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ON A REPRESENTATION OF CONTINUOUS VOLTERRA RIGHT INVERSES TO THE DERIVATIVE IN THE SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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Abstract

Every continuous Volterra right inverse to the derivative in the space of complex-valued infinitely differentiable functions has the form of an integral.

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Let $X = C([0,1], \mathbb{C})$ be the space of all complex-valued continuous functions defined on the interval $[0,1]$. Let $D = d/dt$ be the differentiation operator. In [1] we have proved that every continuous Volterra right inverse R of D is of the form

$$(1) \qquad (Rx)(t) = \int_a^t x(s)\,ds,$$

where $a \in [0,1]$ is arbitrarily fixed.

A natural question arises: what can one say about extensions of this result to other spaces? This note provides an answer to this question.

THEOREM 1. *Let $X = C^\infty([0,1], \mathbb{C})$ be the space of all infinitely differentiable complex-valued functions defined on the interval* $[0,1]$ *with the classical topology of uniform convergence of all derivatives. Then every continuous Voltera right inverse R to the operator d/dt is of the form* (1).

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PROOF. The proof follows along the same lines as in [1]. From the general theory of right invertible operators (cf. [3, Theorem 2]), since the kernel of the operator D is one-dimensional, we find that every continuous right inverse R is of the form

$$(Rx)(t) = \int_a^t x(s)\,dt + \psi(x) \qquad (x \in X), \tag{2}$$

where $a \in [0, 1]$ is arbitrarily fixed, and where ψ is a continuous linear functional defined on X. The classical theory of locally convex spaces and the Riesz representation theorem together imply that

$$\psi(x) = \sum_{i=1}^{k} \int_0^1 x^{(i)}\,d\mu_i \qquad (x \in X), \tag{3}$$

where μ_i $(i = 0, 1, \ldots, k)$ are complex-valued Borel measures (cf. for instance [4, page 24]).

Write $x_z(t) = e^{zt}$, where z plays the role of a parameter. Let

$$\begin{aligned} G(z) &= [(I - zR)x_z](t) = e^{zt} - z\left[\frac{1}{z}e^{zt} - \frac{1}{z}e^{za}\right] - z\psi(x_z) \\ &= e^{za} - z\psi(x_z). \end{aligned} \tag{4}$$

Observe that the function G does not vanish for a Volterra right inverse. Since ψ is of the form (3), we conclude that G is an entire function of order 1 and of type 1. Thus, by the great Picard theorem (cf. for instance, [5, page 341]), G is of the form $Ce^{\alpha z}$ for some complex number α. Hence

$$z\psi(x_z) = e^{za} - Ce^{z\alpha}. \tag{5}$$

Putting $z = 0$, we find that $C = 1$, and so

$$z\psi(x_z) = e^{za} - e^{z\alpha} = z\int_\alpha^a e^{zt}\,dt.$$

The form (3) of ψ and the definition of x_z together imply that

$$\limsup_{\xi \to -\infty} \psi(x_{\xi + i\eta}) < +\infty \quad \text{and} \quad \limsup_{\xi \to +\infty} \psi(x_{\xi + i\eta})e^{-\xi} < +\infty \qquad (\xi, \eta \in \mathbb{R}).$$

Therefore $0 \leqslant \operatorname{Re}\alpha \leqslant 1$. Moreover,

$$\lim_{\eta \to \pm\infty} \psi(x_{\xi + i\eta}) < +\infty,$$

which implies that $\operatorname{Im}\alpha = 0$.

Thus

$$\psi(x_z) = \int_\alpha^a e^{zt}\,dt \qquad (0 \leqslant \alpha \leqslant 1). \tag{6}$$

Since the set $\operatorname{lin}\{e^{zt}\}_{z \in \mathbf{C}}$ is dense in the space $C^\infty([0, 1], \mathbf{C})$, we conclude that

$$\psi(x) = \int_\alpha^a x(s)\,ds \qquad \text{for all } x \in X,$$

and, by (2), that

$$(Rx)(t) = \int_a^t x(s)\,ds + \int_\alpha^a x(s)\,ds = \int_\alpha^t x(s)\,ds. \tag{7}$$

Consider now a locally convex space X consisting of functions. Suppose that $C^\infty([0,1],\mathbf{C})$ is continuously embedded in X and is dense in X. Let f be a continuous linear functional defined on X. It is clear that f is also continuous on $C^\infty([0,1],\mathbf{C})$, and hence is of the form (3). Since the set $\operatorname{lin}\{e^{zt}\}_{z\in\mathbf{C}}$ is dense in $C^\infty([0,1],\mathbf{C})$, we obtain

THEOREM 2. *Let X be a locally convex space consisting of functions, and suppose that $C^\infty([0,1],\mathbf{C})$ is continuously embedded in X and is dense in X. Then every continuous Volterra right inverse to the operator d/dt is of the form* (1).

Let $X = C^k([0,1],\mathbf{C})$ $(k = 0,1,2,\dots,+\infty)$ be the space of all complex-valued k-times continuously differentiable functions with the topology of uniform convergence of all derivatives up to order k (or of all derivatives, if $k = +\infty$). The space $C^k([0,1],\mathbf{C})$ is a Banach space for $k = 0,1,2,\dots$ and a locally convex metrizable complete space for $k = +\infty$.

For $k = 0,1,2,\dots,+\infty$, for $x \in X$, and for $0 < n \leqslant k$, write

$$(Rx)(t) = \int_a^t x(s)\,ds + b_0x(a) + \cdots + b_nx^{(n)}(a). \tag{8}$$

By definition, R is a continuous right inverse to the operator d/dt in the space $C^k([0,1],\mathbf{C})$, and we have

$$\begin{aligned}(I - zR)e^{zt} &= e^{zt} - e^{zt} + e^{az} - b_0ze^{az} - \cdots - b_nz^{n+1}e^{az}\\ &= e^{az}\left(1 - b_0z - \cdots - b_nz^{n+1}\right).\end{aligned}$$

Observe that there exist $b_0,\dots,b_n$ such that the right hand side is equal to 0 at $n+1$ different points $z_0,\dots,z_n$. These points constitute eigenvalues of the operator R. However, if R is not of the form (8), then the function $G(z)$ defined by formula (4) is of order 1, and it is not of the form $G(z) = p(z)C^{P(z)}$, where $p(z)$ and $P(z)$ are polynomials. Thus, by the great Picard theorem, G has an infinite number of zeros. This implies that R has an infinite number of eigenvalues. We therefore obtain

THEOREM 3. *Let $X = C^k([0,1],\mathbf{C})$ $(k = 0,1,2,\dots)$. If R is a continuous right inverse of the operator d/dt and has at least $k+2$ eigenvalues, then it has an infinite number of eigenvalues.*

Observe that the assumption about the continuity of R is essential, as is shown by the following example.

EXAMPLE 1. Let $X = C([0,1], \mathbf{C})$. By the axiom of choice, there exists a non-trivial linear functional ψ such that $C^1([0,1], \mathbf{C}) \subset \ker \psi$. Write

$$(Rx)(t) = \int_0^t x(s)\,ds + \psi(x) \cdot 1. \tag{9}$$

Clearly, R is a right inverse to the operator d/dt. Since the series $\sum_{n=0}^{\infty} \lambda^n R^n$ is convergent for all differentiable functions, and for all $\lambda \in \mathbf{C}$, we conclude that the operators $I - \lambda R$ are invertible for all $x \in X$ and $\lambda \in \mathbf{C}$, and that $(I - \lambda R)^{-1} = \sum_{n=0}^{\infty} \lambda^n R^n$. Hence R is a Volterra operator.

Theorem 1 cannot be extended to differential operators of higher orders, as shown by the following example.

EXAMPLE 2. Let $X = C([-1,1], \mathbf{C})$ be the space of complex-valued functions defined on the interval $[-1,1]$, and let $D = d^2/dt^2$. For $x \in X$, write $(Fx)(t) = x(1) + x(-1) - x(0) + tx'(0)$. It is easy to verify that F is a projection onto the kernel of D, which comprises the linear functions. Hence F is an initial operator for D. Consider the initial value problem

$$Dx = \lambda x \qquad (\lambda > 0), \tag{10}$$

$$Fx = 0. \tag{11}$$

We shall prove that this initial value problem has only zero as a solution. Indeed, for simplicity only , let us write $\lambda = \mu^2$. Then every solution of Equation (10) is of the form $e(t) = ae^{\mu t} + be^{-\mu t}$, where a, b are arbitrary scalars. The condition $Fx = 0$ implies that $e'(0) = a\mu e^{\mu t} - b\mu e^{-\mu t} = \mu(ae^{\mu t} - be^{-\mu t}) = 0$. Thus $a = b$, and

$$(Fe)(t) = e(1) + e(-1) - e(0) + te'(0) = 2a(e^t + e^{-t} - 1).$$

We therefore conclude that $Fe = 0$ if and only if $a = 0$, i.e. $e = 0$. A right inverse R corresponding to F may be defined by

$$(Rx)(t) = \int_0^t \left(\int_0^u x(s)\,ds \right) du - \int_{-1}^1 \left(\int_0^u x(s)\,ds \right) du,$$

and this is a Volterra operator which is not of the form $\int_\alpha^t \left(\int_\alpha^u x(s)\,ds \right) du$ for any $\alpha \in [-1,1]$.

A trivial consequence of Example 2 is that, in the space $C_2([-1,1], \mathbf{C})$ of all two-dimensional complex-valued continuous vector functions, ther are Volterra right inverses to the d/dt which are not of the form $\int_a^t$.

The approach presented above may be used for other spaces. For instance, let Ω be a simply connected domain on the complex plane and let $H(\Omega)$ be the space of all analytic functions defined on Ω with the topology of uniform convergence on compact sets. This means that the topology in $H(\Omega)$ is defined by a sequence of pseudonorms

$$\|x\|_m = \sup_{z \in \Omega_m} |x(z)|,$$

where the sets Ω_m are compact sets such that $\Omega = \bigcup_{m=1}^{\infty} \Omega$ (see [4, page 355]). Let $D = d/dz$. In the same way as before, we conclude that the only continuous right inverses to the operator D are operators of the form R_a, where $R_a x = \int_a^z x(z)\,dz$, and where the integration is taken along an arbitrary curve in Ω connnecting a and z. Of course, since Ω is simply connected, the integral does not depend on the choice of such a curve.

The same result may be obtained if we consider in $H(\Omega)$ an arbitrary topology which is weaker than the original one, and in which the integration operator $\int_a^z$ is continuous.

If we pass from the space of all entire functions or of all anallytic functions in the unit disc to the space of their coefficients, then we can obtain representations of all continuous Volterra right inverses to the backward shift operator in this sequence space. However, these representations have more complicated form.

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