k} \right)^{1/2} \left(\sum_{r=1}^{N+1} \sum_{j=1}^{r} j |z|^{2j} \right)^{1/2} \\ &= 6 \sum_{n=0}^{N} (N+1-n)(n+1) |z|^{2n} . \end{aligned}$$

∎

https://doi.org/10.1017/S0004972700012053 Published online by Cambridge University Press

The inequalities (24) and (25) are sharp only if $N = 0$. The reasons are two fold: The Grunsky Inequalities are used here in $[(N+2)/2]$ *different* ways; also, the conditions required for equality to hold when using the Cauchy–Schwarz inequality are not fulfilled. Nevertheless, inequality (25) is still worthwhile, since it is an order of magnitude better than that which could have been obtained by using (23).

We now consider the partial sums of the Schwarzian series.

COROLLARY 3.1. *Let $f \in S$ and let s_n $(n = 0, 1, 2, \dots)$ denote the Schwarzian coefficients of f. Then,*

(26)
$$\left|\sum_{n=0}^{N} s_n z^n\right| \leqslant 6\sum_{n=0}^{N}(n+1)|z|^{2n} + \frac{1}{N+1}\cdot\left\{\frac{3\pi}{4}\sum_{n=0}^{N} n(n+2)^2|z|^n - 6\sum_{n=0}^{N} n(n+1)|z|^{2n}\right\}$$

for every integer $N \geqslant 0$ and every complex z. In particular, if $z = 1$, then

$$\left|\sum_{n=0}^{N} s_n\right| \leqslant \frac{3\pi}{16}N^3 + \left(1 + \frac{19\pi}{16}\right)N^2 + (4 + 2\pi)N + 6.$$

Also, if $|z| < 1$, then as $N \to \infty$, we obtain the classical inequality of Kraus [7]

$$|\{f, z\}| \leqslant \frac{6}{\left(1 - |z|^2\right)^2}. \tag{27}$$

PROOF: To obtain (26), divide (24) by $N + 1$, and use (23) to estimate s_n. Since the remainder term is $0(1/N)$, (27) follows immediately. ▯

This next corollary may be used to estimate efficiently moment integrals of the form $\int z^{k-1}\{f, z\}\, dz$.

COROLLARY 3.2. *Let $f \in S$ and let s_n $(n = 0, 1, 2, \dots)$ denote the Schwarzian coefficients of f. For every $N \geqslant 0$ and $k \geqslant 1$,*

$$\left|\sum_{n=0}^{N} \frac{s_n}{n+k} z^n\right| \leqslant 6\sum_{n=0}^{N} \frac{n+1}{2n+k}|z|^{2n} + R_N(|z|) \tag{28}$$

where $R_N(|z|)$ is a finite sum depending on $|z|$ and N alone, and $R_N(|z|) = 0(1/N)$ as $N \to \infty$. If $|z| < 1$, then, for every positive integer $k \geqslant 1$,

$$\left|\sum_{n=0}^{\infty} \frac{s_n}{n+k} z^n\right| \leqslant 6\sum_{n=0}^{\infty} \frac{n+1}{2n+k}|z|^{2n}. \tag{29}$$

PROOF: Multiply (24) by z^{k-1}, and integrate the result. Divide by $N + 1$, rearrange terms and use (26) to obtain (28). Let $N \to \infty$ to obtain (29). ▯

We note finally that the series in (29) may be evaluated in closed form. If we write

$$F_k(|z|) = 6\sum_{n=0}^{\infty} \frac{n+1}{2n+k} |z|^{2n} ,$$

then

$$F_1(|z|) = \frac{3}{\left(1-|z|^2\right)} + \frac{3}{2|z|}\log\left(\frac{1+|z|}{1-|z|}\right),$$

$$F_2(|z|) = \frac{3}{\left(1-|z|^2\right)}$$

and, for $k \geqslant 3$, we give the estimate

$$F_3(|z|) \leqslant \frac{3}{\left(1-|z|^2\right)} - (k-2)\left[\frac{3}{k} - \frac{k\pi^2}{8} + \frac{3}{2}\log\left(\frac{1}{1-|z|^2}\right)\right].$$

References

[1] P.L. Duren, *Univalent functions* (Springer–Verlag, Berlin, Heidelberg and New York, 1983).

[2] G.M. Goluzin, *Geometric theory of functions of a complex variable*, English Translation (Amer. Math. Soc., Providence, R.I., 1969).

[3] R. Harmelin, 'Generalized Grunsky coefficients and inequalities', *Israel J. Math.* **57** (1987), 347-364.

[4] E. Hille, *Analytic Function Theory II* (Chelsea, New York, 1962).

[5] J.A. Hummel, 'The Grunsky coefficients of a schlicht function', *Proc. Amer. Math. Soc.* **15** (1964), 142–150.

[6] J.A. Jenkins, 'On certain coefficients of univalent functions', in *Analytic functions* (Princeton University Press, Princeton, N.J., 1960).

[7] W. Kraus, 'Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereiches mit der Kreisabbildung', *Mitt. Math. Sem. Giessen* **21** (1932), 1–28.

[8] Ch. Pommerenke, *Univalent Functions* (Vandenhoeck and Ruprecht, Göttingen, 1975).

[9] M. Schiffer, 'Sur un problème d'extremum de la représentation conforme', *Bull. Soc. Math. France* **66** (1938), 48–55.

[10] G. Schober, *Univalent Functions–Selected Topics*: *Lecture Notes in Math.* 478 (Springer–Verlag, Berlin, Heidelberg, New York, 1975).

[11] I. Schur, 'On Faber polynomials', *Amer. J. Math.* **67** (1945), 33–41.

[12] P. Todorov, 'Explicit Formulas for the coefficients of Faber polynomials with respect to univalent functions of the class $\sum$'', *Proc. Amer. Math. Soc.* **82** (1981), 431–438.

Department of Mathematics
Penn State University
Mont Alto PA 17237-9799
United States of America