Canad. Math. Bull. Vol. 58 (2), 2015 pp. 306–316
http://dx.doi.org/10.4153/CMB-2015-021-4

On the Largest Dynamic Monopolies of Graphs with a Given Average Threshold

Kaveh Khoshkhah and Manouchehr Zaker

Abstract. Let G be a graph and let τ be an assignment of nonnegative integer thresholds to the vertices of G. A subset of vertices, D, is said to be a τ-dynamic monopoly if $V(G)$ can be partitioned into subsets $D_0, D_1, \ldots, D_k$ such that $D_0 = D$ and for any $i \in \{0, \ldots, k-1\}$, each vertex v in D_{i+1} has at least $\tau(v)$ neighbors in $D_0 \cup \cdots \cup D_i$. Denote the size of smallest τ-dynamic monopoly by $\mathrm{dyn}_\tau(G)$ and the average of thresholds in τ by $\overline{\tau}$. We show that the values of $\mathrm{dyn}_\tau(G)$ over all assignments τ with the same average threshold is a continuous set of integers. For any positive number t, denote the maximum $\mathrm{dyn}_\tau(G)$ taken over all threshold assignments τ with $\overline{\tau} \le t$, by $\mathrm{Ldyn}_t(G)$. In fact, $\mathrm{Ldyn}_t(G)$ shows the worst-case value of a dynamic monopoly when the average threshold is a given number t. We investigate under what conditions on t, there exists an upper bound for $\mathrm{Ldyn}_t(G)$ of the form $c|G|$, where $c < 1$. Next, we show that $\mathrm{Ldyn}_t(G)$ is coNP-hard for planar graphs but has polynomial-time solution for forests.

1 Introduction

In this paper we deal with simple undirected graphs. For any such graph $G = (V, E)$, we denote the cardinality of its vertex set by $|G|$ and the edge density of graph G by $\epsilon(G) := |E|/|G|$. We denote the degree of a vertex v in G by $\deg_G(v)$. For other graph theoretical notations we refer the reader to [2]. By a threshold assignment for the vertices of G we mean any function $\tau\colon V(G) \to \mathbb{N} \cup \{0\}$. A subset of vertices D is said to be a *τ-dynamic monopoly* of G or simply *τ-dynamo* of G, if for some nonnegative integer k, the vertices of G can be partitioned into subsets $D_0, D_1, \ldots, D_k$ such that $D_0 = D$ and for any i, $1 \le i \le k$, the set D_i consists of all vertices v which has at least $\tau(v)$ neighbors in $D_0 \cup \cdots \cup D_{i-1}$. Denote the smallest size of any τ-dynamo of G by $\mathrm{dyn}_\tau(G)$. Dynamic monopolies are in fact modeling the spread of influence in social networks. The spread of innovation or a new product in a community, the spread of opinion in Yes-No elections, the spread of a virus on the internet, and the spread of disease in a population are some examples of these phenomena. Obviously, if for a vertex v we have $\tau(v) = \deg_G(v) + 1$, then v should belong to any dynamic monopoly of (G, τ). We call such a vertex v *self-opinioned* (from another interpretation it can be called *vaccinated vertex*). Irreversible dynamic monopolies and the equivalent concepts target set selection and conversion sets have been the subject of active research in recent years by many authors [3, 4, 6–8, 10–13].

Received by the editors November 19, 2013.
Published electronically March 18, 2015.
AMS subject classification: **05C69**, 05C85.
Keywords: spread of influence in graphs, irreversible dynamic monopolies, target set selection.

In this paper, by (G,τ) we mean a graph G and a threshold assignment for the vertices of G. The average threshold of τ, denoted by $\overline{\tau}$, is $\sum_{v\in V(G)}\tau(v)/|G|$. In Proposition 2.2 we show that the values of $\mathrm{dyn}_\tau(G)$ over all threshold assignments with the same average threshold form a continuous set of integers. The maximum element of this set was studied for first time in [10], where the following notation was introduced. Let t be a non-negative rational number such that $t|G|$ is an integer, then $\mathrm{Dyn}_t(G)$ is defined as $\mathrm{Dyn}_t(G) = \max_{\tau:\overline{\tau}=t} \mathrm{dyn}_\tau(G)$. The smallest size of dynamic monopolies with a given average threshold was introduced and studied in [13]. Dynamic monopolies with given average threshold was also recently studied in [5]. In the definition of $\mathrm{Dyn}_t(G)$, it is assumed that $t|G|$ is integer. In order to consider all values of t, we modify the definition slightly, but we are forced to make a new notation, *i.e.*, $\mathrm{Ldyn}_t(G)$ (which stands for the largest dynamo). The formal definition is as follows.

Definition 1.1 Let G be a graph and let t be a positive number. We define $\mathrm{Ldyn}_t(G) = \max\{\mathrm{dyn}_\tau(G)|\overline{\tau} \le t\}$. Assume that a subset $D \subseteq V(G)$ and an assignment of thresholds τ_0 are such that $\overline{\tau_0} \le t$, $|D| = \mathrm{dyn}_{\tau_0}(G) = \mathrm{Ldyn}_t(G)$ and D is a τ_0-dynamic monopoly of (G,τ_0). Then we say that (D,τ_0) is a t-Ldynamo of G.

$\mathrm{Ldyn}_t(G)$ does in fact show the worst-case value of a dynamic monopoly when the average threshold is a prescribed given number. The following concept is motivated by the concept of dynamo-unbounded family of graphs, defined in [12], concerning the smallest size of dynamic monopolies in graphs.

Definition 1.2 For any $n \in \mathbb{N}$, let G_n be a graph and t_n be a number such that $0 \le t_n \le 2\epsilon(G_n)$. We say $\{(G_n,t_n)\}_{n\in\mathbb{N}}$ is *Ldynamo*-bounded if there exists a constant $\lambda < 1$ such that for any n, $\mathrm{Ldyn}_{t_n}(G_n) \le \lambda|G_n|$.

The outline of the paper is as follows. In Section 2, we show that the values of $\mathrm{dyn}_\tau(G)$ over all assignments τ with the same average threshold is a continuous set of integers (Proposition 2.2). Then we obtain a necessary and sufficient condition for a family of graphs to be Ldynamo-bounded (Propositions 2.4 and 2.5). In Section 3, it is shown that the decision problem $\mathrm{Ldynamo}(k)$ (to be defined later) is coNP-hard for planar graphs (Theorem 3.1) but has polynomial-time solution for forests (Theorem 3.8).

2 Some Results on $\mathrm{Ldyn}_t(G)$

We first show that the values of $\mathrm{dyn}_\tau(G)$ over all threshold assignments τ with the same average threshold are continuous. We need the following lemma from [11].

Lemma 2.1 ([11]) *Let G be a graph and let τ and τ' be two threshold assignments to the vertices of G such that $\tau(u) = \tau'(u)$ for all vertices u of G except for exactly one vertex, say v. Then*

$$\begin{cases} \mathrm{dyn}_\tau(G) - 1 \le \mathrm{dyn}_{\tau'}(G) \le \mathrm{dyn}_\tau(G), & \text{if } \tau(v) > \tau'(v),\\ \mathrm{dyn}_\tau(G) \le \mathrm{dyn}_{\tau'}(G) \le \mathrm{dyn}_\tau(G) + 1, & \text{if } \tau(v) < \tau'(v).\end{cases}$$

The continuity result is as follows.

Proposition 2.2 *Let τ and τ' be two threshold assignments for the vertices of G such that $\overline{\tau} = \overline{\tau}'$. Let also r be an integer such that $\mathrm{dyn}_{\tau}(G) \le r \le \mathrm{dyn}_{\tau'}(G)$. Then there exists τ'' with $\overline{\tau} = \overline{\tau}''$ such that $\mathrm{dyn}_{\tau''}(G) = r$.*

Proof For any two threshold assignments τ and τ' with the same average threshold, define $\delta(\tau, \tau') = \sum_{v:\tau(v)>\tau'(v)}(\tau(v) - \tau'(v))$. We prove the proposition by the induction on $\delta(\tau, \tau')$. If $\delta(\tau, \tau') = 0$, then for any vertex v, $\tau(v) \le \tau'(v)$. But the average thresholds are the same, hence $\tau = \tau'$, and the assertion is trivial. Let $k \ge 1$ and assume that the proposition holds for any two τ and τ' with the same average threshold such that $\delta(\tau, \tau') \le k$. We prove it for $k+1$. Assume that τ and τ' are given such that $\delta(\tau, \tau') = k+1$ and $\tau \ne \tau'$. Define $W = \{v : \tau(v) > \tau'(v)\}$. Let $w \in W$. There exists a vertex u such that $\tau(u) < \tau'(u)$, since otherwise by $\overline{\tau} = \overline{\tau}'$ we would have $\tau = \tau'$. Define a new threshold τ'' as follows. For any vertex v with $v \notin \{u, w\}$ set $\tau''(v) = \tau(v)$. Also, set $\tau''(w) = \tau(w) - 1$ and $\tau''(u) = \tau(u) + 1$. We have $\delta(\tau'', \tau') = k$, and the average threshold of τ'' is the same as that of τ. So the assertion holds for τ'' and τ'. By Lemma 2.1 we have $|\mathrm{dyn}_{\tau}(G) - \mathrm{dyn}_{\tau''}(G)| \le 1$. We conclude that the assertion also holds for τ and τ'. ∎

Let G be a graph and let t be a positive number such that $t|G|$ is integer. Let τ be any assignment with average t such that $\tau(v) \le \deg_G(v)$ for any vertex v. Let $d_1 \le d_2 \le \cdots \le d_n$ be a degree sequence of G in increasing form. It was proved in [10] that the size of any τ-dynamic monopoly of G is at most

$$\max\Big\{ k : \sum_{i=1}^{k}(d_i + 1) \le nt \Big\}.$$

The proof of this result in [10] shows that if we allow $\tau(v) = \deg_G(v) + 1$ for some vertices v of G, then the same assertion still holds. We have the following proposition concerning this fact.

Proposition 2.3 *Let t be a positive number. Assume that the threshold assignments in the definition of $\mathrm{Ldyn}_t(G)$ are allowed to have self-opinioned vertices. Then $\mathrm{Ldyn}_t(G)$ can be easily obtained by a polynomial-time algorithm.*

Proof Let $d_1 \le d_2 \le \cdots \le d_n$ be a degree sequence of G in increasing form. By the argument we made before Proposition 2.3, we have

$$\mathrm{Ldyn}_t(G) \le \max\Big\{ k : \sum_{i=1}^{k}(d_i + 1) \le nt \Big\}.$$

Let $k_0 = \max\{k : \sum_{i=1}^{k}(d_i + 1) \le nt\}$. We obtain a threshold assignment τ as follows:

$$\tau(v_i) = \begin{cases} \deg_G(v_i) + 1 & i \le k_0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = \{v_1, v_2, \ldots, v_{k_0}\}$. It is clear that (D, τ) is a t-Ldynamo of G. ∎

In [10], it was proved that there exists an infinite sequence of graphs $G_1, G_2, \dots$ such that $|G_n| \to \infty$ and $\lim_{n\to\infty} \mathrm{Ldyn}_{\epsilon(G_n)}(G_n)/|G_n| = 1$. In the following proposition, we show that a stronger result holds. In fact we show that not only the same result holds for $\mathrm{Ldyn}_{k\epsilon(G_n)}(G_n)$, where k is any constant with $0 < k \le 2$, but also it holds for any sequence k_n for which $k_n|G_n| \to \infty$. In the opposite direction, Proposition 2.5 shows that if $k_n = \mathcal{O}(1/|G_n|)$, then $\lim_{n\to\infty} \mathrm{Ldyn}_{k_n\epsilon(G_n)}(G_n)/|G_n| \ne 1$.

Proposition 2.4 *There exists an infinite sequence of graphs $\{(G_n, \tau_n)\}_{n=1}^{\infty}$ satisfying $|G_n| \to \infty$ and $\epsilon(G_n)/|G_n| = o(\overline{\tau}_n)$ such that*

$$\lim_{n\to\infty} \frac{\mathrm{Ldyn}_{\overline{\tau}}(G_n)}{|G_n|} = 1.$$

Proof We construct G_n as follows. The vertex set of G_n is disjoint union of a complete graph K_n and n copies of complete graphs K_{n+1}. There exists exactly one edge between each copy of K_{n+1} and K_n. Set $\tau_n(v) = 0$ for each vertex v in K_n and $\tau_n(v) = \deg(v)$ for each vertex v in any copy of K_{n+1}. It is clear that any dynamic monopoly of G_n includes at least n vertices of each copy of K_{n+1} and hence $\mathrm{Ldyn}_{\overline{\tau}}(G_n) \ge n^2$. Then we have

$$1 \ge \lim_{n\to\infty} \frac{\mathrm{Ldyn}_{\overline{\tau}}(G_n)}{|G_n|} \ge \lim_{n\to\infty} \frac{n^2}{n(n+2)} = \lim_{n\to\infty} \frac{n}{n+2} = 1.$$

To complete the proof we show that $\frac{\overline{\tau}_n}{|E(G_n)|/|V(G_n)|^2} \to \infty$:

$$\lim_{n\to\infty} \frac{\overline{\tau}_n}{|E(G_n)|/|V(G_n)|^2} = \lim_{n\to\infty} \frac{(n^2+n+1)/(n+2)}{(n^2+n+n(n+n^2))/2(n^2+2n)^2} = \infty. \qquad \blacksquare$$

Proposition 2.4 shows that if t_n is such that $\epsilon(G_n)/|G_n| = o(t_n)$, then $\{(G_n, t_n)\}_n$ is not necessarily Ldynamo-bounded. In the opposite direction, the next proposition shows that if there exists a positive number c such that t_n satisfies $t_n \le c\epsilon(G_n)/|G_n|$, then any family $\{(G_n, t_n)\}_n$ is Ldynamo-bounded.

Proposition 2.5 *Let G be a graph and let c and t be two constants such that $t \le c\frac{\epsilon(G)}{|G|}$. Then*

$$\mathrm{Ldyn}_t(G) < \frac{c}{c+1}|G|.$$

Proof Let n be the order of G. If $n < c/2$, then $\lceil cn/(c+1) \rceil = n$, and hence the inequality $\mathrm{Ldyn}_t(G) < c|G|/(c+1)$ is trivial. Assume now that $n \ge c/2$. Let $d_1 \le d_2 \le \dots \le d_n$ be a degree sequence of G in increasing form and set $k_0 = \max\{k : \sum_{i=1}^{k}(d_i + 1) \le nt\}$. As we mentioned before, by a result from [10] we have $\mathrm{Ldyn}_t(G) \le k_0$. The assumption $t \le c(\epsilon(G)/n)$ implies that $nt \le (c/2n)\sum_{i=1}^{n} d_i$ and hence $\sum_{i=1}^{k_0}(d_i + 1) \le (c/2n)\sum_{i=1}^{n} d_i$ or equivalently

$$(2n/c) \le \Big(\sum_{i=1}^{n} d_i\Big) \Big/ \sum_{i=1}^{k_0}(d_i + 1).$$

https://doi.org/10.4153/CMB-2015-021-4 Published online by Cambridge University Press

Assume on the contrary that $k_0 \geq cn/(c+1)$. Then

$$\frac{2n}{c} \leq \frac{\sum_{i=1}^{k_0} d_i + \sum_{i=k_0+1}^{n} d_i}{\left(\sum_{i=1}^{k_0} d_i\right) + \frac{c}{c+1}n} \leq \frac{\left(\sum_{i=1}^{k_0} d_i\right) + \frac{n^2}{c+1}}{\left(\sum_{i=1}^{k_0} d_i\right) + \frac{c}{c+1}n}.$$

Therefore,

$$\frac{2n-c}{c} \sum_{i=1}^{k_0} d_i \leq \frac{n^2}{c+1} - \frac{2n^2}{c+1}.$$

The left-hand side of the last inequality is nonnegative, but the other side is negative. This contradiction implies $k_0 < cn/(c+1)$, as required. ■

3 Algorithmic Results

Algorithmic results concerning determining $\mathrm{dyn}_\tau(G)$, with various types of threshold assignments such as constant thresholds or majority thresholds, were studied in [4, 6, 7]. In this section, we first show that to compute the size of D such that (D,τ) is a $k\epsilon(G)$-Ldynamo of G is a coNP-hard problem on planar graphs. Then we prove that the same problem has a polynomial-time solution for forests. The formal definition of the decision problem concerning Ldynamo is the following, where k is any arbitrary but fixed real number with $0 < k \leq 2$.

Name: LARGEST DYNAMIC MONOPOLY (Ldynamo(k))

Instance: A graph G on say n vertices and a positive integer d.

Question: Is there an assignment of thresholds τ to the vertices of G with $n\overline{\tau} = \lfloor nk\epsilon(G) \rfloor$ such that $\mathrm{dyn}_\tau(G) \geq d$?

The following theorem shows coNP-hardness of the above problem. Recall that the Vertex Cover (VC) asks for the smallest number of vertices S in a graph G such that S covers any edge of G. Denote the smallest cardinality of any vertex cover of G by $\beta(G)$. The problem VC is NP-complete for planar graphs [9].

Theorem 3.1 *For any fixed k, where $0 < k \leq 2$,* Ldynamo(k) *is coNP-hard even for planar graphs.*

Proof We make a polynomial time reduction from VC (planar) to our problem. Let $\langle G, l \rangle$ be an instance of VC, where G is planar. Define $s = 4|E(G)| \times \max\{1, 1/k\} + 14$ and set $p = \lfloor (ks-2)/(2-k) \rfloor - |E(G)|$. Construct a graph H from G as follows. To each vertex v of G attach a star graph $K_{1,s-1}$ in such a way that v is connected to the central vertex of the star graph. Consider one of these star graphs and let y be a vertex of degree one in it. Add a path P of length $p-1$ starting from y (see Figure 1). The path P intersects the rest of the graph only in y. Call the resulting graph H. Since G is planar, H is planar too.

We claim that $\langle G, l \rangle$ is a yes-instance of VC if and only if $\langle H, l + \lfloor p/2 \rfloor + 1 \rangle$ is a no-instance of Ldynamo(k). We have $|E(H)| = |E(G)| + s + p$ from the construction

https://doi.org/10.4153/CMB-2015-021-4 Published online by Cambridge University Press

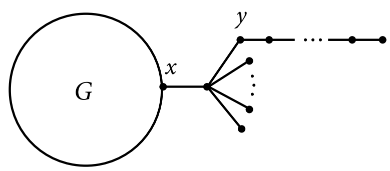

Figure 1: The graph H.

of H. Then since $p = \lfloor (ks-2)/(2-k) \rfloor - |E(G)|$, we have

$$\begin{aligned} p &\le (ks-2)/(2-k) - |E(G)| \\ &\Rightarrow 2p + 2|E(G)| + 2 \le k(s + p + |E(G)|) \\ &\Rightarrow 2p + 2|E(G)| + 2 \le \lfloor k|E(H)| \rfloor. \end{aligned}$$

Also, from the value of p we have

$$\begin{aligned} p &\ge (ks-2)/(2-k) - |E(G)| - 1 \\ &\Rightarrow 2p + 2|E(G)| + 2 + (2-k) > k(s + p + |E(G)|) \\ &\Rightarrow 2p + 2|E(G)| + 2 + \lfloor 2-k \rfloor \ge \lfloor k|E(H)| \rfloor \\ &\Rightarrow 2p + 2|E(G)| + 3 \ge \lfloor k|E(H)| \rfloor. \end{aligned}$$

Assume first that $\langle G, l \rangle$ is a no-instance of VC. Then $\beta(G) \ge l + 1$. We construct a threshold assignment τ for graph H as follows:

$$\tau(v) = \begin{cases} \deg_H(v) & v \in G \cup P, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $\overline{\tau} \le k\epsilon(H)$ and also $\operatorname{dyn}_\tau(H) = \beta(G) + \lfloor p/2 \rfloor$. Therefore, $\langle H, l + \lfloor p/2 \rfloor + 1 \rangle$ is a yes-instance for Ldynamo(k).
Let $\langle G, l \rangle$ be a yes-instance of VC. Then $\beta(G) < l+1$. Assume that (D, τ) is a $(k\epsilon(H))$-Ldynamo of H. The assumption that $s > 4|E(G)| + 14$ implies $|D \cap (H \setminus G)| \le \lfloor p/2 \rfloor$. On the other hand, $|D \cap G| \le \beta(G) < l + 1$. Hence, $|D| < l + \lfloor p/2 \rfloor + 1$. This shows that $\langle H, l + \lfloor p/2 \rfloor + 1 \rangle$ is a no-instance for Ldynamo(k). This completes the proof. ■

In the rest of this section we obtain a polynomial-time solution for forests (Theorem 3.8). We will need some prerequisites. We will make use of the concept of a resistant subgraph, defined in [12] as follows. Given (G, τ), any induced subgraph $K \subseteq G$ is said to be a *τ-resistant subgraph* in G, if for for any vertex $v \in K$ the inequality $\deg_K(v) \ge \deg_G(v) - \tau(v) + 1$ holds, where $\deg_G(v)$ is the degree of v in G. The following proposition in [12] shows the relation between resistant subgraphs and dynamic monopolies.

Proposition 3.2 ([12]) *A set $D \subseteq G$ is a τ-dynamo of graph G if and only if $G \setminus D$ does not contain any resistant subgraph.*

The following lemma provides more information on resistant subgraphs that are also triangle-free.

Lemma 3.3 *Assume that (G,τ) is given. Let also H be a triangle-free τ-resistant subgraph in G and $e = uv$ be any arbitrary edge with $u, v \in H$. Let τ' be defined as follows:*

$$\tau'(w) = \begin{cases} \tau(w) & \text{if } w \notin H, \\ 0 & \text{if } w \in H \setminus \{u, v'\}, \\ \deg_G(v) & \text{if } w = v, \\ \deg_G(u) & \text{if } w = u. \end{cases}$$

Then $\overline{\tau'} \leq \overline{\tau}$.

Proof Since H is triangle-free, then $|H| \geq \deg_H(u) + \deg_H(v)$. From the definition of the resistant subgraphs, for any vertex $w \in H$, one has $\tau(w) \geq \deg_{G \setminus H}(w) + 1$. Hence the following inequalities hold:

$$\begin{aligned} \sum_{w \in H} \tau(w) &\geq \sum_{w \in H} (\deg_{G \setminus H}(w) + 1) \geq |H| + \deg_{G \setminus H}(u) + \deg_{G \setminus H}(v) \\ &\geq \deg_H(u) + \deg_H(v) + \deg_{G \setminus H}(u) + \deg_{G \setminus H}(v) \\ &= \deg_G(u) + \deg_G(v). \end{aligned}$$

It turns out that $\sum_{w \in G} \tau'(w) \leq \sum_{w \in G} \tau(w)$, and hence $\overline{\tau'} \leq \overline{\tau}$. ∎

By a *(zero,degree)-assignment* we mean any threshold assignment τ for the vertices of a graph G such that for each vertex $v \in V(G)$, either $\tau(v) = 0$ or $\tau(v) = \deg_G(v)$. The following remark is useful and easy to prove. We omit its proof.

Remark 3.4 Assume that (G,τ) is given where τ is (zero,degree)-assignment. Let G_1 be the subgraph of G induced on $\{v \in G \mid \tau(v) = \deg_G(v)\}$. Then every minimum vertex cover of G_1 is a minimum τ-dynamo of G, and vice versa.

The following theorem concerning (zero,degree)-assignments in forests is essential in obtaining an algorithm for t-Ldynamo of forests for a given t.

Theorem 3.5 *Let F be a forest and let t be a positive constant. There exists a (zero,degree)-assignment τ' such that $\overline{\tau'} \leq t$ and*

$$\mathrm{Ldyn}_t(F) = \mathrm{dyn}_{\tau'}(F).$$

Proof Let (D,τ) be a t-Ldynamo of F. We prove the theorem by induction on $|D|$. Assume first that $|D| = 1$. Then by Proposition 3.2, F has at least one τ-resistant subgraph, say F'. Let u and v be two adjacent vertices in F'. Let τ' be the threshold assignment constructed in Lemma 3.3 such that $\tau'(u) = \deg_F(u)$ and $\tau'(v) = \deg_F(v)$. Modify τ' so that $\tau'(w) = 0$ for every vertex $w \in F \setminus \{u, v\}$. It is clear that τ' is a (zero,degree)-assignment. The edge uv is a τ'-resistant subgraph in F, and hence $\mathrm{dyn}_\tau(F) = \mathrm{Ldyn}_t(F) = 1$. This proves the induction assertion in this case.

Now assume that the assertion holds for any forest F with $|D| < k$. Let F be a forest with $\mathrm{Ldyn}_t(F) = k$ and let D be a t-Ldynamo of F with $|D| = k$. Also let F_1 be the largest τ-resistant subgraph of F. For any $v \in F_1$, set $\phi(v) = \tau(v) - \deg_{F \smallsetminus F_1}(v)$. By the definition of resistant subgraphs, $\phi(v) \geq 0$. It is clear that $\mathrm{dyn}_\phi(F_1) = k$. We show that there exists a (zero,degree)-assignment τ_1' for F_1 such that (D_1, τ_1') is a $\overline{\phi}$-Ldynamo of F_1 with $|D_1| = k$.

Let T be a connected component of F_1. Consider T as a top-down tree, where the topmost vertex is considered as the root of T. Since T is a ϕ-resistant subgraph in F_1, it implies that $D_1 \cap T$ is not the empty set. We argue that D_1 can be chosen in such a way that it does not contain any vertex $w \in T$ with $\phi(w) = 1$, except possibly the root. The reason is that if $w \in D_1 \cap T$ with $\phi(w) = 1$, then we replace w by its nearest ancestor (with respect to the root of T) whose threshold is not 1, and if there is no such ancestor, then we replace w by the root. Let $v \in D_1 \cap T$ be the farthest vertex from the root of T. Let T_v be the subtree of T consisting of v and its descendants. Obviously, $T_v \cap D_1 = \{v\}$.

Now we show that T_v is a ϕ-resistant subgraph in F_1. For each vertex $w \in T_v \smallsetminus \{v\}$, since $\phi(w) \geq 1$ and $\deg_{F_1 \smallsetminus T_v}(w) = 0$, then $\phi(w) \geq \deg_{F_1 \smallsetminus T_v}(w) + 1$. We also have $\phi(v) \geq \deg_{F_1 \smallsetminus T_v}(v) + 1$. Since $\phi(v) = 1$, v is the root of T and $T_v = T$ and hence $\deg_{F_1 \smallsetminus T_v}(v) = 0$. And if $\phi(v) > 1$, then $\deg_{F_1 \smallsetminus T_v}(v) \leq 1$. This proves that T_v is a ϕ-resistant subgraph in F_1. Let v' be an arbitrary neighbor of v in T_v. We construct the threshold assignment τ_1 for F_1 as follows:

$$\tau_1(w) = \begin{cases} \phi(w) & \text{if } w \notin T_v, \\ 0 & \text{if } w \in T_v \smallsetminus \{v, v'\}, \\ \deg_{F_1}(w) & \text{if } w \in \{v, v'\}. \end{cases}$$

By Lemma 3.3, we have $\overline{\tau_1} \leq \overline{\phi}$. Since edge vv' is a τ_1-resistant subgraph in F_1, then $\mathrm{dyn}_{\tau_1}(F_1) = \mathrm{dyn}_\phi(F_1) = k$, and so D_1 is a minimum τ_1-dynamo of F_1. Set $F_2 = F_1 \smallsetminus T_v$. Let u be the parent of the vertex v. Construct the threshold assignment τ_2 for F_2 as follows:

$$\tau_2(w) = \begin{cases} \tau_1(w) & \text{if } w \in F_2 \smallsetminus \{u\}, \\ \tau_1(w) - 1 & \text{if } w = u. \end{cases}$$

It is easily seen that the union of any τ_2-dynamo of F_2 and $\{v\}$ is a τ_1-dynamo of F_1 and also $D_1 \smallsetminus \{v\}$ is a τ_2-dynamo of F_2. Hence, $\mathrm{dyn}_{\tau_2}(F_2) = \mathrm{dyn}_{\tau_1}(F_1) - 1 = k - 1$. Let ϕ_2 be any threshold assignment for F_2 with $\overline{\phi_2} = \overline{\tau_2}$. Now construct the threshold assignment ϕ_1 for F_1 as follows:

$$\phi_1(w) = \begin{cases} \phi_2(w) & \text{if } w \in F_2 \smallsetminus \{u\}, \\ \tau_1(w) & \text{if } w \in T_v, \\ \phi_2(w) + 1 & \text{if } w = u. \end{cases}$$

Because the union of any ϕ_2-dynamo of F_2 and $\{v\}$, forms a ϕ_1-dynamo of F_1 and also for any ϕ_1-dynamo P of F_1, the set $P \cap F_2$ is a ϕ_2-dynamo of F_2 then $P \nsubseteq F_2$. This result and $\mathrm{dyn}_{\tau_2}(F_2) = k - 1$ imply that $\mathrm{Ldyn}_{\overline{\tau}_2}(F_2) = k - 1$. From the induction hypothesis there exists a (zero,degree)-assignment τ_2' for F_2 with $\overline{\tau_2'} \leq \overline{\tau_2}$ such that

https://doi.org/10.4153/CMB-2015-021-4 Published online by Cambridge University Press

$\mathrm{dyn}_{\tau_2'}(F_2) = k-1$. Now we construct the (zero,degree)-assignment τ_1' for F_1 as follows:

$$\tau_1'(w) = \begin{cases} \tau_2'(w) & \text{if } w \in F_2 \setminus \{u\}, \\ \tau_1(w) & \text{if } w \in T_v, \\ \tau_2'(w)+1 & \text{if } w = u \text{ and } \tau_2'(u) \neq 0, \\ 0 & \text{if } w = u \text{ and } \tau_2'(u) = 0. \end{cases}$$

It is easily seen that $\mathrm{dyn}_{\tau_1'}(F_1) = k$. We finally obtain the desired (zero,degree)-assignment τ' for F as follows:

$$\tau'(w) = \begin{cases} \deg_F(w) & \text{if } w \in F_1, \tau_1'(w) = \deg_{F\setminus F_1}(w), \\ 0 & \text{if } w \in F_1, \tau_1'(w) = 0, \\ 0 & \text{if } w \notin F_1. \end{cases}$$ ∎

In the following proposition we show that for any forest there exists a (zero, degree)-assignment that is zero outside the vertices of a matching.

Proposition 3.6 *Let F be a forest and let t a positive constant. Then there exists a matching M such that for the (zero,degree)-assignment τ defined below, we have $\overline{\tau} \le t$ and* $\mathrm{Ldyn}_t(F) = \mathrm{dyn}_\tau(F) = |M|$,

$$\tau(w) = \begin{cases} \deg_F(w) & \text{if } w \text{ is a vertex saturated by } M, \\ 0 & \text{otherwise.} \end{cases}$$

Proof By Theorem 3.5, there exists a (zero,degree)-assignment τ' such that $\overline{\tau'} \le t$ and $\mathrm{Ldyn}_t(F) = \mathrm{dyn}_{\tau'}(F)$. Let F_1 be a subgraph induced on all vertices w, with $\tau'(w) = \deg_F(w)$. Let D be a minimum vertex cover of F_1. Remark 3.4 implies that D is a minimum τ'-dynamic monopoly of F. Assume that M is a maximum matching of F_1. We show that M satisfies the conditions of the theorem. Each edge of M forms a τ-resistant subgraph in F. Hence $\mathrm{dyn}_\tau(F) \ge |M|$. Using the so-called König Theorem on bipartite graphs, we have $|D| = |M|$. Consequently, $\mathrm{dyn}_\tau(F) \ge |D| = \mathrm{dyn}_{\tau'}(F) = \mathrm{Ldyn}_t(F)$. It is easily seen that $\overline{\tau} \le \overline{\tau'} \le t$. The proof is complete. ∎

To prove Theorem 3.8, we will need the following proposition, whose proof is given in the appendix.

Proposition 3.7 *Let G be a bipartite graph, where each edge e has a cost $c(e) \ge 0$. Let also d be a positive number. Then there is a polynomial time algorithm that finds a maximum matching M in G with* $\mathrm{cost}(M) \le d$, *where* $\mathrm{cost}(M) = \sum_{e\in M} c(e)$.

Theorem 3.8 *Given a forest F and a positive number t, there exists an algorithm that computes* $\mathrm{Ldyn}_t(F)$ *in polynomial-time.*

Proof For each edge $e = uv$ of F define $\mathrm{cost}(e) = \deg_F(u) + \deg_F(v)$, and for each $S \subseteq E(F)$ define $\mathrm{cost}(S) = \sum_{e\in S} \mathrm{cost}(e)$. Let M be any arbitrary matching and let τ be a (zero,degree)-assignment constructed from M as obtained in Proposition 3.6. It is easily seen that $\overline{\tau} \le t$ if and only if $\mathrm{cost}(M) \le t|F|$. Now, if M is a maximum matching satisfying $\mathrm{cost}(M) \le t|F|$, then Proposition 3.6 implies $\mathrm{Ldyn}_t(F) = \mathrm{dyn}_\tau(F) = |M|$.

By Proposition 3.7, there is a polynomial-time algorithm that finds maximum matching M in F with $\mathrm{cost}(M) \le c$ for any value c. Then using Proposition 3.6 for given forest F and constant t, there is a polynomial time algorithm that finds a (zero,degree)-assignment τ such that $\mathrm{Ldyn}_t(F) = \mathrm{dyn}_\tau(F)$. From the other side, finding a minimum vertex cover in bipartite graphs is a polynomial-time problem. Therefore, using Remark 3.4 a minimum τ-dynamic monopoly for F can be found in polynomial-time. ■

For further research, it would be interesting to obtain other families of graphs for which $\mathrm{Ldynamo}(k)$ has polynomial-time solution. Also, we do not yet know whether $\mathrm{Ldynamo}(k) \in \mathrm{NP} \cup \mathrm{coNP}$, but our guess is that it is not.

Appendix A

We prove Proposition 3.7 using the minimum cost flow algorithm. The minimum cost flow problem (MCFP) is as follows (see *e.g.,* [1] for details).

Let $G = (V, E)$ be a directed network with a cost $c(i,j) \ge 0$ for any of its edges (i,j). Also for any edge $(i,j) \in E$ there exists a capacity $u(i,j) \ge 0$. We associate with each vertex $i \in V$ a number $b(i)$ that indicates its source or sink depending on whether $b(i) > 0$ or $b(i) < 0$. The minimum cost flow problem (MCFP) requires the determination of a flow mapping $f: E \to \mathbb{R}$ with minimum cost $z(f) = \sum_{(i,j)\in E} c(i,j) f(i,j)$ subject to the following two conditions:

(a) $0 \le f(i,j) \le u(i,j)$ for all $(i,j) \in E$ (capacity restriction);

(b) $\sum_{\{j:(i,j)\in E\}} f(i,j) - \sum_{\{j:(j,i)\in E\}} f(j,i) = b(i)$ for all $i \in V$ (demand restriction).

In [1], a polynomial-time algorithm is given such that determines if such a mapping f exists. And in case of existence, the algorithm outputs f. Furthermore, if all values $u(i,j)$ and $b(i)$ are integers, then the algorithm obtains an integer-valued mapping f. In the following theorem, we prove Proposition 3.7.

Theorem A.1 *Let $G[X,Y]$ be a bipartite graph with $\mathrm{cost}(ij) \ge 0$ for each edge $ij \in G$ and let d be a positive number. Then there exists a polynomial-time algorithm that finds maximum matching M in G with $\mathrm{cost}(M) \le d$.*

Proof Construct a directed network H from bipartite graph $G[X,Y]$ as follows. Add two new vertices s and t as the source and the sink of H, respectively and directed edges (s,x) for each $x \in X$ and (y,t) for each $y \in Y$. Make all other edges directed from X to Y. For each edge (i,j) set $u(i,j) = 1$ and define $c(i,j)$ as follows:

$$c(i,j) = \begin{cases} 0 & i = s \text{ or } j = t, \\ \mathrm{cost}(ij) & i \in X, j \in Y. \end{cases}$$

For each vertex $i \in X \cup Y$, set $b(i) = 0$ and define $b(s) = -b(t) = k$, where k is an arbitrary positive integer. We now have an instance of MCFP. Assume that there exists a minimum cost flow mapping for this instance (obtained by the above-mentioned algorithm of [1]). Since $u(i,j)$ and $b(i)$ are integers, f is an integer-valued mapping. Therefore, $f(i,j)$ is either 0 or 1. Let M be the set of edges (i,j) with $f(i,j) = 1$, where $i \in X$ and $j \in Y$. Clearly M is a matching of size k having $\mathrm{cost}(M) = z(f)$, where $z(f)$ is as defined in MCFP above.

Conversely, let M' be any arbitrary matching in G with $|M'| = k$. We construct a flow mapping f as follows:

$$f(i,j) = \begin{cases} 1 & i \in X, j \in Y, ij \in M', \\ 1 & i = s, jl \in M' \text{ for some } l \in Y, \\ 1 & j = t, li \in M' \text{ for some } l \in X, \\ 0 & \text{otherwise.} \end{cases}$$

The conditions of MCFP are satisfied for f. Also, $z(f) = \mathrm{cost}(M')$. We conclude that to obtain a matching of size k with the minimum cost is equivalent to obtaining a minimum cost flow mapping for the associated MCFP instance (note that k is a parameter of this instance). We conclude that in order to find a matching M satisfying $\mathrm{cost}(M) \le d$ and with the maximum size, it is enough to run the corresponding algorithm for the above-constructed MCFP instance for each k, where k varies from 1 to $|G|/2$. Note that $|G|/2$ is an upper bound for the size of any matching. This completes the proof. ∎

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network flows. Theory, algorithms, and applications.* Prentice Hall, Englewood Cliffs, NJ, 1993.

[2] J. A. Bondy and U.S.R. Murty, *Graph theory.* Graduate Texts in Mathematics, 244, Springer, New York, 2008.

[3] C-L. Chang and Y-D. Lyuu, *On irreversible dynamic monopolies in general graphs.* arxiv:0904.2306v3

[4] C. C. Centeno, M. C. Dourado, L. D. Penso, D. Rautenbach, and J. L. Szwarcfiter, *Irreversible conversion of graphs.* Theoret. Comput. Sci. **412**(2011), no. 29, 3693–3700. http://dx.doi.org/10.1016/j.tcs.2011.03.029

[5] C. C. Centeno and D. Rautenbach, *Remarks on dynamic monopolies with given average thresholds.* Discuss. Math. Graph Theory **35**(2015), no. 1, 133–140. http://dx.doi.org/10.7151/dmgt.1788

[6] N. Chen, *On the approximability of influence in social networks.* SIAM J. Discrete Math. **23**(2009), no. 3, 1400–1415. http://dx.doi.org/10.1137/08073617X

[7] P. A. Dreyer, Jr. and F. S. Roberts, *Irreversible k-threshold processes: graph-theoretical threshold models of the spread of disease and of opinion.* Discrete Appl. Math. **157**(2009), no. 7, 1615–1627. http://dx.doi.org/10.1016/j.dam.2008.09.012

[8] P. Flocchini, R. Kralovic, A. Roncato, P. Ružička, and N. Santoro, *On time versus size for monotone dynamic monopolies in regular topologies.* J. Discrete Algorithms **1**(2003), no. 2, 129–150. http://dx.doi.org/10.1016/S1570-8667(03)00022-4

[9] M. R. Garey and D. S. Johnson, *Computers and intractability. A guide to theory of NP-completeness.* A Series of Books in the Mathematical Sciences, W. H. Freeman & Co., San Francisco, CA, 1979.

[10] K. Khoshkhah, H. Soltani, M. Zaker, On dynamic monopolies of graphs: The average and strict majority thresholds, Discrete Optim. 9 (2012) 77–83. http://dx.doi.org/10.1016/j.disopt.2012.02.001

[11] H. Soltani and M. Zaker, *Dynamic monopolies of graphs with probabilistic thresholds.* Bull. Aust. Math. Soc. **90**(2014), no. 3, 363–375. http://dx.doi.org/10.1017/S0004972714000604

[12] M. Zaker, *On dynamic monopolies of graphs with general thresholds.* Discrete Math. **312**(2012), no. 6, 1136–1143. http://dx.doi.org/10.1016/j.disc.2011.11.038

[13] ———, *Generalized degeneracy, dynamic monopolies and maximum degenerate subgraphs.* Discrete Appl. Math. **161**(2013), no. 16–17, 2716–2723. http://dx.doi.org/10.1016/j.dam.2013.04.012

Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan 45137-66731, Iran
e-mail: khoshkhah@iasbs.ac.ir mzaker@iasbs.ac.ir