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ON GENERALIZED PÓLYA URN MODELS

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Abstract

We study an urn model introduced in the paper of Chen and Wei (2005), where at each discrete time step m balls are drawn at random from the urn containing colors white and black. Balls are added to the urn according to the inspected colors, generalizing the well known Pólya–Eggenberger urn model, case $m = 1$. We provide exact expressions for the expectation and the variance of the number of white balls after n draws, and determine the structure of higher moments. Moreover, we discuss extensions to more than two colors. Furthermore, we introduce and discuss a new urn model where the sampling of the m balls is carried out in a step-by-step fashion, and also introduce a generalized Friedman's urn model.

Keywords: Urn model; limiting distribution

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1. Introduction

Pólya–Eggenberger urn models are defined as follows. At the start, time 0, the urn contains W_0 white balls and B_0 black balls. The evolution of the urn occurs in discrete time steps. At every step a ball is chosen at random from the urn. The color of the ball is inspected and then the ball is reinserted into the urn. According to the observed color of the ball, balls are added/removed due to the following rules. If we have chosen a white ball, we put into the urn a white balls and b black balls, but if we have chosen a black ball, we put into the urn c white balls and d black balls. The values $a, b, c, d \in \mathbb{Z}$ are fixed integer values and the urn model is specified by the 2×2 ball replacement matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Urn models are simple and useful mathematical tools for describing many evolutionary processes in diverse fields of application such as analysis of algorithms and data structures, statistics, and genetics; see [11], [12], and [13].

One of the most fundamental urn models is the original Pólya–Eggenberger urn model [4], associated with the ball replacement matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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The Pólya–Eggenberger urn is a balanced urn model; the total number of added or removed balls is constant and so independent of the observed color. A parameter of interest is the number W_n of white balls contained in the urn after n draws. Various generalizations of this urn model have been considered and results appear in [1], [6], [7], [9],[14], and [16]; we refer to [2] for a brief discussion of the aforementioned works.

This work is devoted to the study of a generalization of the Pólya–Eggenberger urn model, where *several balls* are drawn at each discrete time step, their colors are inspected, and the balls are reinserted into the urn. The addition/removal of balls depends on the combinations of colors of the drawn balls. Such urn models recently received some attention in the literature; see, e.g. [2], [13], and [15]. Chen and Wei [2] introduced a particular urn model they called model M, where $m \geq 1$ balls are drawn from the urn at each discrete time step. Say $m - \ell$ white balls and ℓ black balls have been drawn, $0 \leq \ell \leq m$, their colors are noted, and the drawn balls are returned to the urn together with addition $c(m-\ell)$ white balls and $c\ell$ black balls. The ball replacement matrix of this urn model is a rectangular matrix M, given by

$$M = \begin{bmatrix} mc & 0 \\ (m-1)c & c \\ \dots & \dots \\ c & (m-1)c \\ 0 & mc \end{bmatrix}, \tag{1}$$

with parameter $c \in \mathbb{N}$ and $m \geq 1$. The rows of the rectangular replacement matrix encode the sampling scheme in the obvious way; the ℓth row corresponds to the case of drawing a combination of $(m-\ell)$ white balls and ℓ black balls, $0 \leq \ell \leq m$, where $c(m-\ell)$ white balls and $c\ell$ black balls are being added. Note that in model M the drawing of the m balls occurs without replacement, in other words the distribution of the number of white balls in the sample of size m follows a hypergeometric distribution.

Chen and Wei studied the distribution of the number of white balls W_n after n draws and showed the almost sure convergence of W_n, suitably normalized, to a continuous distribution by using martingales. The aim of this note is to provide further insight into the limiting distribution of the number of white balls by providing exact expressions for the expectation and the variance of W_n, from which one obtains the expectation and variance of the limit law. Moreover, we also obtain, in principle, exact expressions for arbitrary moments of the limit law. Note that the case $m = 1$ corresponds to the well known Pólya–Eggenberger urn

$$M = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix},$$

which is completely understood using different arguments such as counting arguments, or stochastic processes; see, e.g. [10]. It is known that the proportion of white balls after n draws is a martingale and has a beta distribution as the limit law with parameters b/c and w/c. Therefore, the case $m = 1$ is excluded from our study.

Throughout this work we use the notations $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$, where $\left[{n \atop k}\right]$ denotes the unsigned Stirling numbers of the first kind (also called the Stirling cycle numbers), and $\left\{{n \atop k}\right\}$ denotes the Stirling numbers of the second kind (see [8]). Moreover, we denote with $x^{\underline{\ell}}$ the ℓth falling factorial, $x^{\underline{\ell}} = x(x-1)\dots(x-\ell+1)$, $\ell \geq 0$, with $x^{\underline{0}} = 1$.

2. The distributional equation

We consider the urn model called model M of [2], specified by ball replacement matrix (1). Let W_n and B_n denote the random variables counting the numbers of white balls and black balls respectively after n draws, $n \geq 0$. We assume that the initial numbers B_0 of black balls and W_0 of white balls satisfy $B_0 > 0$ and $W_0 > 0$, and the total number of balls at time 0, T_0, satisfies $T_0 = B_0 + W_0 \geq m$, in order to avoid any degenerate cases. Since the urn model is balanced, regardless of the inspected color combination a total of mc balls are added to the urn at every discrete time step; the total number T_n of balls after m draws is a deterministic quantity, and given by

$$T_n = T_0 + nmc = W_n + B_n, \qquad n \geq 0. \tag{2}$$

We are interested in the random variable W_n, counting the number of white balls contained in the urn after n draws, $n \geq 0$. The starting point of our considerations is the distributional equation,

$$W_n \stackrel{\mathrm{D}}{=} W_{n-1} + \sum_{k=0}^{m} kc\, \mathbf{1}_n(W^k B^{m-k}), \tag{3}$$

which says that the number of white balls after n draws can be decomposed as the number of white balls after $n-1$ draws, plus the additional balls added when the colors of the nth draw have been inspected, $n \geq 1$. Here the random variables $\mathbf{1}_n(W^k B^{m-k})$, $0 \leq k \leq m$ denote the indicators of drawing k white balls and $m-k$ black balls from the urn at the nth draw, $n \geq 1$. Let $\mathcal{F}_{n-1}$ denote the σ-field generated by the first $n-1$ draws. By (2), we have $B_{n-1} = T_{n-1} - W_{n-1}$, and further

$$\mathbb{P}\{\mathbf{1}_n(W^k B^{m-k}) = 1 \mid \mathcal{F}_{n-1}\} = \frac{\binom{W_{n-1}}{k}\binom{B_{n-1}}{m-k}}{\binom{T_{n-1}}{m}} = \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}}{\binom{T_{n-1}}{m}}, \tag{4}$$

where $0 \leq k \leq m$, $n \geq 1$. In order to study the moments $\mathbb{E}(W_n^s)$, $s \geq 1$, of the random variable W_n, we first have to derive a distributional equation for W_n^s. In order to do so, we take the sth power in (3) and use the fact that indicator variables are mutually exclusive. We obtain the distributional equation, valid for $n \geq 1$, $s \geq 1$,

$$W_n^s \stackrel{\mathrm{D}}{=} W_{n-1}^s + \sum_{\ell=1}^{s} \binom{s}{\ell} W_{n-1}^{s-\ell} c^\ell \sum_{k=1}^{m} k^\ell\, \mathbf{1}_n(W^k B^{m-k}). \tag{5}$$

3. Results for the moments

Theorem 1. *The expected value of the random variable W_n, counting the numbers of white balls after n draws, is given by $\mathbb{E}(W_n) = (W_0/T_0)(nmc + T_0)$, and the variance $\mathbb{V}(W_n) = \mathbb{E}(W_n^2) - \mathbb{E}(W_n)^2$ is determined via the second moment*

$$\mathbb{E}(W_n^2) = \frac{\binom{n-1+\lambda_1}{n}\binom{n-1+\lambda_2}{n}}{\binom{n-1+T_0/mc}{n}\binom{n-1+(T_0-1)/mc}{n}} \times \left(W_0^2 + \frac{W_0 c^2 m}{T_0} \sum_{\ell=0}^{n-1} \frac{\ell + (T_0 - m)/mc}{\ell + (T_0-1)/mc} \frac{\binom{\ell+T_0/mc}{\ell+1}\binom{\ell+(T_0-1)/mc}{\ell+1}}{\binom{\ell+\lambda_1}{\ell+1}\binom{\ell+\lambda_2}{\ell+1}} \right),$$

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where the values λ_1, λ_2 *are given by*

$$\lambda_{1,2} = \frac{-1 + 2mc + 2T_0 \pm \sqrt{1 + 4mc(1 + c)}}{2mc}.$$

Concerning higher moments, we obtain the following recursive characterization.

Theorem 2. *The sth moment* $\mathbb{E}(W_n^s)$ *is, for* $s \geq 1$*, given by*

$$\mathbb{E}(W_n^s) = \biggl(\prod_{j=0}^{n-1} \alpha_{j,s}\biggr)\biggl(W_0^s + \sum_{\ell=0}^{n-1} \frac{\beta_{\ell,s}}{\prod_{j=0}^{\ell} \alpha_{j,s}}\biggr), \tag{6}$$

where the quantities $\alpha_{n,s}$ *and* $\beta_{n,s}$ *are defined as*

$$\alpha_{n,s} = \sum_{\ell=0}^{s} c^\ell \frac{\binom{s}{\ell}\binom{m}{\ell}}{\binom{T_n}{\ell}}, \tag{7}$$

$$\beta_{n,s} = \sum_{i=2}^{s} \mathbb{E}(W_n^{s+1-i}) \sum_{\ell=i}^{s} \binom{s}{\ell} c^\ell \sum_{j=\ell+1-i}^{\ell} (-1)^{j+i-\ell-1} \frac{\left[{\ell \atop j}\right]\left\{{j \atop \ell+1-i}\right\}\binom{m}{j}}{\binom{T_n}{j}}.$$

Here $\left[{n \atop k}\right]$ *denotes an (unsigned) Stirling number of the first kind, and* $\left\{{n \atop k}\right\}$ *denotes a Stirling number of the second kind.*

The result above enables us to obtain in principle arbitrary high moments, since the moment $\mathbb{E}(W_n^s)$ can be expressed in terms of the moments $\mathbb{E}(W_h^r)$, with $0 \leq h \leq n-1$, $0 \leq r \leq s-1$.

Remark 1. As mentioned in the introduction, the case $m = 1$ corresponds to the Pólya-Eggenberger urn

$$M = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

From Theorem 2, we have, for $s \geq 1$, the sth moment $\mathbb{E}(W_n^s)$ given by (6) with $\alpha_{n,s} = 1 + (cs/T_n) = T_{n+s}/T_n$ and $\beta_{n,s} = \sum_{i=2}^{s} c^i \binom{s}{i}\mathbb{E}(W_n^{s+1-i})/T_n$.

Thus, $\mathbb{E}(W_n) = W_0 T_n / T_0$ and

$$\mathbb{E}(W_n^2) = \biggl(\prod_{j=0}^{n-1} \frac{T_{j+2}}{T_j}\biggr)\biggl(W_0^2 + \sum_{\ell=0}^{n-1} \frac{c^2\mathbb{E}(W_n)/T_n}{\prod_{j=0}^{\ell} T_{j+2}/T_j}\biggr) = \frac{W_0 T_n (c^2 n + T_{n+1} W_0)}{T_0 T_1}.$$

These imply that $\mathbb{V}(W_n) = c^2 n T_n W_0 B_0 / (T_0^2 T_1)$, where $B_0 = T_0 - W_0$. Note that, for $m = 1$, the probability mass function can be directly obtained using different techniques, leading to explicit expression for all moments, and the limiting distribution (see, e.g. [5]).

The following results concern the moments of the normalized random variable W_n/n.

Corollary 1. *The limits* $\lim_{n\to\infty} \mathbb{E}(W_n^s/n^s)$ *exist; in particular, we obtain for the expected value* $\lim_{n\to\infty} \mathbb{E}(W_n)/n = W_0 mc/T_0$*, and for the normalized second moment*

$$\lim_{n\to\infty} \frac{\mathbb{E}(W_n^2)}{n^2} = \frac{\Gamma(T_0/mc)\Gamma((T_0-1)/mc)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} \times \biggl(W_0^2 + \frac{W_0 c^2 m}{T_0} \sum_{\ell=0}^{\infty} \frac{\ell + (T_0 - m)/mc}{\ell + (T_0 - 1)/mc} \frac{\binom{\ell + T_0/mc}{\ell+1}\binom{\ell+(T_0-1)/mc}{\ell+1}}{\binom{\ell+\lambda_1}{\ell+1}\binom{\ell+\lambda_2}{\ell+1}}\biggr).$$

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Moreover, for arbitrary $s \geq 1$ the limit of the normalized sth moment can be expressed in terms of an infinite sum

$$\lim_{n\to\infty} \frac{\mathbb{E}(W_n^s)}{n^s} = \frac{\prod_{\ell=1}^{s} \Gamma((T_0+1-\ell)/mc)}{\prod_{\ell=1}^{s} \Gamma(\lambda_{\ell,s})} \left(W_0^s + \sum_{\ell=0}^{\infty} \frac{\beta_{\ell,s}}{\prod_{j=0}^{\ell} \alpha_{j,s}} \right)$$

with $\beta_{\ell,s}$ as defined above in Theorem 2, involving the moments of the form $\mathbb{E}(W_h^r)$ with $0 \leq h < \infty$ and $0 \leq r \leq s-1$. Here the $\lambda_{\ell,s}$ denote the roots (times by minus one) of the equation in the variable x

$$\frac{s!}{(mc)^s} \sum_{\ell=0}^{s} c^\ell \binom{m}{\ell} \binom{xmc+T_0-\ell}{s-\ell} = \prod_{\ell=1}^{s} (x+\lambda_{\ell,s}).$$

Remark 2. As mentioned in the introduction Chen and Wei [2] proved almost sure (a.s.) convergence of W_n, namely $\mathcal{W}_n = W_n/T_n = W_n/(nmc+T_0) \to \mathcal{W}_\infty$ a.s. Hence, we get the expectation and the variance of $\mathcal{W}_\infty$ by the relations $\mathbb{E}(\mathcal{W}_\infty) = \lim_{n\to\infty} \mathbb{E}(W_n)/nmc$, and $\mathbb{V}(\mathcal{W}_\infty) = \lim_{n\to\infty} \mathbb{V}(W_n)/(nmc)^2$. Our results show that the distribution of $\mathcal{W}_\infty$ is not an ordinary beta law, in contrast to the case $m = 1$. Note that the moments of W_n do not grow very fast, since they satisfy the trivial bounds $\mathbb{E}(W_n^s) \leq (nmc+T_0)^s$. Hence, by Carleman's condition the limit law $\mathcal{W}_\infty$ is uniquely determined by its moments $\mathbb{E}(\mathcal{W}_\infty^s)$, which are given by $\mathbb{E}(\mathcal{W}_\infty^s) = \lim_{n\to\infty} \mathbb{E}(W_n^s)/(nmc)^s$.

Example 1. For the case $m = 2$ and $c = 1$, applying Theorem 1, the expected value of the random variable W_n is given by $\mathbb{E}(W_n) = (W_0/T_0)(2n+T_0)$. Moreover, by Theorem 2, we see that, for $s \geq 2$, the sth moment $\mathbb{E}(W_n^s)$ is given by (6) with

$$\alpha_{n,s} = 1 + \frac{2s}{T_n} + \frac{\binom{s}{2}}{\binom{T_n}{2}}, \qquad \beta_{n,s} = \sum_{i=2}^{s} \mathbb{E}(W_n^{s+1-i}) \left(\frac{2\binom{s}{i}}{T_n} - \frac{\binom{s}{i}\left\{{i \atop 2}\right\}}{\binom{T_n}{2}} + \frac{\binom{s}{i+1}\left\{{i+1 \atop 2}\right\}}{\binom{T_n}{2}} \right).$$

For instance, if $s = 2$, then

$$\alpha_{n,2} = 1 + \frac{4}{T_n} + \frac{1}{\binom{T_n}{2}} = \frac{T_n^2 + 3T_n - 2}{T_n(T_n-1)},$$

$$\beta_{n,2} = \mathbb{E}(W_n) \left(\frac{2}{T_n} - \frac{1}{\binom{T_n}{2}} \right) = \frac{2W_0}{T_0} \frac{T_n-2}{T_n-1},$$

and so

$$\mathbb{E}(W_n^2) = \frac{W_0 T_n^2}{T_0} - \frac{4^n W_0 B_0 \Gamma(n+\lambda_1)\Gamma(n+\lambda_2)\Gamma(T_0-1)}{\Gamma(2n+T_0-1)\Gamma(\lambda_1)\Gamma(\lambda_2)},$$

where $B_0 = T_0 - W_0$, $\lambda_1 = (2T_0 + 3 + \sqrt{17})/4$ and $\lambda_2 = (2T_0 + 3 - \sqrt{17})/4$. The above results imply that $\mathbb{E}(\mathcal{W}_\infty) = \lim_{n\to\infty} \mathbb{E}(W_n/T_n) = W_0/T_0$ and

$$\mathbb{E}(\mathcal{W}_\infty^2) = \lim_{n\to\infty} \frac{\mathbb{E}(W_n^2)}{T_n^2} = \frac{W_0}{T_0} - \frac{\sqrt{\pi} W_0 B_0 \Gamma(T_0-1)}{2^{T_0}\Gamma(\lambda_1)\Gamma(\lambda_2)},$$

and so

$$\mathbb{V}(\mathcal{W}_\infty^2) = \mathbb{E}(\mathcal{W}_\infty^2) - \mathbb{E}(\mathcal{W}_\infty)^2 = \frac{W_0 B_0}{T_0^2} \left(1 - \frac{\sqrt{\pi} T_0^2 \Gamma(T_0-1)}{2^{T_0}\Gamma(\lambda_1)\Gamma(\lambda_2)} \right).$$

4. Determining the structure of the moments

In order to prove Theorems 1 and 2, we need the following result.

Lemma 1. *The moments $\mathbb{E}(W_n^s)$ satisfy the recurrence relation*

$$\mathbb{E}(W_n^s) = \alpha_{n-1,s}\mathbb{E}(W_{n-1}^s) + \beta_{n-1,s}, \qquad n, s \geq 1, \tag{8}$$

where the quantities $\alpha_{n,s}$ and $\beta_{n,s}$ are as defined in Theorem 2.

Proof. Our starting point for the proof of Lemma 1 is the distributional equation for W_n^s, (5), and we take the conditional expectation with respect to $\mathcal{F}_{n-1}$. This leads to

$$\mathbb{E}(W_n^s \mid \mathcal{F}_{n-1}) = W_{n-1}^s + \sum_{\ell=1}^{s}\binom{s}{\ell}W_{n-1}^{s-\ell}c^\ell\sum_{k=1}^{m}k^\ell\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}}{\binom{T_{n-1}}{m}}, \tag{9}$$

where we have used (3) and (4). In order to simplify the stated expression we have to use some combinatorial identities. We convert the ordinary powers of k into falling factorials using the Stirling numbers of the second kind,

$$x^\ell = \sum_{j=1}^{\ell}\left\{ {\ell \atop j} \right\}x^{\underline{j}},$$

where $x^{\underline{j}} = x(x-1)(x-2)\dots(x-(j-1))$, $\ell \geq 1$. We have

$$\begin{aligned}\sum_{k=1}^{m}k^\ell\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k} &= \sum_{k=1}^{m}\sum_{j=1}^{\ell}\left\{ {\ell \atop j} \right\}k^{\underline{j}}\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}\\ &= \sum_{j=1}^{\ell}\sum_{k=j}^{m}\left\{ {\ell \atop j} \right\}k^{\underline{j}}\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}\\ &= \sum_{j=1}^{\ell}\left\{ {\ell \atop j} \right\}W_{n-1}^{\underline{j}}\sum_{k=0}^{m-j}\binom{W_{n-1}-j}{k}\binom{T_{n-1}-W_{n-1}}{m-j-k}.\end{aligned}$$

Next we use the Vandermonde convolution formula

$$\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n},$$

in order to obtain the expression

$$\sum_{k=1}^{m}k^\ell\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k} = \sum_{j=1}^{\ell}\left\{ {\ell \atop j} \right\}W_{n-1}^{\underline{j}}\binom{T_{n-1}-j}{m-j}.$$

This leads to

$$\sum_{k=1}^{m}k^\ell\frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}}{\binom{T_{n-1}}{m}} = \sum_{j=1}^{\ell}\left\{ {\ell \atop j} \right\}W_{n-1}^{\underline{j}}\frac{\binom{T_{n-1}-j}{m-j}}{\binom{T_{n-1}}{m}} = \sum_{j=1}^{\ell}\left\{ {\ell \atop j} \right\}W_{n-1}^{\underline{j}}\frac{\binom{m}{j}}{\binom{T_{n-1}}{j}}. \tag{10}$$

Next we convert the falling factorials into ordinary powers, and obtain

$$W_{n-1}^{\underline{j}} = \sum_{i=1}^{j} \begin{bmatrix} j \\ i \end{bmatrix} (-1)^{j-i} W_{n-1}^{i}. \tag{11}$$

Applying (11) to (10), we have

$$\begin{aligned}
\sum_{k=1}^{m} k^{\ell} \frac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}}{\binom{T_{n-1}}{m}} &= \sum_{j=1}^{\ell} \begin{Bmatrix} \ell \\ j \end{Bmatrix} \left(\sum_{i=1}^{j} \begin{bmatrix} j \\ i \end{bmatrix} (-1)^{j-i} W_{n-1}^{i} \right) \frac{\binom{m}{j}}{\binom{T_{n-1}}{j}} \\
&= \sum_{i=1}^{\ell} W_{n-1}^{i} \sum_{j=i}^{\ell} (-1)^{j-i} \frac{\left[{j \atop i}\right]\left\{{\ell \atop j}\right\}\binom{m}{j}}{\binom{T_{n-1}}{j}} \\
&= \sum_{i=1}^{\ell} W_{n-1}^{\ell+1-i} \sum_{j=\ell+1-i}^{\ell} (-1)^{j+i-\ell-1} \frac{\left[{j \atop \ell+1-i}\right]\left\{{\ell \atop j}\right\}\binom{m}{j}}{\binom{T_{n-1}}{j}}.
\end{aligned} \tag{12}$$

Note that the result above is an explicit expression for the ℓth moment of a hypergeometric distributed random variable with parameters W_{n-1}, T_{n-1}, and m. Applying (12) to the right of (9), we get

$$\begin{aligned}
\mathbb{E}(W_n^s \mid \mathcal{F}_{n-1}) &= W_{n-1}^s + \sum_{\ell=1}^{s} \binom{s}{\ell} c^{\ell} \sum_{i=1}^{\ell} W_{n-1}^{s+1-i} \sum_{j=\ell+1-i}^{\ell} (-1)^{j+i-\ell-1} \frac{\left[{j \atop \ell+1-i}\right]\left\{{\ell \atop j}\right\}\binom{m}{j}}{\binom{T_{n-1}}{j}} \\
&= W_{n-1}^s + \sum_{i=1}^{s} \sum_{\ell=i}^{s} \binom{s}{\ell} c^{\ell} W_{n-1}^{s+1-i} \sum_{j=\ell+1-i}^{\ell} (-1)^{j+i-\ell-1} \frac{\left[{j \atop \ell+1-i}\right]\left\{{\ell \atop j}\right\}\binom{m}{j}}{\binom{T_{n-1}}{j}} \\
&= \left(1 + \sum_{\ell=1}^{s} c^{\ell} \frac{\binom{s}{\ell}\binom{m}{\ell}}{\binom{T_{n-1}}{\ell}}\right) W_{n-1}^s \\
&\quad + \sum_{i=2}^{s} W_{n-1}^{s+1-i} \sum_{\ell=i}^{s} \binom{s}{\ell} c^{\ell} \sum_{j=\ell+1-i}^{\ell} (-1)^{j+i-\ell-1} \frac{\left[{j \atop \ell+1-i}\right]\left\{{\ell \atop j}\right\}\binom{m}{j}}{\binom{T_{n-1}}{j}} \\
&= \alpha_{n-1,s} W_{n-1}^s + \beta_{n-1,s},
\end{aligned}$$

where the quantities $\alpha_{n,s}$ and $\beta_{n,s}$ are as defined in Theorem 2. Now the stated result easily follows by taking the expectation on both sides.

Remark 3. Note that, in the case $c = 1$, a simpler expression exists for the factorial moments $\mathbb{E}(W_n^{\underline{s}}) = \mathbb{E}(W_n(W_n - 1)\dots(W_n - s + 1))$ of W_n and consequently also for the ordinary moments of W_n.

$$\mathbb{E}(W_n^{\underline{s}}) = \mathbb{E}(W_{n-1}^{\underline{s}}) \sum_{\ell=0}^{s} \frac{\binom{s}{\ell}\binom{m}{\ell}}{\binom{T_n}{\ell}} + \sum_{i=1}^{s} i!\,\mathbb{E}(W_{n-1}^{\underline{s-i}}) \sum_{\ell=i}^{s} \frac{\binom{s}{\ell}\binom{m}{\ell}\binom{s-\ell}{i}\binom{\ell}{i}}{\binom{T_{n-1}}{\ell}}.$$

Next we use Lemma 1 to prove Theorem 2.

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Proof of Theorem 2. Repeatedly using (8) yields, for any $n \geq 1$, $s \geq 1$,

$$\begin{aligned}\mathbb{E}(W_n^s) &= \alpha_{n-1,s}\mathbb{E}(W_{n-1}^s) + \beta_{n-1,s}\\ &= \alpha_{n-1,s}\alpha_{n-2,s}\mathbb{E}(W_{n-2}^s) + \alpha_{n-1,s}\beta_{n-2,s} + \beta_{n-1,s}\\ &\ \vdots\\ &= \bigg(\prod_{j=0}^{n-1}\alpha_{j,s}\bigg)\mathbb{E}(W_0^s) + \sum_{\ell=0}^{n-1}\bigg(\beta_{\ell,s}\prod_{j=\ell+1}^{n-1}\alpha_{j,s}\bigg)\\ &= \bigg(\prod_{j=0}^{n-1}\alpha_{j,s}\bigg)\bigg(W_0^s + \sum_{\ell=0}^{n-1}\frac{\beta_{\ell,s}}{\prod_{j=0}^{\ell}\alpha_{j,s}}\bigg),\end{aligned}$$

which implies the stated result.

Proof of Theorem 1. In order to obtain the result for the expected value and the variance (second moment), as stated in Theorem 1, we proceed as follows. In the case of the expected value we observe that $\beta_{j,1} = 0$ and

$$\alpha_{j,1} = 1 + \frac{cm}{T_j} = \frac{T_j + cm}{T_{n-1}} = \frac{T_{j+1}}{T_j},$$

since $T_{j+1} = T_j + mc$; the total number of balls contained in the urn increases by mc after each draw. Consequently,

$$\prod_{j=0}^{n-1}\alpha_{j,1} = \prod_{j=0}^{n-1}\frac{T_{j+1}}{T_j} = \frac{T_n}{T_0}$$

and by (6), the stated result follows.

Remark 4. As already mentioned before $\mathcal{W}_n = W_n/T_n \to \mathcal{W}_\infty$ a. s. This can easily be seen as follows: we readily note that, for $s = 1$,

$$\mathbb{E}(W_n \mid \mathcal{F}_{n-1}) = W_{n-1}\frac{T_n}{T_{n-1}}, \qquad n \geq 1$$

holds. Thus, $\mathcal{W}_n = W_n/T_n$ is a positive martingale with respect to the filtration $\mathcal{F}_n$, as previously observed in [2], which directly leads to the proof of the almost sure convergence of $\mathcal{W}_n$.

In order to obtain the variance $\mathbb{V}(W_n) = \mathbb{E}(W_n^2) - \mathbb{E}(W_n)^2$, we study the second moment $\mathbb{E}(W_n^2)$, given by

$$\mathbb{E}(W_n^2) = \bigg(\prod_{j=0}^{n-1}\alpha_{j,2}\bigg)\bigg(W_0^2 + \sum_{\ell=0}^{n-1}\frac{\beta_{\ell,2}}{\prod_{j=0}^{\ell}\alpha_{j,2}}\bigg).$$

The value $\alpha_{n,2}$ is given by

$$\begin{aligned}\alpha_{n,2} &= 1 + \frac{2cm}{T_n} + \frac{c^2\binom{m}{2}}{\binom{T_n}{2}}\\ &= \frac{T_{n+2}(T_n - 1) + c^2 m(m-1)}{T_n(T_n - 1)}\\ &= \frac{(n + 2 + T_0/cm)(n + (T_0 - 1)/cm) + 1 - 1/m}{(n + T_0/cm)(n + (T_0 - 1)/cm)}.\end{aligned}$$

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We factor the numerator of $\alpha_{n,2}$, by determining the 0s of the quadratic equation with respect to n, and get

$$\alpha_{n,2} = \frac{(n+\lambda_1)(n+\lambda_2)}{(n+T_0/cm)(n+(T_0-1)/cm)}, \tag{13}$$

with λ_1 and λ_2 as stated in Theorem 1. Concerning $\beta_{n,2}$ we have

$$\beta_{n,2} = \mathbb{E}(W_n)c^2\left(\frac{m}{T_n} - \frac{\binom{m}{2}}{\binom{T_n}{2}}\right) = \frac{W_0c^2m}{T_0}\left(1 - \frac{m-1}{T_n-1}\right) = \frac{W_0c^2m}{T_0}\,\frac{T_n-m}{T_n-1}.$$

This readily leads to the stated exact result for the second moment.

4.1. Asymptotic expansions

We finally turn to the proof of Corollary 1. In order to get the results for $\lim_{n\to\infty}\mathbb{E}(W_n^2)/n^2$ we proceed as follows: first, by (13), we write $\prod_{j=0}^{n-1}\alpha_{j,2}$ in terms of Gamma functions,

$$\prod_{j=0}^{n-1}\alpha_{j,2} = \frac{\binom{n-1+\lambda_1}{n}\binom{n-1+\lambda_2}{n}}{\binom{n-1+T_0/mc}{n}\binom{n-1+(T_0-1)/mc}{n}} = \frac{\Gamma(n+\lambda_1)\Gamma(n+\lambda_2)\Gamma(T_0/mc)\Gamma((T_0-1)/mc)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(n+T_0/mc)\Gamma(n+(T_0-1)/mc)}.$$

Using Stirling's formula for the Gamma function

$$\Gamma(z) = \left(\frac{z}{\mathrm{e}}\right)^z\frac{\sqrt{2\pi}}{\sqrt{z}}\left(1+\frac{1}{12z}+\frac{1}{288z^2}+\mathcal{O}\left(\frac{1}{z^3}\right)\right), \tag{14}$$

and, by the fact that $\lambda_1+\lambda_2 = 2+(2T_0-1)/mc$, we obtain the asymptotic expansion

$$\prod_{j=0}^{n-1}\alpha_{j,2} = n^2\frac{\Gamma(T_0/mc)\Gamma((T_0-1)/mc)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \tag{15}$$

The asymptotic expansion (15) of $\prod_{j=0}^{n-1}\alpha_{j,2}$ also implies that the sum $\sum_{\ell=0}^{n-1}\beta_{\ell,2}/\prod_{j=0}^{\ell}\alpha_{j,2}$ converges as $n\to\infty$ by the comparison test,

$$\sum_{\ell=0}^{n-1}\frac{\beta_{\ell,2}}{\prod_{j=0}^{\ell}\alpha_{j,2}} \le K\sum_{\ell=1}^{n-1}\frac{1}{\ell^2},$$

for some suitable constant $K > 0$.

In order to provide the asymptotics of $\mathbb{E}(W_n^s)$, we will proceed by induction. Before doing so, we derive an asymptotic expansion of $\prod_{j=0}^{n-1}\alpha_{j,s}$. From (7), we have

$$\begin{aligned}\alpha_{j,s} &= \frac{\sum_{\ell=0}^{s}c^\ell\binom{m}{\ell}\binom{T_j-\ell}{s-\ell}}{\binom{T_j}{s}}\\ &= s!\frac{\sum_{\ell=0}^{s}c^\ell\binom{m}{\ell}\binom{T_j-\ell}{s-\ell}}{(mc)^s\prod_{\ell=0}^{s-1}(j+(T_0-\ell)/mc)}\\ &= s!\frac{\sum_{\ell=0}^{s}c^\ell\binom{m}{\ell}\binom{jmc+T_0-\ell}{s-\ell}}{(mc)^s\prod_{\ell=1}^{s}(j+(T_0+1-\ell)/mc)}.\end{aligned} \tag{16}$$

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Let $\lambda_{\ell,s}$ denote the roots (times minus one) of the equation

$$\frac{s!}{(mc)^s}\sum_{\ell=0}^{s} c^\ell \binom{m}{\ell}\binom{xmc+T_0-\ell}{s-\ell} = \prod_{\ell=1}^{s}(x+\lambda_{\ell,s}).$$

Then we can rewrite (16) as

$$\alpha_{j,s} = s!\frac{\sum_{\ell=0}^{s} c^\ell \binom{m}{\ell}\binom{jmc+T_0-\ell}{s-\ell}}{(mc)^s\prod_{\ell=1}^{s}(j+(T_0+1-\ell)/mc)} = \frac{\prod_{\ell=1}^{s}(j+\lambda_{\ell,s})}{\prod_{\ell=1}^{s}(j+(T_0+1-\ell)/mc)},$$

and consequently

$$\prod_{j=0}^{n-1}\alpha_{j,s} = \prod_{\ell=1}^{s}\frac{\Gamma(n+\lambda_{\ell,s})\Gamma((T_0+1-\ell)/mc)}{\Gamma(\lambda_{\ell,s})\Gamma(n+(T_0+1-\ell)/mc)}.$$

By Stirling's formula (14) we obtain the asymptotic expansion

$$\prod_{j=0}^{n-1}\alpha_{j,s} = \frac{n^{\sum_{\ell=1}^{s}\lambda_{\ell,s}}}{n^{(sT_0-\binom{s}{2})/mc}}\prod_{\ell=1}^{s}\frac{\Gamma((T_0+1-\ell)/mc)}{\Gamma(\lambda_{\ell,s})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \tag{17}$$

Let $[x^k]$ denote the extraction of coefficients operator. Since

$$\begin{aligned}\sum_{\ell=1}^{s}\lambda_{\ell,s} &= [x^{s-1}]\prod_{\ell=1}^{s}(x+\lambda_{\ell,s})\\ &= [x^{s-1}]\frac{s!}{(mc)^s}\sum_{\ell=0}^{s}c^\ell\binom{m}{\ell}\binom{xmc+T_0-\ell}{s-\ell}\\ &= \frac{s!}{(mc)^s}\left(\frac{(cm)^{s-1}(sT_0-\binom{s}{2})}{s!}+\frac{cm(cm)^{s-1}}{(s-1)!}\right)\\ &= \frac{sT_0-\binom{s}{2}}{mc}+s,\end{aligned}$$

it follows that $n^{\sum_{\ell=1}^{s}\lambda_{\ell,s}} = n^s n^{(sT_0-\binom{s}{2})/mc}$ and so (17) becomes

$$\prod_{j=0}^{n-1}\alpha_{j,s} = n^s\prod_{\ell=1}^{s}\frac{\Gamma((T_0+1-\ell)/mc)}{\Gamma(\lambda_{\ell,s})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \tag{18}$$

Now note that $\mathbb{E}(W_n^s) = \kappa_s n^s(1+\mathcal{O}(1/n))$ is true for $s=1,2$. By Theorem 2 and (18), we have

$$\begin{aligned}\mathbb{E}(W_n^s) &= \left(\prod_{j=0}^{n-1}\alpha_{j,s}\right)\left(W_0^s+\sum_{\ell=0}^{n-1}\frac{\beta_{\ell,s}}{\prod_{j=0}^{\ell}\alpha_{j,s}}\right)\\ &= n^s\prod_{\ell=1}^{s}\frac{\Gamma((T_0+1-\ell)/mc)}{\Gamma(\lambda_{\ell,s})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(W_0^s+\sum_{\ell=0}^{n-1}\frac{\beta_{\ell,s}}{\prod_{j=0}^{\ell}\alpha_{j,s}}\right).\end{aligned}$$

By our induction hypothesis, $\mathbb{E}(W_\ell^k) = \kappa_k\ell^k(1+\mathcal{O}(1/\ell))$ for sufficiently large ℓ, $1\le k\le s-1$, and consequently $\beta_{\ell,s}$ satisfies $\beta_{\ell,s}\le K\ell^{s-2}$, where K denotes a sufficiently large constant only depending on s. Using our result for $\prod_{j=0}^{\ell}\alpha_{j,s}$, the sum $\sum_{\ell=0}^{n-1}\beta_{\ell,s}/\prod_{j=0}^{\ell}\alpha_{j,s}$ is convergent according to the comparison test with $\sum_{\ell=1}^{\infty}1/\ell^2$. Hence, $\mathbb{E}(W_n^s) = \kappa_s n^s(1+\mathcal{O}(1/n))$, and we have proven our stated results.

5. The case of three or more colors

The urn model M can be readily generalized to $r \geq 2$ different colors. As in the case of two colors, m balls are drawn at random from the urn, say k_i balls of color i, $1 \leq i \leq r$, their colors are noted, and they are returned to the urn together with ck_i balls of color i. Let $\mathbf{X}_n = (X_{n,1}, \dots, X_{n,r})$ denote the random vector counting the number of balls of type i contained in the urn after n draws, with initial values $\mathbf{X}_0 = (X_{0,1}, \dots, X_{0,r})$. The urn is again balanced, so the number of balls after n draws is given by $T_n = T_0 + nmc$, with $T_0 = \sum_{i=1}^{r} X_{0,i}$. One gets the distributional equation

$$\mathbf{X}_n \stackrel{\mathrm{D}}{=} \mathbf{X}_{n-1} + \sum_{\substack{k_1+\cdots+k_r=m \\ k_\ell \geq 0}} c\mathbf{k}\,\mathbf{1}_n\bigg(\prod_{\ell=1}^{r} X_\ell^{k_\ell}\bigg),$$

where $\mathbf{k} = (k_1, \dots, k_r)$, and $\mathbf{1}_n(\prod_{\ell=1}^{r} X_\ell^{k_\ell})$ denote the indicators of drawing k_ℓ balls of color ℓ, $1 \leq \ell \leq r$ at the nth draw. Note that each individual random variable $X_{n,\ell}$ satisfies $X_{n,\ell} \stackrel{\mathrm{D}}{=} W_n$, where W_n denotes the previously considered random variable from the two color case. The distributional equation above can be used to study the mixed moments of $X_{n,\ell}$ similar to the results of Theorems 1, 2, and Corollary 1. We refrain from going into details since the resulting expressions get very involved; we only mention our findings for the covariance of two different colors i, j, with $r \geq 3$ and $1 \leq i < j \leq m$.

$$\operatorname{cov}(X_{n,i}, X_{n,j}) = \frac{\binom{n-1+\lambda_1}{n}\binom{n-1+\lambda_2}{n}}{\binom{n-1+T_0/mc}{n}\binom{n-1+(T_0-1)/mc}{n}} X_{0,i}X_{0,j} - \frac{(mc)^2(n+T_0/mc)^2}{T_0^2} X_{0,i}X_{0,j}.$$

Moreover, in the limit we obtain

$$\lim_{n\to\infty} \operatorname{cov}\bigg(\frac{X_{n,i}}{n}, \frac{X_{n,j}}{n}\bigg) = X_{0,i}X_{0,j}\bigg(\frac{\Gamma(T_0/mc)\Gamma((T_0-1)/mc)}{\Gamma(\lambda_1)\Gamma(\lambda_2)} - \frac{(mc)^2}{T_0^2}\bigg),$$

with λ_1, λ_2 as given in Theorem 1.

6. Urn model R-drawing with replacement

We consider another urn model which we call model R. It can be considered as a variant of model M. The dynamics of model R are identical to model M: $m \geq 1$ balls are drawn from the urn containing balls of colors black and white. Their colors are inspected, and they are returned to the urn. According to the observed colors we add new balls: if ℓ black balls and $m - \ell$ white balls have been observed, we add $c(m - \ell)$ white balls and ℓ black balls; the ball replacement matrix coincides with (1). The main difference in model R is that the sampling of the m balls occurs *with replacement*, i.e. the m balls are drawn one by one from the urn, the colors are observed, and then put back into the urn. Hence, the distribution of the number of white balls in the sample of size m follows a binomial distribution instead of a hypergeometric distribution in model M. The distributional equation for the number of white balls W_n after n draws is identical to (3),

$$W_n \stackrel{\mathrm{D}}{=} W_{n-1} + \sum_{k=0}^{m} kc\,\mathbf{1}_n(W^k B^{m-k}),$$

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but the distribution of the indicator variables $\mathbf{1}_n(W^k B^{m-k})$ changes to

$$\mathbb{P}\{\mathbf{1}_n(W^k B^{m-k}) = 1 \mid \mathcal{F}_{n-1}\} = \frac{\binom{m}{k} W_{n-1}^k B_{n-1}^{m-k}}{T_{n-1}^m} = \frac{\binom{m}{k} W_{n-1}^k (T_{n-1} - W_{n-1})^{m-k}}{T_{n-1}^m},$$

$0 \le k \le m, n \ge 1$.

Concerning a limit law for the number of white balls W_n after n draws, we have similar results to model M of Chen and Wei [2].

Theorem 3. *The random variable $\mathcal{W}_n = W_n/T_n$ is a positive martingale with respect to the natural filtration $\mathcal{F}_n$, $\mathbb{E}(\mathcal{W}_n \mid \mathcal{F}_{n-1}) = \mathcal{W}_{n-1}$. Consequently,*

$$\mathcal{W}_n \xrightarrow{\text{a.s.}} \mathcal{W}_\infty.$$

Furthermore, for fixed W_0, B_0, c, and m, $\mathcal{W}_\infty$ is absolutely continuous.

Moreover, the moments of W_n satisfy recurrence relations similar to model M.

Theorem 4. *The expected value is given by $\mathbb{E}(W_n) = (W_0/T_0)(nmc + T_0)$ and the variance $\mathbb{V}(W_n) = \mathbb{E}(W_n^2) - \mathbb{E}(W_n)^2$ is determined by the second moment,*

$$\mathbb{E}(W_n^2) = \frac{\binom{n-1+\mu_1}{n}\binom{n-1+\mu_2}{n}}{\binom{n-1+T_0/mc}{n}^2}\left(W_0^2 + \frac{W_0 c^2 m}{T_0}\sum_{\ell=0}^{n-1}\frac{\binom{\ell+T_0/mc}{\ell+1}^2}{\binom{\ell+\mu_1}{\ell+1}\binom{\ell+\mu_2}{\ell+1}}\right),$$

where the values μ_1 and μ_2 are given by $\mu_{1,2} = (T_0 + mc \pm c\sqrt{m})/mc$. The sth moment satisfies the recurrence relation

$$\mathbb{E}(W_n^s) = \left(\prod_{j=0}^{n-1}\gamma_{j,s}\right)\left(W_0^s + \sum_{\ell=0}^{n-1}\frac{\delta_{\ell,s}}{\prod_{j=0}^{\ell}\gamma_{j,s}}\right),$$

where the quantities $\gamma_{n,s}$ and $\delta_{n,s}$ are defined as

$$\gamma_{n,s} = \sum_{\ell=0}^{s} c^\ell \frac{\binom{s}{\ell} m^{\underline{\ell}}}{T_n^\ell}, \qquad \delta_{n,s} = \sum_{j=2}^{s}\mathbb{E}(W_n^{s+1-j})\sum_{\ell=j}^{s}\binom{s}{\ell} c^\ell \frac{\left\{{\ell \atop \ell+1-j}\right\} m^{\underline{\ell+1-j}}}{T_n^{\ell+1-j}}.$$

In the following we present recurrence relations for the limits of the moments of the normalized random variable W_n/n, with $\lim_{n\to\infty}\mathbb{E}(W_n^s/n^s) = (mc)^s\mathbb{E}(\mathcal{W}_\infty^s)$.

Corollary 2. *The limits of the normalized moments $\mathbb{E}(W_n^s/n^s)$ exist, and satisfy*

$$\lim_{n\to\infty}\frac{\mathbb{E}(W_n^s)}{n^s} = \frac{\Gamma(T_0/mc)^s}{\prod_{\ell=1}^{s}\Gamma(\mu_{\ell,s})}\left(W_0^s + \sum_{\ell=0}^{\infty}\frac{\delta_{\ell,s}}{\prod_{j=0}^{\ell}\gamma_{j,s}}\right)$$

with $\gamma_{j,s}$ and $\delta_{\ell,s}$ as defined above. Here the $\mu_{\ell,s}$ denote the roots (times minus one) of the equation

$$\frac{1}{(mc)^s}\sum_{\ell=0}^{s}\binom{s}{\ell}\binom{m}{\ell}c^\ell \ell!(xmc + T_0)^{s-\ell} = \prod_{\ell=1}^{s}(x + \mu_{\ell,s}).$$

In the following we first sketch the proofs of Theorem 4 and Corollary 2. Since the proofs are very similar to the proofs for model M, we will be very brief. Then we discuss the proof of Theorem 3.

6.1. The structure of the moments

Our starting point is again the distributional equation for W_n, which leads to a distributional equation for W_n^s. We take the conditional expectation with respect to $\mathcal{F}_{n-1}$, and obtain

$$\mathbb{E}(W_n^s \mid \mathcal{F}_{n-1}) = W_{n-1}^s + \sum_{\ell=1}^{s} \binom{s}{\ell} W_{n-1}^{s-\ell} c^\ell \sum_{k=1}^{m} k^\ell \binom{m}{k} \frac{W_{n-1}^k (T_{n-1} - W_{n-1})^{m-k}}{T_{n-1}^m}. \tag{19}$$

The sum appearing on the right-hand side is the ℓth moment of a binomial distributed random variable with parameters m and W_{n-1}/T_{n-1}. We get the result

$$\sum_{k=1}^{m} k^\ell \binom{m}{k} \frac{W_{n-1}^k (T_{n-1} - W_{n-1})^{m-k}}{T_{n-1}^m} = \sum_{j=1}^{\ell} \left\{ {\ell \atop j} \right\} m^{\underline{j}} \frac{W_{n-1}^j}{T_{n-1}^j}.$$

This implies that (19) becomes

$$\mathbb{E}(W_n^s \mid \mathcal{F}_{n-1}) = \gamma_{n-1,s} W_{n-1} + \delta_{n-1,s},$$

with $\gamma_{n,s}$ and $\delta_{n,s}$ as stated in Theorem 4. This recurrence relation can be solved in a similar way to that used in the proof of Theorem 2 for model M, and the stated result for $\mathbb{E}(W_n^s)$ follows. Moreover, we easily obtain the stated formula for the expected value, and also the result for the second moment of W_n. The results for the higher moments can be obtained in a similar way to that used for model M. Concerning the asymptotic expansions we can proceed as we did in the proof of Corollary 1; we omit the details.

6.2. Martingales and absolute continuity

Since $\delta_{n-1,1} = 0$ we obtain, for $\mathcal{W}_n = W_n/T_n$,

$$\mathbb{E}(\mathcal{W}_n \mid \mathcal{F}_{n-1}) = \mathcal{W}_{n-1}.$$

Hence, $\mathcal{W}_n$ is a martingale. Since it is a positive martingale, it converges almost surely to a limit $\mathcal{W}_\infty$. Note that we have $\mathcal{W}_n \sim W_n/cmn$ provided $W_n \to \infty$ and, thus, we can obtain the moments of $\mathcal{W}_\infty$ via the moments of W_n/n.

Concerning the absolute continuity of the distribution of $\mathcal{W}_\infty$ we can adapt the argumentation of Chen and Wei [2]. For the convenience of the reader we outline the main steps, quote the main results of [2], and present the new ingredient of the proof for model R in Lemma 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space. In order to prove that the distribution of $\mathcal{W}_\infty$ has a density, we introduce a sequence of events $(\Omega_\ell)_{\ell \ge 1}$ with $\Omega_\ell \subset \Omega_{\ell+1}$ and $\mathbb{P}\{\cup_{\ell=1}^{\infty} \Omega_\ell\} = 1$. Then, $\mathcal{W}_\infty$ is restricted to Ω_ℓ and it is shown that it has a density f_ℓ. The proof is then finished by proving that $f = \lim_{\ell \to \infty} f_\ell$ exists and that f is the density of $\mathcal{W}_\infty$.

Proposition 1. ([2].) *Let* $(\Omega_\ell)_{\ell \ge 1}$ *be a sequence of increasing events with* $\mathbb{P}\{\cup_{\ell=1}^{\infty} \Omega_\ell\} = 1$. *If there exist nonnegative Borel measurable functions* $(f_\ell)_{\ell \ge 1}$ *satisfying* $\mathbb{P}(\Omega_\ell \cap \mathcal{W}_\infty^{-1}(B)) = \int_B f_\ell(x)\,\mathrm{d}x$ *for all Borel sets* B, *then* $f = \lim_{\ell \to \infty} f_\ell$ *exists almost everywhere, and* f *is the density of* $\mathcal{W}_\infty$.

In order to construct the sequence of events $(\Omega_\ell)_{\ell \ge 1}$ we can follow [2].

Proposition 2. *For fixed* W_0, B_0, c, *and* m *let*

$$\Omega_\ell = \{\omega\colon W_\ell(\omega) \ge cm,\ B_\ell(\omega) \ge cm\}, \qquad \ell \ge 1.$$

Then, $\Omega_\ell \subset \Omega_{\ell+1}$ *and* $\mathbb{P}\{\bigcup_{\ell=1}^{\infty} \Omega_\ell\} = 1$.

In order to show that $\mathcal{W}_\infty$ has a density by restricting $\mathcal{W}_\infty$ to Ω_ℓ, it suffices to show that the restriction of $\mathcal{W}_\infty$ to $\Omega_{\ell,j} = \{\omega : W_\ell(\omega) = j\}$ has a density for each j with $cm \le j \le T_{\ell-1}$. For this the following result is needed, which is the main new ingredient in our argument (compared to the argument used in the case of model M).

Lemma 2. *The sum* $\sum_{i=0}^{m} \mathbb{P}\{W_{n+1} = j + ck \mid W_n = j + c(k-i)\}$ *satisfies*

$$\sum_{i=0}^{m} \mathbb{P}\{W_{n+1} = j + ck \mid W_n = j + c(k-i)\} = \frac{1}{T_n^m} \sum_{\ell=0}^{m} T_n^\ell \sum_{i=0}^{m-\ell} \binom{m}{i}\binom{m-i}{\ell} (j + c(k-i))^i (-j - c(k-i))^{m-i-\ell}.$$

Consequently, for a fixed ℓ*, for all* $n \ge \ell$*,* $cm \le j \le T_{\ell-1}$*, and* $k < m(n+1)$*, and a suitably chosen constant* $\kappa > 0$*, we have*

$$\sum_{i=0}^{m} \mathbb{P}\{W_{n+1} = j + ck \mid W_n = j + c(k-i)\} \le 1 - \frac{1}{n} + \frac{\kappa}{n^2}.$$

Remark 5. Note that the corresponding result of Chen and Wei for model M (Lemma 4.1 and Lemma 4.2 of [2]) can be extended and largely simplified noting that

$$\sum_{i=0}^{m} \binom{j + c(k-i)}{i}\binom{T_n - j - c(k-i)}{m-i} = \sum_{f=0}^{m} T_n^f \sum_{i=0}^{m-f} \binom{j + c(k-i)}{i} \sum_{\ell=f}^{m-i} \frac{\left[{\ell \atop f}\right](-1)^{\ell-f}\binom{-j-c(k-i)}{m-i-\ell}}{\ell!}.$$

Proof of Lemma 2. We have

$$\begin{aligned}
\sum_{i=0}^{m} &\mathbb{P}\{W_{n+1} = j + ck \mid W_n = j + c(k-i)\} \\
&= \frac{1}{T_n^m} \sum_{i=0}^{m} \binom{m}{i} (j + c(k-i))^i (T_n - j - c(k-i))^{m-i} \\
&= \frac{1}{T_n^m} \sum_{i=0}^{m} \binom{m}{i} (j + c(k-i))^i \sum_{\ell=0}^{m-i} \binom{m-i}{\ell} T_n^\ell (-j - c(k-i))^{m-i-\ell} \\
&= \frac{1}{T_n^m} \sum_{\ell=0}^{m} T_n^\ell \sum_{i=0}^{m-\ell} \binom{m}{i}\binom{m-i}{\ell} (j + c(k-i))^i (-j - c(k-i))^{m-i-\ell},
\end{aligned}$$

giving

$$\sum_{i=0}^{m} \mathbb{P}\{W_{n+1} = j + ck \mid W_n = j + c(k-i)\} = 1 - \frac{mc}{T_n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Using $T_n = nmc + T_0$, we further have

$$\frac{mc}{T_n} = \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Hence, the stated result follows.

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The proof of Theorem 3 can be finished by combining the results of Propositions 1 and 2, and Lemma 2; it is identical to the proof in [2] (Proof of Theorem 4.2) and therefore the details are omitted.

7. Outlook

An interesting line of research would be to extend the results for models M and R to nonbalanced urns with ball replacement matrix given by

$$\begin{bmatrix} ma & 0 \\ (m-1)a & b \\ \dots & \dots \\ a & (m-1)b \\ 0 & mb \end{bmatrix},$$

with $a, b \in \mathbb{N}$.

For both nonbalanced models M and R, by simple calculation, we have that the random variable $\mathcal{W}_n = W_n/T_n$ is a strict submartingale with respect to the natural filtration $\mathcal{F}_n$, $\mathbb{E}(\mathcal{W}_n \mid \mathcal{F}_{n-1}) \geq \mathcal{W}_{n-1}$ if $a > b$, and $\mathcal{W}_n$ is a strict supermartingale with respect to $\mathcal{F}_n$ if $a < b$. Since $0 \leq \mathcal{W}_n \leq 1$, by the martingale convergence theorem, $\mathcal{W}_n$ converges a.s. to a random variable $\mathcal{W}_\infty$. However, in contrast to the balanced versions it seems much more difficult to obtain closed-form expressions for the moments of W_n.

One can analyze a generalized Friedman's urn, with ball replacement matrix

$$M = \begin{bmatrix} 0 & mc \\ c & (m-1)c \\ \dots & \dots \\ (m-1)c & c \\ mc & 0 \end{bmatrix}$$

and parameters $c \in \mathbb{N}$ and $m \geq 1$. If, say, $m - \ell$ white balls and ℓ black balls have been drawn from the urn, $0 \leq \ell \leq m$, then the drawn balls are returned to the urn together with additional $c\ell$ white balls and $c(m-\ell)$ black balls. It is possible to look at drawing the m balls without replacement, which we call model FM, or to look at drawing the m balls with replacement, or simply model FR. In any case, the distributional equation, for the number of white balls W_n after n draws, is given by

$$W_n \overset{\mathrm{D}}{=} W_{n-1} + \sum_{k=0}^{m} (m-k)c\,\mathbf{1}_n(W^k B^{m-k}), \tag{20}$$

with

$$\mathbb{P}\{\mathbf{1}_n(W^k B^{m-k}) = 1 \mid \mathcal{F}_{n-1}\} = \begin{cases} \dfrac{\binom{W_{n-1}}{k}\binom{T_{n-1}-W_{n-1}}{m-k}}{\binom{T_{n-1}}{m}} & \text{model } FM, \\[2ex] \dfrac{\binom{m}{k} W_{n-1}^k (T_{n-1}-W_{n-1})^{m-k}}{T_{n-1}^m} & \text{model } FR, \end{cases} \tag{21}$$

$0 \leq k \leq m$, and $n \geq 1$. We can obtain the moments of W_n, for both models FM and FR, in a similar way to that used to obtain our previous results for models M and R. In particular, the expectation and the variance can be obtained; we get, for example, the result

$$\mathbb{E}(W_n) = \begin{cases} \dfrac{(mc)^2\binom{n}{2} + mcT_0 n + (T_0 - mc)W_0}{mc(n-1) + T_0}, & mc \neq T_0, \\ \dfrac{mc(n+1)}{2}, & mc = T_0, \end{cases}$$

$n \geq 0$, being valid for both models FM and FR. In contrast to the generalized Pólya urn models M and R the simple martingale structure of W_n/T_n is not present anymore. However, we can use supermartingale theory to prove that W_n/T_n converges to 1/2 almost surely. Here we only consider the model FM since the proof of the model FR is similar. Let

$$Z_n = \frac{W_n}{T_n} - \frac{1}{2}.$$

We find that

$$\mathbb{E}(Z_n \mid \mathcal{F}_{n-1}) = \frac{W_{n-1} + cm - cmW_{n-1}/T_{n-1}}{T_n} - \frac{1}{2} = (1 - 2\lambda_n)Z_{n-1}, \tag{22}$$

where $\lambda_n = cm/T_n$, and

$$\begin{aligned} \mathbb{E}(Z_n^2 \mid \mathcal{F}_{n-1}) &= \mathbb{E}\bigg(\bigg(Z_n + \frac{1}{2}\bigg)^2 - Z_n - \frac{1}{4} \,\bigg|\, \mathcal{F}_{n-1}\bigg) \\ &= \frac{\mathbb{E}(W_n^2 \mid \mathcal{F}_{n-1})}{T_n^2} - \mathbb{E}(Z_n \mid \mathcal{F}_{n-1}) - \frac{1}{4}. \end{aligned} \tag{23}$$

Using (20), (21), and $Z_n = W_n/T_n - \frac{1}{2}$, we have

$$\begin{aligned} \frac{\mathbb{E}(W_n^2 \mid \mathcal{F}_{n-1})}{T_n^2} &= \bigg((1 - 2\lambda_n)^2 - \frac{\lambda_n^2(T_{n-1} - m)}{m(T_{n-1} - 1)}\bigg)Z_{n-1}^2 \\ &\quad + (1 - 2\lambda_n)Z_{n-1} + \frac{1}{4} + \frac{\lambda_n^2(T_{n-1} - m)}{4m(T_{n-1} - 1)}. \end{aligned} \tag{24}$$

Thus, by (22)–(24), we obtain

$$\begin{aligned} \mathbb{E}(Z_n^2 \mid \mathcal{F}_{n-1}) &= \bigg((1 - 2\lambda_n)^2 - \frac{\lambda_n^2(T_{n-1} - m)}{m(T_{n-1} - 1)}\bigg)Z_{n-1}^2 + \frac{\lambda_n^2(T_{n-1} - m)}{4m(T_{n-1} - 1)} \\ &\leq (1 - 2\lambda_n)^2 Z_{n-1}^2 + \frac{\lambda_n^2}{4m}. \end{aligned} \tag{25}$$

Let $a_n = (1 - 2\lambda_{n+1})^2$ and $b_n = \lambda_{n+1}^2/(4m)$. Then we see that $0 < a_n < 1$ and $0 < b_n < \lambda_n^2$ for $n \geq 1$. Thus, from (25),

$$\begin{aligned} \mathbb{E}(Z_n^2 + \lambda_n \mid \mathcal{F}_{n-1}) &\leq Z_{n-1}^2 + \lambda_n^2 + \lambda_n \\ &= Z_{n-1}^2 + \frac{cm(cm + T_n)}{T_n^2} \\ &\leq Z_{n-1}^2 + \frac{cm}{T_n - cm} \\ &= Z_{n-1}^2 + \lambda_{n-1}. \end{aligned}$$

This implies $Z_n^2 + \lambda_n$ is a positive supermartingale. Applying the supermartingale convergence theorem, $Z_n^2 + \lambda_n$ converges almost surely and so does Z_n^2 since $\lambda_n = cm/T_n$ converges to 0.

Let $\lim_{n\to\infty} Z_n^2 = Z$ almost surely. If we can claim $\mathbb{E}(Z_n^2) \to 0$ then, by the dominated convergence theorem, $\mathbb{E}(Z) = 0$ and so $Z = 0$ almost surely, which implies that Z_n converges to 0 almost surely. Hence, W_n/T_n converges to $\frac{1}{2}$ almost surely.

Next, taking the expectation on both sides of (25), we have that $\mathbb{E}(Z_n^2) \le a_{n-1}\mathbb{E}(Z_{n-1}^2) + b_{n-1}$. Note that

$$\prod_{i=1}^n a_i = \prod_{i=1}^n (1 - 2\lambda_{i+1})^2 = \left(\frac{T_0 T_1}{T_n T_{n+1}}\right)^2 \to 0 \quad \text{as } n \to \infty$$

and

$$\sum_{n=1}^\infty b_n = \sum_{n=1}^\infty \frac{\lambda_{n+1}^2}{4m} = \frac{c^2 m}{4} \sum_{n=1}^\infty \frac{1}{(T_0 + cm + cmn)^2} < \infty.$$

Therefore, $\mathbb{E}(Z_n^2) \to 0$ now follows by applying the following lemma which can also be found in [3].

Lemma 3. *Suppose $\{x_n\}_{n\ge1}$, $\{a_n\}_{n\ge1}$, and $\{b_n\}_{n\ge1}$ are nonnegative real sequences satisfying $x_{n+1} \le a_n x_n + b_n$, where $0 < a_n < 1$ for $n \ge 1$. If $\prod_{i=1}^n a_i \to 0$ and $\sum_{n=1}^\infty b_n < \infty$, then $x_n \to 0$.*

Proof. First, note that $x_{n+1} \le x_n + b_n$ since $0 < a_n < 1$. Thus, $x_{n+1} \le x_1 + \sum_{i=1}^n b_i \le x_1 + \sum_{i=1}^\infty b_i$, which implies that $\{x_n\}_{n\ge1}$ is uniformly bounded by a positive constant M.

Given $\varepsilon > 0$, choose n_0 satisfying $\sum_{n=n_0}^\infty b_i < \varepsilon/(1+M)$, and then choose $n_1 > n_0$ for which $\prod_{i=n_0}^n a_i < \varepsilon/(1+M)$ whenever $n > n_1$. Now, for each $n > n_1$, we have

$$\begin{aligned} x_{n+1} &\le a_n x_n + b_n \\ &\le a_n a_{n-1} x_{n-1} + b_{n-1} + b_n \\ &\;\;\vdots \\ &\le \left(\prod_{i=n_0}^n a_i\right) x_{n_0} + \sum_{i=n_0}^n b_i \\ &\le \frac{\varepsilon M}{1+M} + \frac{\varepsilon}{1+M} \\ &= \varepsilon. \end{aligned}$$

Hence, $x_n \to 0$ as in the assertion.

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