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KRONECKER PRODUCTS AND LOCAL JOINS OF GRAPHS

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1. Introduction. When studying the category $\mathscr{G}raph$ of finite graphs and their morphisms, we can exploit the fact that this category has *products*, [we define these ideas in detail in § 2]. This categorical product of graphs is usually called their *Kronecker product*, though it has been approached by various authors in various ways and under various names, including tensor product, cardinal product, conjunction and of course categorical product (see for example [**6**; **7**; **11**; **14**; **17** and **23**]).

Another 'product' of graphs, the *lexicographic product*, although not 'categorically correct', has been investigated by several authors. In this paper we shall show that the lexicographic product can be studied using some methods which are appropriate to the Kronecker product. The coordination of approach thus permitted, is mainly due to the 'quotient structure' of the projections of these two products onto their factors, which is shown to have nice functorial properties.

In particular, the 'local structure' of these projections is preserved under pullbacks. The abstraction of this property leads naturally to the other main object of study in this paper, viz. the *local join* of graphs.

2. The Kronecker product of graphs. We are primarily concerned throughout this paper with finite graphs without loops or multiple edges.

Notation. $V(G)$ and $E(G)$ denote as usual the *vertex set* and *edge set* of a graph G, $v \sim_G w$ (or $v \sim w$ if no ambiguity) denotes *adjacency* of the vertices v, w, i.e. there is an edge (denoted $[v, w]$) joining v to w in G. $\bar{G}$ denotes the *complement* of the graph G. Any notation not explained will be standard (see, for example, Harary [**10**]).

A *morphism* (i.e. a graph-homomorphism) is a function $f : V(G) \to V(H)$ which preserves adjacency, i.e. $v \sim_G w$ implies that $f(v) \sim_H f(w)$.

$\mathscr{G}raph$ denotes the category of finite graphs and their morphisms.

$\mathscr{S}et$ denotes the category of sets and set-functions.

We shall use only elementary ideas concerning categories and functors, as given in standard books, for example [**13**].

We now define the Kronecker product of graphs as a binary operation. In any product of graphs we shall denote a product vertex (v_1, v_2) by v_1v_2.

Definition. The *Kronecker product* $G_1 \wedge G_2$ of the graphs G_1 and G_2 has

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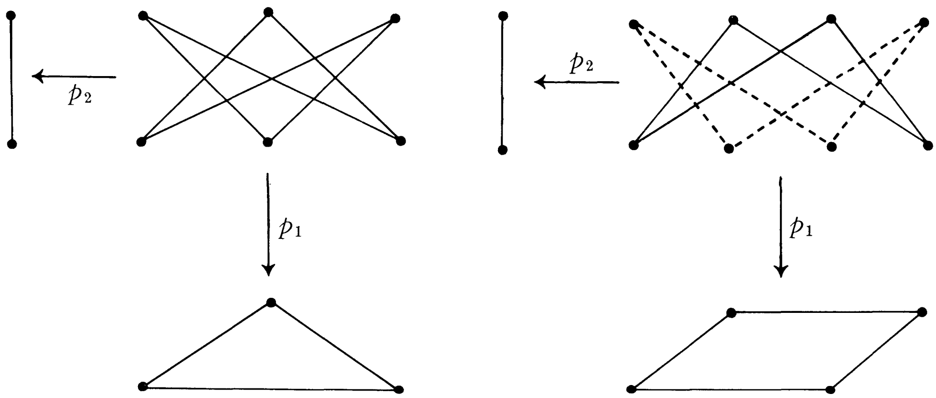
vertex set $V(G_1 \wedge G_2) = V(G_1) \times V(G_2)$ with adjacency in $G_1 \wedge G_2$ given by $v_1v_2 \sim w_1w_2$ if and only if $v_1 \sim_{G_1} w_1$ and $v_2 \sim_{G_2} w_2$.

The *projection* maps $p_i : G_1 \wedge G_2 \to G_i$ $(i = 1, 2)$ are given by $v_1v_2 \mapsto v_i$ and are in fact graph-epimorphisms, with the property that both edges $[v_1v_2, w_1w_2]$ and $[v_1w_2, w_1v_2]$ project to the corresponding edge $[v_i, w_i]$ in G_i, $(i = 1, 2)$.

Example. The Kronecker product of the circuit graph C_n and complete graph K_2 is C_{2n} if n is odd and a disjoint union $C_n \coprod C_n$ if n is even.

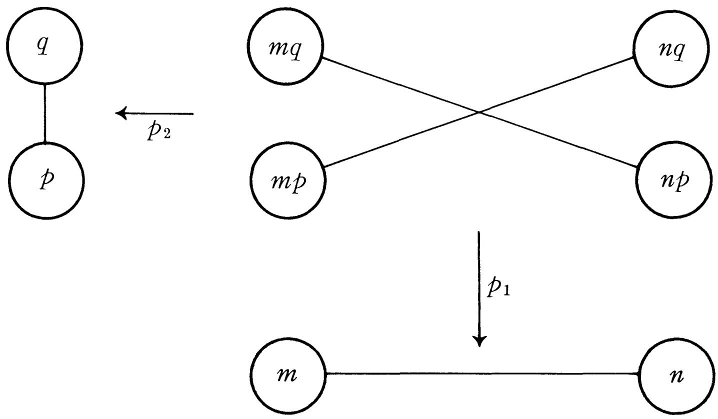


The Kronecker product of any two bipartite graphs is disconnected. We take as our basic result:

2.1 THEOREM. $K_{m,n} \wedge K_{p,q} = K_{mp,nq} \coprod K_{mq,np}$.

Proof. Let $\bar{K}_m$, $\bar{K}_n$ be the two maximal discrete induced subgraphs of $K_{m,n}$, and $\bar{K}_p$, $\bar{K}_q$ those of $K_{p,q}$. Thus $V(K_{m,n} \wedge K_{p,q})$ can be conveniently partitioned as $\bar{K}_m \wedge \bar{K}_p$, $\bar{K}_m \wedge \bar{K}_q$, $\bar{K}_n \wedge \bar{K}_p$ and $\bar{K}_n \wedge \bar{K}_q$, i.e. $\bar{K}_{mp}$, $\bar{K}_{mq}$, $\bar{K}_{np}$ and $\bar{K}_{nq}$ respectively.

The given edges between $\bar{K}_m$ and $\bar{K}_n$ 'multiply' those between $\bar{K}_p$ and $\bar{K}_q$ according to the definition of Kronecker product, to give (all) joining lines between $\bar{K}_{mp}$ and $\bar{K}_{nq}$ and between $\bar{K}_{mq}$ and $\bar{K}_{np}$ as indicated by the following figure:



We can apply Theorem 2.1 using the following obvious lemma to give a short proof (Proposition 2.3 below) of Weichsel's Theorem 1 [**23**]:

2.2 LEMMA. *Suppose* $x, y \in V(G_1 \wedge G_2)$. *If there is a path of length* l_i *from* $p_i x$ *to* $p_i y$ *in* G_i $(i = 1, 2)$ *such that* $l_1 - l_2$ *is even, then there is a path from* x *to* y *in* $G_1 \wedge G_2$.

2.3 PROPOSITION. *The Kronecker product* $G_1 \wedge G_2$ *of two connected graphs is disconnected if and only if both are bipartite.*

Proof. Let v_1v_2, w_1w_2 be vertices of $G_1 \wedge G_2$. If G_1 is not bipartite then there exists both even and odd paths between v_1 and w_1 in G_1.

Given a path from v_2 to w_2 in G_2, we therefore choose a path of the same 'parity' in G_1, and then apply Lemma 2.2 to produce a path from v_1v_2 to w_1w_2, as required.

Conversely, if both G_1 and G_2 are bipartite, we can apply Theorem 2.1 to the complete bipartite graphs $K_{m,n}$ and $K_{p,q}$, say, of which G_1, G_2 are spanning subgraphs. $G_1 \wedge G_2$ is clearly a spanning subgraph of $K_{m,n} \wedge K_{p,q}$ and so must be disconnected.

3. Local joins. Graphs indexed by a graph X were introduced by Sabidussi [**15**] and called *X-joins*. This concept (especially its particular case, the 'lexicographic product' of graphs) has subsequently received much attention, (see for example [**4**; **12**; **15**; **18** and **19**]).

Some of our results on local joins in this and the following sections were announced in [**7**].

With a view to studying the functorial properties of local joins in § 4, we begin as follows.

By a *projection* $p : G \to X$ of a graph G onto a graph X is meant a map sending vertices to vertices, such that

(i) if $p(v) = p(w)$, then $p([v, w])$ is the vertex $p(v)$; and

(ii) if $p(v) \neq p(w)$, then $p([v, w])$ is the edge $[p(v), p(w)]$.

Thus a projection is a simplicial map (not necessarily a morphism in $\mathscr{G}raph$). The preimage $p^{-1}(x)$ of a vertex $x \in V(X)$ is called the *fibre over x*.

A projection $p : G \to X$ is called a *local join* if

(i) for each vertex x in X, the fibre over x is an induced subgraph of G; and

(ii) if $[x, y]$ is an edge of X, then $p^{-1}([x, y])$ consists of edges between each vertex of $p^{-1}(x)$ and each vertex of $p^{-1}(y)$.

Note. It is implicit in the definition of projection that if x and y are *not* joined by an edge in X, then there can be *no* edges between $p^{-1}(x)$ and $p^{-1}(y)$.

Recall that a graph G is called a *join*, $G_1 * G_2$ if each vertex of G_1 is joined to each vertex of G_2. There is then an obvious projection $p : G \to K_2$ with $p(G_i) = v_i$, $(i = 1, 2)$, and p sending each 'joining edge' to the edge $[v_1, v_2]$.

A local join occurs whenever a graph projection has this property over each edge of X.

Some basic n-ary operations on graphs, viz. *disjoint union* $\coprod$ and (*n-fold*) *join* $*$ (some call it *sum*; see, for example, [**20**]) can be coordinated as both are examples of local joins:

1. There is a natural projection $\coprod_{i=1}^{n} G_i \rightarrow \bar{K}_n$, which is a local join with the connected components G_i as fibres.

2. There is a natural projection $*_{i=1}^{n} H_i \rightarrow K_n$ which is a local join with the H_i as fibres. In particular, the *complete n-partite graph* $K_{r_1, r_2, \ldots, r_n}$ has a natural structure as an n-fold join, with discrete fibres, projecting to K_n.

3. The *lexicographic product* (*composition graph*) $G[H]$, a binary operation on graphs (see Sabidussi [**15**]), is a local join projecting to G with all fibres isomorphic to H. If H has m vertices, then for each edge e of G, $p^{-1}e$ consists of the edges of the complete bipartite graph $K_{m,m}$. In particular $K_n[\bar{K}_m] = K_{m,m,\ldots m}$($n$-fold).

Decompositions of graphs with respect to local join are best viewed in terms of the binary operation of composition of the projections:

3.1 PROPOSITION. *The composite of local joins is a local join. That is, if* $p_1 : W \rightarrow X$, $p_2 : X \rightarrow Y$ *are local joins then so is* $p_2 \circ p_1 : W \rightarrow Y$.

Proof. (i) If $y \in V(Y)$ then $(p_2 \circ p_1)^{-1} y = p_1^{-1}(p_2^{-1}(y))$.

(ii) If $[y_1, y_2] \in E(Y)$ then for each $x_1 \in p_2^{-1}y_1$, $x_2 \in p_2^{-1}y_2$ we have $[x_1, x_2] \in E(X)$ and so for each $w_i \in p_1^{-1}x_i = (p_2 \circ p_1)^{-1}y_i$, $i = 1, 2$ we have $[w_1, w_2] \in E(W)$ and the result follows.

3.2 COROLLARY. (i) *If* $G \xrightarrow{p} X_1 \coprod X_2$ *is a local join then G is a disjoint union* $(= p^{-1}X_1 \coprod p^{-1}X_2)$.

(ii) *If* $G \xrightarrow{p} X_1 * X_2$ *is a local join then* $G = (p^{-1}X_1) * p^{-1}(X_2)$.

Proof. (i) Compose p with $X_1 \coprod X_2 \rightarrow \bar{K}_2$ in 2.3.

(ii) Compose p with $X_1 * X_2 \xrightarrow{p} K_2$ in 2.3.

3.3 COROLLARY. *In the special case of composition graphs we have*

(i) $\left(\coprod_i G_i \right)[H] = \prod_i (G_i[H])$,

$\left(\underset{i}{*} G_i \right)[H] = \underset{i}{*} (G_i[H])$.

3.4 PROPOSITION. *The complement of a local join is a local join. In fact, complementation c gives the double commutative diagram:*

$$\begin{array}{ccc} G & \overset{c}{\leftrightarrow} & \bar{G} \\ \downarrow p & & \downarrow \bar{p}. \\ X & \overset{c}{\leftrightarrow} & \bar{X} \end{array}$$

For given p, the projection $\bar{p}$ is given by $\bar{p}(v) = p(v)$ and $\bar{p}[v, w] = [p(v), p(w)]$, (defined if and only if $[v, w] \in E(\bar{G})$. Note that the fibres of the complement are the complements of the given fibres).

In particular, taking $X = K_n$ in 3.4 we have:

3.5 COROLLARY

$$\overline{\mathop{*}_{i=1}^{n} G_i} = \coprod_{i=1}^{n} \overline{G_i}.$$

The concept of local join is useful when the vertex set of a graph can be partitioned into subsets $V_1, V_2, \ldots, V_p$ such that there are either no joining lines or all possible joining lines between V_i and V_j $(i \neq j)$. For example, it clarifies the structure of any Kronecker product of complete k-partite graphs:

3.6 THEOREM. *$K_{n_1, n_2, \ldots, n_r} \wedge K_{m_1, m_2, \ldots, m_s}$ has the structure of a local join over $K_r \wedge K_s$, with the fibre F_{ij} over the vertex $v_i v_j$ equal to $\bar{K}_{n_i + m_j}$ $(i = 1, 2, \ldots, r;$ $j = 1, 2, \ldots, s)$ and F_{ij} being (fully) joined to F_{kl} if and only if $i \neq k$ and $j \neq l$.*

In particular, denoting a complete regular r-partite graph by $K_{r(n)}$ we relate Kronecker product to lexicographic product by

3.7 COROLLARY $K_{r(n)} \wedge K_{s(m)} \cong (K_r \wedge K_s)[\bar{K}_{m+n}]$.

These are particular cases of a more general relationship between Kronecker product and local joins:

3.8 THEOREM. *Given local joins $p_i : G_i \rightarrow X_i$ with fibre over x_{ij} equal to F_{ij} $(i = 1, 2)$, the local join L over $X_1 \wedge X_2$ whose fibre over $x_{1j}x_{2k}$ is $F_{1j} \wedge F_{2k}$, $j = 1, 2, \ldots, |V(X_1)|$, $k = 1, 2, \ldots, |V(X_2)|$, is a spanning subgraph of $G_1 \wedge G_2$.*

Proof. $V(G_1 \wedge G_2) = V(G_1) \times V(G_2) = \cup_j V(F_{1j}) \times \cup_k V(F_{2k}) = V(L)$. Clearly $G_1 \wedge G_2$ will have $F_{1j} \wedge F_{2k}$ over $x_{1j}\, x_{2k}$. Also an edge $e_i = [x_{ij}, x_{ij'}]$ in X_i indicates the presence of all joining lines between F_{ij} and $F_{ij'}$ in G_i $(i = 1, 2)$. In L these multiply to give (all) joining lines between $F_{1j} \wedge F_{2k}$ and $F_{1j'} \wedge F_{2j'}$ (and between $F_{1j} \wedge F_{2j'}$ and $F_{1j'} \wedge F_{2k}$) as occur in $G_1 \wedge G_2$.

Sketch of proof of Theorem 3.6. This induced subgraph is the whole of $G_1 \wedge G_2$ if both G_1 and G_2 have discrete fibres, for then, these are the only edges L has. Thus Theorem 3.6 follows easily.

4. Functorial properties. The fact that the Kronecker product is *the* product in the category $\mathcal{G}raph$ has some interesting consequences. The basic property is that if (G, H) denotes the set of all morphisms from G to H, then there is a natural bijection:

$$(G, H_1 \wedge H_2) \overset{\approx}{\leftrightarrow} (G, H_1) \times (G, H_2).$$

[It is easily verified that the Kronecker product does have this property and that other 'graph-products' do not.]

The distributive law $G_1 \wedge (G_2 \coprod G_3) = (G_1 \wedge G_2) \coprod (G_1 \wedge G_3)$ shows that for many problems it suffices to consider connected factors G_i. Morphisms $f_i : G_i \to H_i$, $(i = 1, 2)$ induce a morphism $f_1 \wedge f_2 : G_1 \wedge G_2 \to H_1 \wedge H_2$ defined by $v_1v_2 \mapsto f_1(v_1)f_2(v_2)$. (Since $\wedge$ is an associative binary operation, such constructions have an obvious n-ary analogue.)

Fixing the graph G_2 here we obtain a unary operation:

$$(\wedge\, G_2) : G_1 \mapsto G_1 \wedge G_2$$

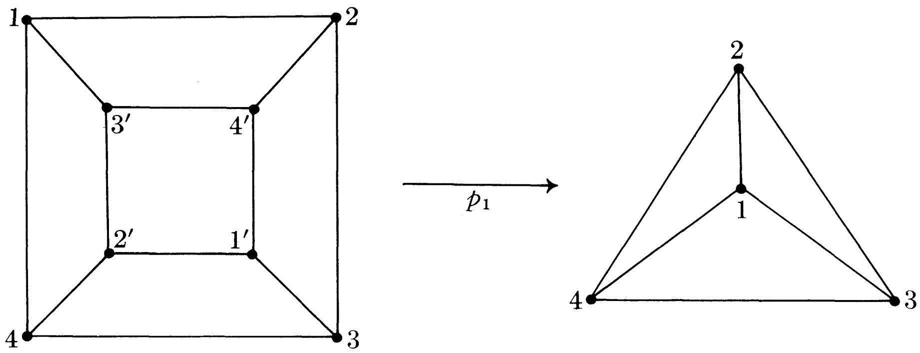
and this gives:

4.1 PROPOSITION. *$(\wedge\, G_2) : \mathcal{G}raph \to \mathcal{G}raph$ is a covariant functor.*

Proof. A morphism $f : G_1 \to H_1$ induces a morphism $f \wedge 1_{G_2} : G_1 \wedge G_2 \to H_1 \wedge G_2$ defined by $v_1v_2 \mapsto f(v_1)v_2$. It is easily seen that this construction respects identity maps and composites.

The *Kronecker double cover* $\tilde{G}$ of a graph G has vertices v_i, v_i' for each vertex v_i of G, with adjacency: $v_i \sim_G v_j$ if and only if $v_i \sim v_j'$ and $v_i' \sim v_j$ in $\tilde{G}$. (Such double covers are studied in [**22**]).

Clearly $\tilde{G} \cong G \wedge K_2$, and the Kronecker product projection $p_1 : G \wedge K_2 \to G$ is the $(2:1)$ covering projection (see § 19 of Biggs [**1**]). For example, $K_4 \wedge K_2$ is the 3-cube Q_3.



As a result of 4.1, a morphism $f : G \to H$ induces a covering morphism $\tilde{f} : \tilde{G} \to \tilde{H}$ of their double covers. In particular an automorphism a of G induces an automorphism $\tilde{a}$ of $\tilde{G}$ defined by $v \mapsto a(v)$ and $v' \mapsto (a(v))'$.

For both this and the following section we need the concept of *induced projection.* All projections concerning Kronecker products are graph-morphisms, but not so those concerning local joins [not even the (non-categorical) lexicographic product].

Definition. If $p : G \to X$ is any projection [as in § 3, i.e. not necessarily a morphism], and $\alpha : Y \to X$ is any graph-morphism, then *the graph induced by α from p* is

$$G_\alpha = \{(y, g) : y \in Y, g \in G \text{ and } \alpha(y) = p(g)\}.$$

[*Notation.* Here $g \in G$ denotes $g \in V(G) \cup E(G)$, i.e. g is either a vertex or an edge.]

The set of vertices and edges defined are joined in the obvious way, i.e.

$$V(G_\alpha) = \{(y, g) \in V(Y) \times V(G) : p(g) = \alpha(y)\} \quad \text{with}$$
$$[(y, g) \sim (y', g')] \in E(G_\alpha) \quad \text{if and only if}$$
$$[g, g'] \in E(G) \quad \text{and} \quad [y, y'] \in E(Y).$$

G_α has an induced projection onto Y, given by $p_\alpha : (y, g) \mapsto y$.

When the projection p is a morphism, so is the induced projection p_α, and this construction is simply the '*pullback*' in the category $\mathcal{G}raph$. The projection $\tilde{\alpha} : G_\alpha \to G$ given by $(y, g) \mapsto g$ $(y \in V(Y) \coprod E(Y))$ is a morphism in any case.

In the case where $p_1 : G_1 \wedge G_2 \to G_1$ is a Kronecker product projection, it is easily shown that the graph induced by $\alpha : Y \to G_1$ from $p_1 : G_1 \wedge G_2 \to G_1$ is isomorphic to the Kronecker product $Y \wedge G_2$ (with its projection to Y).

Note in particular that if $\alpha : Y \to G_1$ is a subgraph-inclusion, then α induces $Y \wedge G_2$ which is isomorphic to the subgraph $\alpha(Y) \wedge G_2$ of $G_1 \wedge G_2$.

Taking $Y = G_1$ and α an automorphism of G_1, we obtain an automorphism $\tilde{\alpha} : G_1 \wedge G_2 \to G_1 \wedge G_2$ given by $v_1 v_2 \mapsto \alpha(v_1) v_2$, $v_i \in V(G_i)$.

At this stage, we introduce an alternative notation, $*_X(G_x)$ denoting the local join $p : G \to X$ where the G_x are the fibres, indexed by vertices of a graph X.

4.2 Proposition. *If $\{\alpha_x : G_x \to H_x\}_{x \in V(X)}$ is any collection of maps indexed by the vertices of a graph X, then we obtain an induced map*

$$*_X(\alpha_x) : *_X(G_x) \to *_X(H_x)$$

given by $v_x \mapsto \alpha_x(v_x)$, $[v_x, w_x] \mapsto [\alpha_x(v_x), \alpha_x(w_x)]$ and $[v_x, v_y] \mapsto [\alpha_x(v_x), \alpha_y(v_y)]$ (this is well defined by the definition of local joins).

4.3 Corollary. *If $\{\alpha_i : G_i \to H_i\}$ is any collection of maps, then we get induced maps*:

(i) $\coprod_i \alpha_i : \coprod_i G_i \to \coprod_i H_i$,

(ii) $\ast_i \alpha_i : \ast_i G_i \to \ast_i H_i$.

Proof. Take X as $\bar{K}_n$ in (i) and as K_n in (ii).

4.4 Theorem. *$*_X$ is a covariant functor of n variables* $(|V(X)| = n)$,

$$*_X : \mathcal{G}raph \to \mathcal{G}raph.$$

i.e. (i) *$*_X(1_X : G_x \to G_x)$ equals the identity on $*_X G_x$*;

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(ii) *if* $\alpha_x : G_x \to H_x$, $p_x : F_x \to G_x$; *then*

$$*_X(\alpha_x) \circ *_X(\beta_x) = *_X(\alpha_x \circ \beta_x).$$

4.5 PROPOSITION. *If* $\{p_x : G_x \to H_x\}$ *is a collection of local joins then* $*_X(p_x) : *_X(G_x) \to *_X(H_x)$ *is a local join where* $(*_X(p_x))^{-1}h = p*^{-1}h$ *for* $h \in H_x$.

4.6 COROLLARY. *The disjoint union and join of local joins* $p_i : G_i \to H_i$ *are themselves local joins.*

Proof. $(\coprod p_i)^{-1}h = p_i^{-1}h$ for $h \in H_i$, and similarly for $*$.

4.7 PROPOSITION. *If* $p : G \to X$ *is a local join then so is* $p_\alpha : G_\alpha \to Y$ *with* $G_\alpha = *_Y(p^{-1}(\alpha(Y)))$.

Remark. p is a local join if and only if for every edge-inclusion map $e : K_2 \to X$, the induced projection $p_e : G_e \to K_2$ is a join.

Definition. If $p : G \to X$, $p' : G' \to X$ are local joins, then a *map of local joins* is a map $f : G \to G'$ such that $p'f = p$, i.e. the following diagram commutes.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "p"'] & & G' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

The collection L_X of local joins over X together with maps of local joins over X form a category $(\mathscr{L})_X$.

4.8 PROPOSITION. (i) *A morphism* $\alpha : Y \to X$ *induces from the map* $f : G \to G'$ *of local joins, a map* f_α ; $G_\alpha \to G_\alpha'$ *of local joins, defined by* $f_\alpha : (y, g) \mapsto (y, f(g))$.

(ii) *There is a covariant functor* $\alpha_* : \mathscr{L}_X \to \mathscr{L}_Y$, *defined by* $G \mapsto G_\alpha$, $f \mapsto f_\alpha$.

Proof. (i) Let $p_\alpha : G_\alpha \to Y$, $p_\alpha' : G_\alpha' \to Y$ be the local joins induced by α from $p : G \to X$, $p' : G' \to X$ respectively. Then $p_\alpha(y, g) = y$ by definition and $p_\alpha(y, f(g)) = y$, since $\alpha(g) = p(g) = (pf)(g) = p(f(g))$.

Hence

$$\begin{tikzcd} G_\alpha \arrow[rr, "f_\alpha"] \arrow[dr, "p_\alpha"'] & & G_\alpha' \arrow[dl, "p_\alpha'"] \\ & Y & \end{tikzcd}$$

is commutative.

(ii) α induces from the maps $G \xrightarrow{f} G' \xrightarrow{h} G''$ the maps $G_\alpha \xrightarrow{f_\alpha} G_\alpha' \xrightarrow{h_\alpha} G_\alpha''$ such that for each $(y, g) \in G_\alpha$:

$$h_\alpha f_\alpha(y, g) = h_\alpha(y, f(g)) = (y, hf(g)) = (hf)_\alpha(y, g).$$

Therefore, $h_\alpha f_\alpha = (hf)_\alpha$ and $\alpha * h.\alpha * f = \alpha*(hf)$; also $(\alpha*1)(y, g) = 1(y, g)$, by definition of $\alpha *$.

4.9 THEOREM. *The correspondence*

$$L: \begin{cases} X \mapsto L_X \\ \alpha \mapsto L_\alpha \end{cases}$$

gives a contravariant functor $L : \mathscr{G}raph \to \mathscr{S}et$ *where* $L_\alpha : L_X \to L_Y$ *is the set-function given by* $L_\alpha(p : G \to X) = (p_\alpha : G_\alpha \to Y)$.

Proof. (i) In the case $\alpha = 1_X : X \to X$, we get $L_{1_X}(p) = p$.

(ii) If $Z \xrightarrow{\alpha'} Y \xrightarrow{\alpha} X$ then $L_{\alpha'}L_\alpha(p) = L_{\alpha'}(p_\alpha) = (p_\alpha)_{\alpha'} = p_{\alpha\alpha'} = L_{\alpha\alpha'}(p)$.

5. Planarity of Kronecker products. In this section we show that if a graph is decomposable as a Kronecker product, this helps in deciding whether the graph is planar. First we deal with circuit-graphs C_k $(k > 2)$.

5.1 LEMMA. *The Kronecker product of any two circuits is non-planar.*

Proof. In order to utilise the pullback construction of § 4, we observe that there is a morphism $f : C_n \to C_3$ defined by

$$v_0 \mapsto v_0,\ v_r \mapsto v_1 \text{ if } r \text{ is odd},\ v_r \mapsto v_2 \text{ if } r\ (\neq 0) \text{ is even},$$

if and only if n is odd. This morphism of circuits has "winding number 1" in the sense of maps of circles. Similarly we can define morphism $f : C_n \to C_4$ of "winding number 1" if and only if n is even.

The pullback diagram

$$\begin{tikzcd} C_{2k+i} \wedge C_n \arrow[r, "\bar{f}"] \arrow[d, "(p_1)_f"] & C_i \wedge C_n \arrow[d, "p_1"] \\ C_{2k+i} \arrow[r, "f"] & C_i \end{tikzcd} \qquad (i = 3 \text{ or } 4)$$

shows that $C_{2k+i} \wedge C_n$ is non-planar if $C_i \wedge C_n$ is non-planar. Thus it suffices to show non-planarity of $C_3 \wedge C_3$, $C_3 \wedge C_4$ and $C_4 \wedge C_4$. The first two of these are easily shown to have a subgraph contractible to K_5. Finally $C_4 \wedge C_4$ is $K_{2,2} \wedge K_{2,2}$ and so by Theorem 2.1, this is $K_{4,4} \coprod K_{4,4}$. The result then follows by Kuratowski's Theorem.

In order to obtain our main characterisation theorem (5.3) for planarity of Kronecker products, we need other sufficient conditions for non-planarity.

5.2 LEMMA. *The Kronecker product* $G_1 \wedge G_2$ *is non-planar if either*

(i) *both* G_1 *and* G_2 *contain* $K_{1,3}$ *as a subgraph, or*

(ii) *one of the* G_1 *and* G_2 *contains one of the graphs* X_i *shown below, and the other one has the path-graph* P_5 *or the complete graph* K_3 *as a subgraph.*

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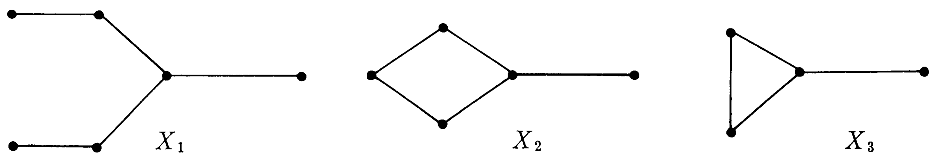
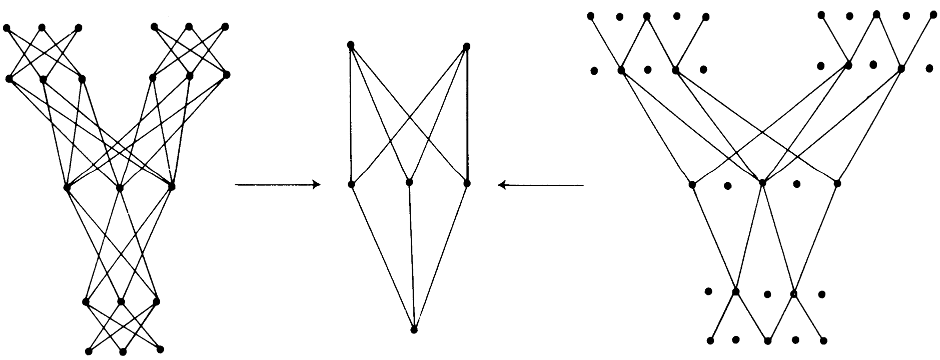
Proof. (i) Theorem 2.1 implies that $K_{1,3} \wedge K_{1,3} = K_{3,3} \coprod K_{1,9}$, and the result follows by Kuratowski's Theorem.

(ii) Our 'forbidden subgraphs' are



The following diagram illustrates the fact that $X_1 \wedge K_3$ and $X_1 \wedge P_5$ each have a subgraph contractible to $K_{3,3}$. The other cases are similar, and are left to the reader.



$X_1 \wedge K_3$ $K_{3,3}$ one of the components of $X_1 \wedge P_5$

(The arrows indicate the obvious contractions).

These two lemmas enable us to characterise planar Kronecker products. For convenience, graphs with less than five vertices are dealt with separately in 5.4. By a 1-*contraction* of G we mean the removal from G of each vertex of degree 1 (and its incident edge).

5.3 THEOREM. *Let G_1 and G_2 be connected graphs with more than four vertices. Then $G_1 \wedge G_2$ is planar if and only if either*

(i) *one of the graphs is a path and the other one is* 1-*contractible to a path or a circuit, or*

(ii) *one of them is a circuit and the other is* 1-*contractible to a path.*

Proof. Let $G_1 \wedge G_2$ be planar. By 5.2(i), not both the graphs contain $K_{1,3}$ as a subgraph. It follows that at least one of them, say G_1, is a path or a circuit. If G_1 is a path, then $P_5 \subset G_1$ and so G_2 cannot contain any of the graphs X_1, X_2, X_3 in 5.2. Therefore G_2 is either a path or a circuit or 1-contractible to a path or a circuit.

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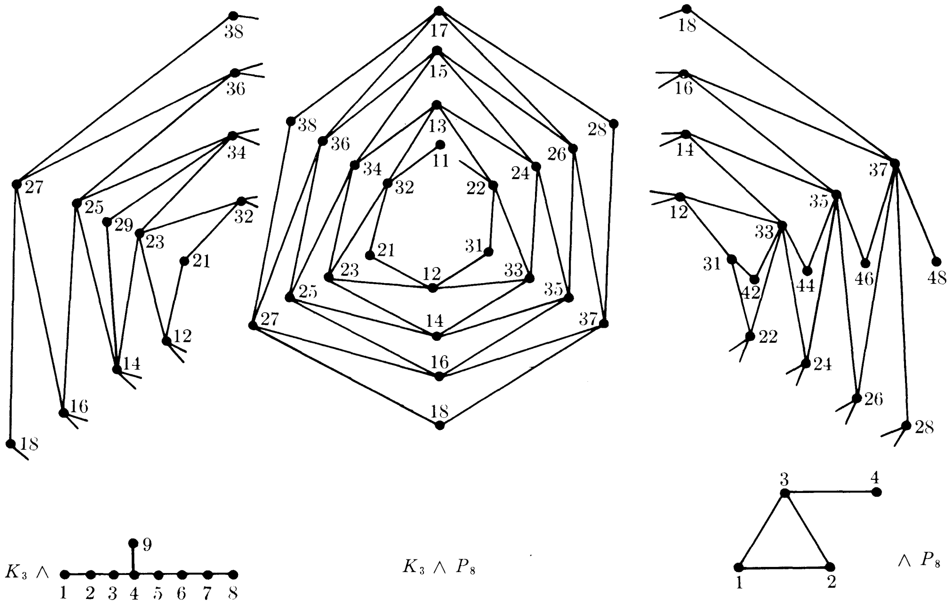
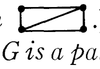
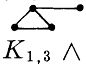
If G_1 is a circuit then G_2 cannot contain a circuit [for Lemma 5.1 would contradict planarity] or the graph X_1, so G_2 is either a path or is 1-contractible to a path, and the necessity is complete.

Sufficiency is easily established. We shall just give the following diagrams of three typical cases of conditions (i) and (ii). Planarity of all such cases is self-evident.



Clearly the conditions (i), (ii) in 5.3 are sufficient for planarity of *all* graphs. However for graphs with less than five vertices, the conditions are not in general necessary. We can supplement Theorem 5.3 by considering the cases where (at least) one of the graphs has less than five vertices. Results can be summarised as follows:

5.4 PROPOSITION. (i) *Each of the graphs $K_4 \wedge G$ and $K_{4/1} \wedge G$ is planar if and only if G is K_2.* [*$K_{4/1}$ denotes the graph* .]

(ii)  $\wedge$ *G is planar if and only if G is a path.*

(iii) $K_{1,3} \wedge$ *G is planar if and only if G is a path or a circuit.*

(iv) $C_4 \wedge$ *G is planar if and only if G is a tree.*

(v) $C_3 \wedge$ *G is planar if and only if G is a path or* 1*-contractible to a path.*

Proof. Lemmas 5.1 and 5.2 are applied with similar arguments to those used in the proof of 5.3.

6. Complexity of local joins and Kronecker products. The *complexity* $\kappa(G)$ of a graph G is the number of spanning trees of G (see, for example, Biggs [**1**]).

For a regular graph G, $\kappa(G)$ can be evaluated using the *spectrum* $S(G)$ of the graph, i.e. the set of eigenvalues of the *adjacency matrix* $\mathbf{G} = [g_{ij}]$ $(i, j = 1, \ldots, n = |V(G)|)$ where g_{ij} is the number of edges between the vertices labelled v_i and v_j.

6.1 PROPOSITION (Cvetkovic [**2**]). *If G is regular of degree d, then $S(G) = \{d, \lambda_2, \ldots, \lambda_n\}$ and $\kappa(G) = n^{-1} \prod_{j=2}^{n} (d - \lambda_j)$.*

We can instead represent G by another matrix $M(G) = [m_{ij}]$ defined by $m_{ij} = g_{ij}$, $i \neq j$; $m_{ii} = n - d_i$, $i =$ degree of v_i. This matrix is row-regular of degree n, i.e. all row-sums are equal to n, and this enables us to generalise to arbitrary graphs some matrix-properties of regular graphs (see [**21**]). In particular, we can generalise to arbitrary graphs a theorem of Finck and Grohmann [**8**, Satz 3] for regular graphs, characterising decomposability of graphs with respect to the join operation $*$. It is an unsolved problem to characterise graphs G by the rank of $\mathbf{G}$; it appears that the rank of $M(G)$ is more relevant.

6.2 THEOREM. (i) *The following propositions are equivalent:*

(a) *$M(G)$ has rank $n - k$,*

(b) *G is a $(k + 1)$-fold join,*

(c) *n has multiplicity $k + 1$ in the spectrum $SM(\bar{G})$ of $M(\bar{G})$.*

(ii) *G is $*$-indecomposable if and only if $0 \in SM(\bar{G})$.*

The matrix $M(G)$ can be interpreted as the adjacency matrix of a graph ρG obtained from G by the adjoining of $n - d_i$ loops at each vertex of degree d_i. Since $\kappa(\rho G) = \kappa(G)$, we can find the complexity of any graph using the following generalisation of 6.1 (see [**20**]):

6.3 PROPOSITION. *If $S(\rho G) = \{n, \lambda_2', \ldots, \lambda_n'\}$ then $\kappa(G) = n^{-1} \prod_{j=2}^{n} (n - \lambda_j')$.*

Similarly 6.1 holds for any *row-regular graph of degree d* (i.e. whose adjacency matrix has all row-sums equal to d).

With a view to computing complexities of local joins and Kronecker products, we first consider their eigenvalues. Let $p : G \to X$ be a local join, where X is a (labelled) graph with $V(X) = \{x_1, \ldots, x_n\}$ and adjacency matrix $\mathbf{X} = [x_{ij}]$. Suppose the fibre $G_i = p^{-1}x_i$ has m_i vertices, adjacency matrix $\mathbf{G}_i$, and spectrum $S(G_i) = \{\mu_j{}^i | j = 1, \ldots, m_i\}$.

6.4 THEOREM. *If the fibres G_i of the local join $p : G \to X$ are row-regular of degree d_i then the eigenvalues of G are:*

(i) *$\mu_j{}^i$, $j = 2, 3, \ldots, m_i$, $i = 1, 2, \ldots, n$;*

(ii) *the n eigenvalues of the matrix $A = \mathbf{X}[m_1, \ldots, m_n]^t + \operatorname{diag}[d_1, \ldots, d_n]$.*

Proof. (i) Each G_i has largest eigenvalues d_i with $[1, 1, \ldots, 1]^t$ as an associated eigenvector. Whether G_i is connected or not, the other eigenvalues $\mu_2{}^i, \ldots, \mu_{m_i}{}^i$ have associated eigenvectors $\mathbf{x}_2{}^i, \ldots, \mathbf{x}_{m_i}{}^i$ whose sum of coordinates in each case is zero.

The adjacency matrix of G can be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{X}_{12} & \cdots & & \mathbf{X}_{1n} \\ \mathbf{X}_{21} & \mathbf{G}_2 & \cdots & & \mathbf{X}_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ & & & & \mathbf{X}_{n-1,n} \\ \mathbf{X}_{n1} & \mathbf{X}_{n2} & \cdots & \mathbf{X}_{n,n-1} & \mathbf{G}_n \end{bmatrix}$$

where $\mathbf{G}_i$ is the submatrix consisting of the matrix of G_i and $\mathbf{X}_{ij}$ is the $m_i \times m_j$-matrix whose every entry is x_{ij}, i.e. a block of 1's or 0's according as x_{ij} is 1 or 0.

The eigenvector equations for the fibres are

$$\mathbf{G}_{ij}\mathbf{x}^i = \mu_j{}^i\mathbf{x}_j{}^i,$$

Also we have

$$\mathbf{X}_{ij}\mathbf{x}^i = \mathbf{0} \quad \text{for } i \neq j.$$

It follows that $x = [\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{x}_j{}^i, \mathbf{0}, \ldots, \mathbf{0}]^t$ satisfies

$$\begin{aligned} \mathbf{G}\mathbf{x} &= [\mathbf{0}, \ldots, \mathbf{0}, \mu_j{}^i\mathbf{x}_j{}^i, \mathbf{0}, \ldots, \mathbf{0}]^t \\ &= p_j{}^i\mathbf{x}, \quad j = 2, \ldots, m_i, i = 1, \ldots, n. \end{aligned}$$

Thus $\mu_j{}^i$ is an eigenvalue for $\mathbf{G}$ with $\mathbf{x}$ as eigenvector.

(ii) It remains to derive the other n eigenvalues of G from the given eigenvalues d_i for G_i where d_i has associated eigenvector $\xi_i = [1, \ldots, 1]^t$ (m_i-fold), $i = 1, 2, \ldots, n$.

If r is such an eigenvalue then $|G - \mu I| = 0$.

This gives a system of linear equations:

$$\begin{aligned} &(d_1 - \mu) + x_{12}m_2 + x_{13}m_3 + \ldots + x_{1n}m_n = 0 \quad (m_1 \text{ times}), \\ &x_{21}m_1 + (d_2 - \mu) + x_{23}m_3 + \ldots + x_{2n}m_n = 0 \quad (m_2 \text{ times}), \ldots, \\ &xn_1m_1 + x_{n2}m_2 + x_{n3}m_3 + \ldots + x_{n,n-1}m_{n-1} + (d_n - \mu) = 0 \quad (m_n \text{ times}). \end{aligned}$$

Equivalently, μ is an eigenvalue of the matrix A required.

Denoting by $\varphi(G)$ the characteristic polynomial of G, 6.4 gives in particular:

6.5 Corollary. (i) *If the fibres G_i of $p : G \to X$ each have cardinality m, and each is row-regular of degree d, then*

$$\varphi(G) = \prod_{i=1}^{n} \left[(x - m\lambda_i - d) \prod_{j=2}^{m} (x - \mu_j{}^i) \right].$$

(ii) *If Y is row-regular of degree d, with $S(Y) = \{d, \mu_2, \ldots, \mu_m\}$, then for the lexicographic product we have*

$$\varphi(X[Y]) = \prod_{i=1}^{n} (x - m\lambda_i - d) \prod_{j=2}^{m} (x - \mu_j).$$

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Finally we exploit the fact that κ ignores loops to derive from 6.4 a complexity theorem for local joins, with fibres having no regularity restriction:

6.7 THEOREM. *Suppose $p : G \to X$ is a local join whose fibres G_i each have m vertices, with $|V(X)| = n$, and $x_i = p(G_i)$ of degree r_i. If $S(\rho G) = \{m, \mu_2{}^i, \ldots, \mu_m{}^i\}$ then $\kappa(G) = m^{n-2}\kappa(X) \prod_{i=1}^n \prod_{j=2}^m (mr_i + m - \mu_j{}^i)$.*

Proof. Construct a local join $H \to X$ whose fibre H_i over x_i is obtained by adjoining $d_i = m(n - r_i - 1)$ loops to each vertex of ρG_i. Such adjoining increases each eigenvalue by d_i (see [**20**, 2.2]), thus

$$S(H_i) = \{m + d_i, \mu_2{}^i + d_i, \ldots, \mu_m{}^i + d_i\}.$$

The matrix A in 6.4 becomes (for H) equal to

$$m\mathbf{X} + \operatorname{diag}(d_1 + m, d_2 + m, \ldots, d_n + m),$$

i.e. $m(\mathbf{X} + \operatorname{diag}(n - r_1, \ldots, n - r_m))$, which is $m.M(X)$.

Thus by 6.4, the eigenvalues ξ_k, $k = 1, \ldots, mn$, of H are:

(i) $d_i + \mu_j{}^i$, $j = 2, \ldots, m$, $i = 1, \ldots, n$.

(ii) $m\lambda_i$, $\lambda_i \in S(\rho X)$.

For any $h \in V(H)$ in the fibre H_i we have

$$\deg_H h = \deg_{H_i} h + m \deg_X x_i = m(n - r_i) + mr_i = mn.$$

Therefore $H = \rho G$, and 6.3 gives:

$$\begin{aligned}
\kappa(G) &= \frac{1}{mn} \prod_{k=2}^{mn} (mn - \xi_k) \\
&= \frac{1}{mn} \prod_{i=2}^{n} (mn - m\lambda_i) \prod_{i=1}^{n} \prod_{j=2}^{m} (mn - (\mu_j{}^i + d_i)) \\
&= m^{n-2} \frac{1}{n} \prod_{i=2}^{n} (n - \lambda_i) \prod_{i=1}^{n} \left[\prod_{j=2}^{m} (mr_i + m - \mu_j{}^i) \right], \quad \text{(using 6.3)} \\
&= m^{n-2} \kappa(X) \prod_{i=1}^{n} \prod_{j=2}^{m} (mr_i + m - \mu_j{}^i).
\end{aligned}$$

In particular if Y is any graph with m vertices, and $S(\rho Y) = \{m, \mu_2, \ldots, \mu_n\}$, we have

6.8 COROLLARY. *The complexity of the lexicographic product $X[Y]$ is*

$$\kappa(X[Y]) = m^{n-2}\kappa(X) \prod_{i=1}^{n} \prod_{j=2}^{m} (mr_i + m - \mu_j).$$

Finally we compute the complexity of any Kronecker product of regular graphs. Since the adjacency matrix of a Kronecker product is the tensor product of the adjacency matrices of the graphs involved, we have a well-known result:

6.9 LEMMA. *For any graphs G_1 and G_2,*

$$S(G_1 \wedge G_2) = \{\lambda\mu : \lambda \in S(G_1),\ \mu \in S(G_2)\}.$$

Applying this result, using similar techniques to those in 6.7, the following result is easily obtained:

6.10 THEOREM. *Let G_i be regular of degree d_i with n_i vertices ($i = 1, 2$). If $S(G_1) = \{\lambda_1, \ldots, \lambda_{n_1}\}$ and $S(G_2) = \{\mu_1, \ldots, \mu_{n_2}\}$, then*

$$\kappa(G_1 \wedge G_2) = d_1^{n_2-1} d_2^{n_1-1} \kappa(G_1)\kappa(G_2) \prod_{i=2}^{n_1} \prod_{j=2}^{n_2} (d_1 d_2 - \lambda_i \mu_j).$$

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