A mathematical promenade along parallel paths

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Introduction

Imagine that you go for a walk with your friend. You walk along a closed loop around a university campus, finishing at the point where you started. You and your friend keep the same distance throughout your walk. You walk on the 'outside'.

How much longer would your path be than your friend's? What about if you two go for a 15 km hike around a big lake? The answer is quite simple and for a non-mathematician possibly unexpected: it does not matter how long is your loop, you would always walk just as much more as if you circled once around your friend at a given distance. The statement sensibly generalises to paths that are not closed and even not smooth. And we could say, these beautiful realisations intuitively and essentially follow from the definition of π, indeed from the elementary fact that the circumference and radius of a circle are directly proportional.

The result is well known to mathematicians and especially to differential geometry specialists (compare for example [1, p. 47, exercise 6]), but, contrary to its beauty, intuitive accessibility and motivational power, remain widely unknown to many curious mathematics teachers and students.

1. *Parallel paths and their length*

Parallel lines are a well-known concept from elementary school. Parallel paths are mathematically more sophisticated to define but their intuitive meaning is pretty obvious. For example, two promenaders may walk along parallel paths even when the paths are winding. Also intuitively obvious are the parallel paths that left and right wheels of a car travel (when a car moves along a winding road). For simplicity, we shall consider only parallel paths in a plane. While it is obvious that the travelled distance is the same if we observe two parallel lines, it is interesting to ask the following question: What is the difference in lengths between the two paths travelled by the right and left wheels of a car? The idea of comparing the lengths of parallel paths might be encountered already within a very simple question in elementary geometry:

> Assume the planet Earth is a perfect sphere and we put a ring around the equator which is 100 m longer than the equator. The ring is positioned equidistantly all around the equator. Is there enough space for a cat to slip through under the ring? What about for a mouse?

For a 'conscious mathematician', the answer is clearly affirmative: even a giraffe could easily walk under the ring. The answer follows from the elementary geometric equation

$$2\pi(R + d) = 2\pi R + 100,$$

where R is the radius of the sphere (Earth) and d is the distance between the

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equator and the ring. From the above equation we get $d \approx 16$ m, which is mathematically no surprise as the radius (R) and the circumference (C) of a circle are proportional and $C = 2\pi R$. We can see our equator and the ring as two parallel paths, which would be travelled by two wheels moving on an imaginary plane, the inside wheel along a circle (of equator size) and the outer at a distance d. The outer wheel would travel only for $2\pi d$ longer distance independently of the size of the circle. That seems obvious for the two wheels travelling along any circle. What about if the parallel paths were more complicated? If the two parallel paths consist of four straight lines and four quarter of circles, as in Figure 1, we get the same result. The outer path is $2\pi d$ longer than the inner path. Note, that this idea basically explains the start lines in athletic stadiums.

Could the above observation be true for any (simple) closed parallel plane path (see Figure 2)? We will see that the answer is in the affirmative. For the proof of this statement, we will need some (basic) differential geometry (see for example [1], [2]).

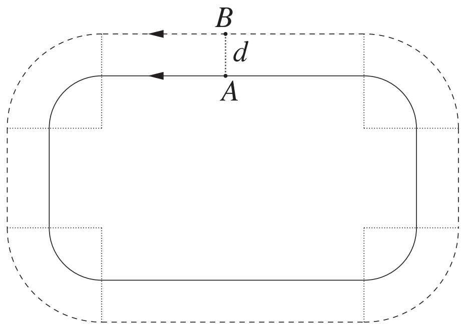

FIGURE 1: Parallel paths of four straight lines and four quarter circles

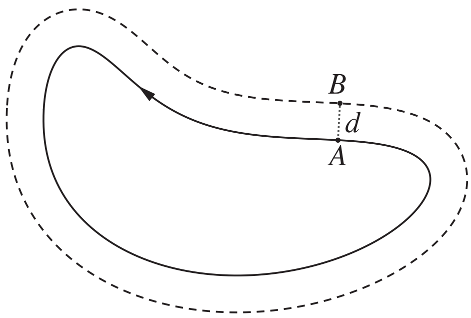

FIGURE 2: Random (simple) closed parallel path

2. *Smooth closed curves*

We start with smooth closed curves in xy-plane. Let $c : [0,1] \rightarrow \mathbb{R}^2$, $c(0) = c(l)$ be a smooth closed curve, given by arc length parameter s.

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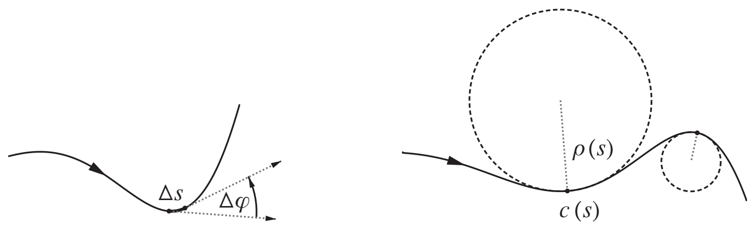
Since the curve's $c(s) = (x(s), y(s))$ parameter is its arc length, we have $c'(s) = (x'(s), y'(s))$ as a unit tangent vector and $\eta(s) = (y'(s), -x'(s))$ is a normal (perpendicular to the curve) vector. The normal vector $\eta(s)$ is perpendicular to the right of the direction of the movement along the curve, which is determined by a parameter s.

For a smooth curve c we define its curvature at point $c(s)$ (Figure 3a) as

$$\kappa(s) = \lim_{\Delta s \to 0} \frac{\Delta \varphi}{\Delta s} = \frac{d\varphi}{ds}.$$

The reciprocal of the curvature is called the radius of curvature: $\rho(s) = 1/\kappa(s)$. Note that $\kappa(s) = 0$ and $\rho(s) = \pm\infty$ correspond to a straight line. If the curvature is positive, the curve is 'turning left' and if it is negative, it is 'turning right', with regard to the direction of travel determined by the parameter s. The radius of curvature can be nicely visualised by imagining the circle around which a car would travel if the steering wheel were to get locked at a specific point, while the car was travelling along a smooth curve (Figure 3b).



(a) (b)

FIGURE 3: Radius of curvature

We define $C(s) = c(s) + d \cdot \eta(s)$, where d determines the distance between the two paths. Curve $C(s)$ is obviously parallel to $c(s)$. Let the function $L : [0, l] \to \mathbb{R}$ measure the length of the curve $C(s)$. A small increment $dL(s)$ of L can be described in terms of $\rho(s)$ and small increments ds of s. Let the point A move along the curve c and on its right point B along the curve C. We consider three different cases:

1. Turn left: B moves faster then A, $\rho(s) > 0$, $dL(s) = \frac{\rho(s) + d}{\rho(s)} ds$ (Figure 4).
2. Wide turn right: B moves slower then A, $\rho(s) < 0$ and $-\rho(s) \geqslant 0$, $dL(s) = \dfrac{-\rho(s) - d}{-\rho(s)} ds = \dfrac{\rho(s) + d}{\rho(s)} ds$ (Figure 5a).
3. Sharp turn right: B moves backwards, $\rho(s) < 0$ and $-\rho(s) < d$, $dL(s) = \dfrac{\rho(s) + d}{\rho(s)} ds$ (Figure 5b).

The first two options are determined by $\kappa(s) \geqslant -\frac{1}{d}$, when $dL(s) = \frac{\rho(s) + d}{\rho(s)} ds$. The third option is determined by $\kappa(s) < -\frac{1}{d}$,when $dL(s) = \frac{\rho(s) + d}{-\rho(s)} ds$. Note that in the first two options, B moves along a smooth curve, while the third option forces B to move along a piecewise smooth curve with cusps. And as in Figure 5b, B moves backwards between the two cusps.

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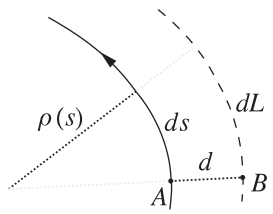

FIGURE 4: Path differential when $\rho(s) > 0$

The first two options are determined by $\kappa(s) \geqslant -\frac{1}{d}$, when $dL(s) = \frac{\rho(s)+d}{\rho(s)}ds$. The third option is determined by $\kappa(s) < -\frac{1}{d}$,when $dL(s) = \frac{\rho(s)+d}{-\rho(s)}ds$. Note that in the first two options, B moves along a smooth curve, while the third option forces B to move along a piecewise smooth curve with cusps. And as in Figure 5b, B moves backwards between the two cusps.

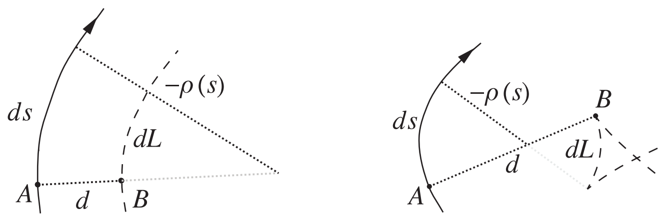

FIGURE 5: Path differential when $\rho(s) < 0$, (a) $-\rho(s) \geqslant d$, (b) $-\rho(s) < d$

We define $dL^r(s) = \frac{\rho(s)+d}{\rho(s)}ds$. Therefore $dL^r(s) = dL(s)$ for every s, where $\kappa(s) \geqslant -\frac{1}{d}$. The differential $dL^r(s) = -dL(s)$ whenever $\kappa(s) < -\frac{1}{d}$. Let us calculate

$$\int_0^l dL^r(s) = \int_0^l \left(1 + \frac{d}{\rho(s)}\right) ds = l + d\int_0^l \frac{1}{\rho(s)}\,ds.$$

The equation

$$\int_0^l \frac{ds}{\rho(s)} = \int_0^l \kappa(s)\,ds = 2\pi n \tag{1}$$

gives the 'total curvature' of the (plane) curve c, where n tells us how many times the tangent vector $c'(s)$ (or normal vector $\eta(s)$) rotates for 360° in the positive direction while point $c'(s)$ travels along the whole path of the curve c. If c is a closed path, n is an integer, sometimes called 'index' or 'turning number' (of a closed plane curve). With a simple closed path as in Figures 2, 7 and 8, we get $n = 1$, while for the path in Figure 6, $n = 0$.

Note: The quantity $\int_0^l dL^r(s)$ does not represent the length of the curve $C(s)$ in the strict sense. Recall that those parts of the curve $C(s)$ where the curvature was negative and its absolute value was bigger than $\frac{1}{d}\left(\kappa(s) < -\frac{1}{d}\right)$, were subtracted and not added to the full length of the curve.

In fact the expression $\int_0^l dL^r(s)$ is exactly what we want to describe, namely, the difference of the lengths of the paths c and C. Summarising:

- If $\kappa(s) > 0$, A moves slower than B (left part of path in Figure 6).
- If $\kappa(s) = 0$, A and B move straight (with the same speed).
- If $\kappa(s) < 0$ and $-\frac{1}{d} < \kappa(s)$, A moves faster than B (right part of path in Figure 6).
- If $\kappa(s) = -\frac{1}{d}$, A moves (in a circle) around the (rotating) B (part of path in Figure 7).
- If $\kappa(s) < -\frac{1}{d}$, A is making a sharp right turn while B moves backwards (part of path c in Figure 8).

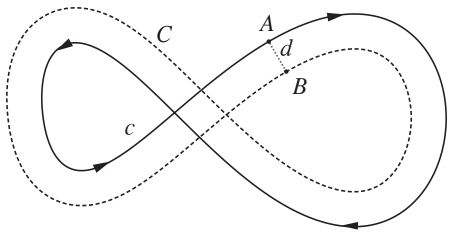

FIGURE 6: Path c with curvature $\kappa(s) > -\dfrac{1}{d}$

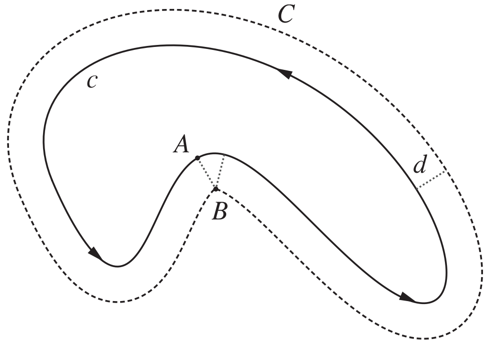

FIGURE 7: Path c with mostly positive curvature and a portion where curvature $\kappa(s) = -\frac{1}{d}$

Summarising the main result we get the following theorem.

Theorem 1: If A moves along a smooth closed path c, with B on the right at a constant distance d and if n (the turning number of the closed curve c) is a count of how many times B has rotated around A (with respect to the relative centre at the point A), then point B has moved exactly a $2\pi nd$ longer path than point A (independently of the length of the curve c).

The number n tells us also how many times the segment AB has revolved (in a positive direction) around A throughout the whole move of A

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along the curve c. With very sharp turns of A to the right, B moves backwards. Moves backwards are counted negatively in the whole piecewise smooth path of B.

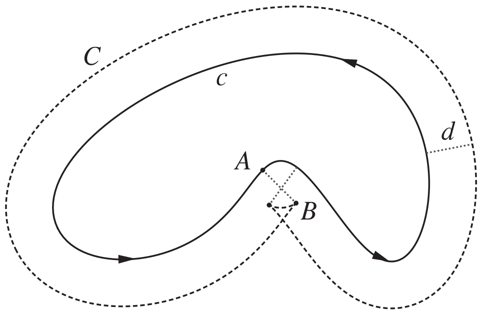

FIGURE 8: Path c with mostly positive curvature and a portion where the curvature is $\kappa(s) < -\frac{1}{d}$

Corollary 1: As in Theorem 1, let A and B move along parallel closed paths at a constant distance d. If the curvature of the path of A remains greater or equal to $-1/d$, then $l(B) = l(A) + 2\pi nd$, where $l(A)$ denotes the length of the path of A and $l(B)$ denotes the length of the path of B.

Corollary 2: If, similarly, A and B move along simple parallel closed paths at a constant distance d (with B on either side of A), then $l(B) = l(A) \pm 2\pi d$, where the sign is determined so that the outer path is longer.

Corollary 3: If, similarly, A and B move along simple parallel closed convex paths at a constant distance d (with B on either side of A), then $l(B) = l(A) \pm 2\pi d$, where the sign is determined so that the outer path is longer.

Definition 2: A simple closed plane curve is convex, if the plane area which it bounds is convex.

Proof (or rather commentary on the last corollary): It is an obvious consequence, which (because of the convexity assumption) can be proved independently with very elementary geometric reasoning (see for example [3]; 14.12).

3. *Smooth curves which are not closed*

The result can be generalised nicely to curves that are not closed. In (1), we had

$$\int_0^l \frac{ds}{\rho(s)} = \int_0^l \kappa(s)\, ds = 2\pi n,$$

where the total curvature of the closed curve c, was $2\pi n$. If we denote the

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total curvature of a possibly not closed smooth curve c by $\Theta(c)$, that is

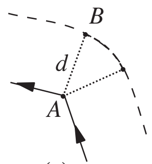
$$\int_0^l \frac{ds}{\rho(s)} = \int_0^l \kappa(s)\, ds = \Theta(c),$$

then $\Theta(c)$ simply measures the angle by which the tangent vector has rotated with respect to the positive direction, along the whole curve c.

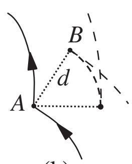
Corollary 4: Under the same assumptions as in Theorem 1 and in Corollary 1, except that the path of A might not be closed, we get an exact analogue of Theorem 1 and Corollary 1 where the difference in paths length becomes $\Theta(c)\, d$ rather than $2\pi nd$.

4. *Piecewise smooth curves*

The path of A might not be smooth at some points. This would mean that A would, at some points, momentarily change the direction of movement by rotating through a certain angle. In other words, at that point the direction of the movement (derivative/gradient) would not be determined. Intuitively, it is easy to picture how we walk, smoothly, until some point where we rotate around our axis to change the direction of our movement. In the language of differential geometry, the curve is not smooth at this point but we say that the curve is piecewise smooth. Visualising our two points A and B, we imagine A moving along a piecewise smooth curve and point B on the right side of A at distance d. At points where A suddenly changes the direction, rotating (at the spot) in a positive direction (to left) for an angle φ (measured in radians), point B circles around A covering arc of length $d \cdot \varphi$ in a positive direction (Figure 9a). If A rotates in a negative direction (to right), B moves backwards on an arc of the appropriate length (Figure 9b). It might be worth mentioning that at the kinks of the piecewise smooth curve, we should consider the speed with which A turns. If A were to move with constant speed also along these kinks, that would mean that A turns momentarily (in zero time) and that B moves along the appropriate loop with very high (infinite) speed. In this situation we could redefine the total curvature of a piecewise smooth curve by adding or subtracting the appropriate angle of rotation of A at the cusps of the piecewise smooth curve.



(a)



(b)

FIGURE 9: Piecewise smooth paths

Theorem 3: With the above-described sensible interpretation, Theorem 1 and all its corollaries hold also in the case of piecewise smooth paths.

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5. *About the area between parallel paths*

We have seen that the difference of lengths of two (at distance d) parallel plane paths is directly related to the circumference of a circle with radius d if the curve is closed, and to the length of an appropriate circular arc if the curve is not closed. Analogous reasoning can be applied to show that the area between two parallel paths is also closely related to the area of a circle with the same radius.

If A moves along a simple smooth convex closed curve c in a positive direction and B along a parallel path C at a distance d on the right of A, then it can be proved using only elementary geometry that the area between the two curves equals to $l(c) \cdot d + \pi d^2$ (see for example [3], 14.12), where $l(c)$ is the length of the curve c. But we also have the following result for a non-convex curve.

Theorem 4: Let c be a simple smooth closed curve of length l such that the curvature satisfies the inequality $\kappa(s) = \dfrac{1}{\rho(s)} \geqslant -\dfrac{1}{d}$. If C is the path parallel to the path c at distance d (to the right side regarding the positive orientation of the curve c), then the area between the two curves equals $d \cdot l + \pi d^2$.

Proof: We observe a (differentially small) part of the path c and the area between ds and dL. When $\kappa(s) = \dfrac{1}{\rho(s)}$ is positive (Figure 10a), we have the central angle corresponding to ds equal to $d\varphi = \dfrac{ds}{\rho(s)}$, and we calculate the corresponding area as

$$dS(s) = \left((\rho(s) + d)^2 - \rho(s)^2\right)\frac{ds}{2\rho(s)} = d \cdot ds + \frac{d^2}{2}\frac{ds}{\rho(s)}.$$

When $\kappa(s) = \dfrac{1}{\rho(s)}$ is negative but greater than or equal to $-\dfrac{1}{d}$ (Figure 10b), we get the corresponding area as

$$dS(s) = \left(\rho(s)^2 - (-\rho(s) - d)^2\right)\frac{ds}{-2\rho(s)} = d \cdot ds + \frac{d^2}{2}\frac{ds}{\rho(s)}.$$

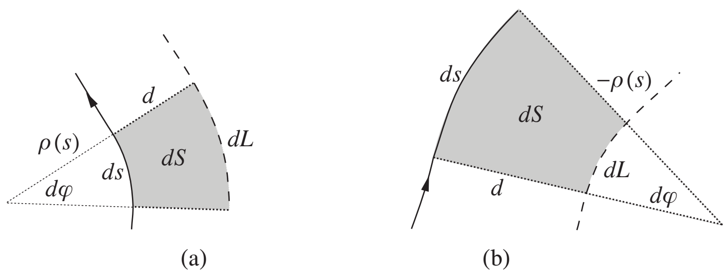

FIGURE 10: Area between parallel paths

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In both cases we compute the area between the two curves as

$$S = \int_0^l dS(s) = \int_0^l \left(d + \frac{d^2}{2} \frac{1}{\rho(s)} \right) ds = d \cdot l + \frac{d^2}{2} \int_0^l \frac{1}{\rho(s)} ds$$

$$= d \cdot l + \frac{d^2}{2} 2\pi = d \cdot l + \pi d^2.$$

Remark: The condition $\kappa(s) = \dfrac{1}{\rho(s)} > -\dfrac{1}{d}$ applies to all smooth curves c along which A moves as far as the curve C, along which B is forced to move, remains smooth. The equality $\kappa(s) = \dfrac{1}{\rho(s)} = -\dfrac{1}{d}$ means that B rotates about its own axis, while $\kappa(s) = \dfrac{1}{\rho(s)} < -\dfrac{1}{d}$ would mean B is moving backwards. The latter would mess up the notion of the area between the curves.

In the consideration of the lengths of parallel paths we have seen that the result can be generalised to piecewise smooth (not necessary closed) curves. A very similar generalisation can be obtained for the area (as defined above) between piecewise smooth parallel curves c and C (see for example Figure 11). In such a case the area between parallel paths c and C is given by $d \cdot l + \Theta(c)\left(\frac{1}{2}d^2\right)$ where l denotes the length of the curve c and $\Theta(c)$ is the total curvature of the curve c.

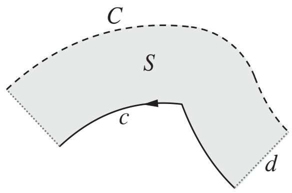

FIGURE 11: Area between parallel paths – piecewise smooth curve

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