NONIMMERSIONS OF COMPLEX GRASSMANN MANIFOLDS

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1. Introduction

If an oriented manifold M immerses in codimension k, then the normal bundle has dimension k such that its Euler class $\chi \in H^k(M; \mathbf{Z})$ and $\chi^2 \in H^{2k}(M; \mathbf{Z})$. (Cf. (**3**)).

If M is the complex Grassmann manifold $G_2(\mathbf{C}^n)$ of 2-planes in $\mathbf{C}^n$ ($n = 4, 5, \ldots, 15, 17$), then $\dim M = 4n - 8 \equiv d$ and we shall show that although M immerses in $\mathbf{R}^{2d-1}$ by classical results (**3**), M does not immerse in $\mathbf{R}^{d+d/2}$.

The same result was obtained for $n = 4$ and 5 by Connell (**2**) and for $n = 6$ and 7 by the author (**6**). The nonimmersion results of this paper are new for $n = 8, 9, \ldots, 15, 17$ and they are an improvement over the result for the general $G_2(\mathbf{C}^n)$ obtained in (**5**). In this paper, we use generators of the cohomology ring of $G_2(\mathbf{C}^n)$ different from those used in (**2**) and (**6**) and this simplifies the calculations considerably.

2. $G_2(\mathbf{C}^n)$

Since the dimension of $G_2(\mathbf{C}^n)$ is $4n - 8$, it follows that $\chi \in H^{2n-4}(G_2(\mathbf{C}^n); \mathbf{Z})$. We shall first investigate the group, $H^{2n-4}(G_2(\mathbf{C}^n); \mathbf{Z})$. We denote the Schubert variety, $\Omega_{a_0 a_1}$ by $[a_1, a_2]$ and the corresponding Schubert class by $[a_0, a_1]^*$.

Lemma 2.1. *The cohomology group $H^{2n-4}(G_2(\mathbf{C}^n); \mathbf{Z})$ is freely generated either by the $(n-1)/2$ Schubert classes*

$$\left\{[0, n-1]^*, [1, n-2]^*, \ldots, \left[\frac{n-3}{2}, \frac{n+1}{2}\right]^*\right\}, \quad \textit{if } n \textit{ is odd}$$

or by the $n/2$ Schubert classes

$$\left\{[0, n-1]^*, [1, n-2]^*, \ldots, \left[\frac{n-2}{2}, \frac{n}{2}\right]^*\right\}, \quad \textit{if } n \textit{ is even.}$$

Moreover, the square of each generator is equal to $[0, 1]^*$, the generator of $H^{4n-8}(G_2(\mathbf{C}^n); \mathbf{Z})$ and any mixed product of two different generators is equal to zero.

Proof. From Theorem II p. 352 of (**4**), and since

$$\dim [a_0, a_1] = a_0 + a_1 - 1,$$

it follows that $H^{2n-4}(G_2(\boldsymbol{C}^n); \boldsymbol{Z})$ is generated by all Schubert classes

$$\{[a_0, a_1]^* \mid a_0 + a_1 - 1 = n - 2\}.$$

Hence, the first part of the lemma follows. Also from Theorem I p. 327 and Theorem II p. 331 of (**4**), we have that Schubert varieties of complementary dimension intersect if and only if

$$a_1 + b_0 \geqq n - 1 \quad \text{and} \quad a_0 + b_1 \geqq n - 1,$$

and that the intersection is simple and is at a unique point. Thus the second part of the lemma also follows.

The result of this paper can now be stated as

Proposition 2.2. *If $d = 4n - 8 = \dim(G_2(\boldsymbol{C}^n))$, and $n = 4, \ldots, 15, 17$ we have that*

$$G_2(\boldsymbol{C}^n) \subseteq \boldsymbol{R}^{2d-1} \quad \text{and} \quad G_2(\boldsymbol{C}^n) \nsubseteq \boldsymbol{R}^{d+1/2d}.$$

Proof. The fact that $G_2(\boldsymbol{C}^n) \subseteq \boldsymbol{R}^{2d-1}$ follows from (**3**). Now the generator $[0, 1]^*$ of the cohomology group $H^{4n-8}(G_2(\boldsymbol{C}^n); \boldsymbol{Z})$ is given by

$$[0, 1]^* = w_{2,n-2}(1; F) = [\sigma_2(\gamma_0, \gamma_1)]^{n-2}.$$

(Cf. p. 328 of (**7**), where $\sigma_2(\gamma_0, \gamma_1) = c_2(\gamma)$, the second Chern class of the canonical bundle γ over $G_2(\boldsymbol{C}^n)$). Hence from Section 3 of (**5**), it follows that the generator y can be identified as

$$y = -\sigma_2(\gamma_0, \gamma_1).$$

Hence a generator of $H^{4n-8}(G_2(\boldsymbol{C}^n); \boldsymbol{Z})$ is

$$[0, 1]^* = [\sigma_2(\gamma_0, \gamma_1)]^{n-2} = (-y)^{n-2}.$$

Now consider the case when n is odd and assume M immerses in $\boldsymbol{R}^{d+d/2}$. Let

$$\chi = a_1[0, n-1]^* + a_2[1, n-2]^* + \ldots + a_{(n-1)/2}\left[\frac{n-3}{2}, \frac{n+1}{2}\right]^*$$

be the Euler class of the normal bundle. Therefore, χ is an element of $H^{d/2}(G_2(\boldsymbol{C}^n); \boldsymbol{Z})$ and by Lemma 2.1 above,

$$\begin{aligned} \chi^2 &= (a_1^2 + a_2^2 + \ldots + a_{(n-1)/2}^2)[0, 1]^* \\ &= -(a_1^2 + a_2^2 + \ldots + a_{(n-1)/2}^2)y^{n-2}. \end{aligned}$$

Then by (**3**), $\chi^2 = p_{n-2}$ where p_r is the r-th Pontrjagin class, and from Section 5 of (**5**) we have

$$a_1^2 + a_2^2 + \ldots + a_{(n-1)/2}^2 = 3\binom{3t-1}{t-1}\binom{3t-2}{t-2}\frac{t^2 - 9t + 6}{(t-1)(3t-2)},$$

where $n = 2t + 1$. Now the right hand side of the above quadratic equation is negative if and only if $2 \leqq t \leqq 8$, i.e. if and only if $n = 5, 7, 9, 11, 13, 15, 17$. In these cases, the quadratic equation has no integral solution. This is a contradiction and so nonimmersion in codimension $2n - 4 = d/2$ is established.

Similarly, we obtain the following quadratic equation when n is even.

$$a_1^2+a_2^2+\ldots+a_{n/2}^2=\binom{3t-3}{t-2}^2\frac{t^2-8t+6}{(t-1)^2},$$

where $n=2t$. The right hand side is negative if and only if $2\leqq t\leqq 7$, i.e. if and only if $n=4, 6, 8, 10, 12, 14$. In these cases, the quadratic equation has no integral solution and so nonimmersion in codimension $2n-4=d/2$ is also established. This completes the proof of the proposition.

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