European Journal of Applied Mathematics (2023), **34**, pp. 1133–1169
doi:10.1017/S0956792523000220

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PAPER

Spreading dynamics of a diffusive epidemic model with free boundaries and two delays

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Received: 13 July 2021; **Revised:** 02 January 2022; **Accepted:** 16 July 2023; **first published online** 11 August 2023

Keywords: Free boundary; semi-wave; spreading speed; time delay; epidemic model

2020 Mathematics Subject Classification: 35K57, 35R35 (Primary); 35K61, 34K30 (Secondary)

Abstract
A delayed reaction-diffusion system with free boundaries is investigated in this paper to understand how the bacteria spread spatially to larger area from the initial infected habitat. Under the assumptions that the nonlinearities are of monostable type and the initial values satisfy some compatible condition, we show that the free boundary problem is well-posed and discuss the long-time behaviour of solution (including spreading and vanishing) in terms of the spatial-temporal risk index. Furthermore, to determine the spreading speed of free boundaries when spreading occurs, we first study the distribution of roots of a transcendental equation containing a polynomial of degree four and then establish the existence and uniqueness of monotone solution to a delay-induced nonlocal semi-wave problem by employing the approximation method, lower-upper solutions technique and Schauder fixed point theorem. It is shown that time delays slow down the spreading of bacteria.

1. Introduction

Each infectious disease usually has its own specific route of transmission, such as contact transmission, droplet transmission, faecal-oral transmission and so on. Faecal-oral transmitted diseases, including hand foot mouth disease, cholera, poliomyelitis and viral hepatitis A, spread mostly through unapparent faecal contamination of food, water and hands. Researchers have estimated that each year, there are 1.3–4.0 million cases of cholera, and 21,000 to 143,000 deaths worldwide due to cholera [2]. To model the cholera epidemic which spread in the European Mediterranean regions in 1973, Capasso and Paveri-Fontana [6] proposed the following system of two ordinary differential equations

$$\begin{cases} u'(t) = -b_1 u + av, \\ v'(t) = -b_2 v + g(u), \end{cases} \tag{1.1}$$

which describes the positive feedback interaction between the infective human population v and the concentration of bacteria u in the environment. Here the constants b_i $(i = 1, 2)$ respectively represent the intrinsic decay rates of the two populations, av is the contribution of the infective humans to the growth rate of bacteria and $g(u)$ is the infection rate of humans under the assumption that the total susceptible human population is constant. The qualitative analysis shows that there exist threshold dynamics for (1.1) with suitable monotonicity assumptions on the nonlinearity $g(u)$ [7]. Moreover, the model (1.1) can also be used to describe the spread of other faecal-oral transmitted diseases, including typhoid fever and infectious hepatitis, under suitable modification [8].

For more realistic models, spatial diffusions of the bacteria and the infective humans should be considered. Capasso and Maddalena [9] studied the following random diffusive system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \dfrac{\partial^2 u}{\partial x^2} - b_1 u + av, \\ \dfrac{\partial v}{\partial t} = d_2 \dfrac{\partial^2 v}{\partial x^2} - b_2 v + g(u), \end{cases} \tag{1.2}$$

where d_i $(i = 1, 2)$ are diffusion coefficients. If the diffusion of the infective humans is relatively smaller than that of the bacteria, we can ignore it by setting $d_1 > 0$ and $d_2 = 0$ (in this case, the model is *partially degenerate*). In [9], the corresponding Robin boundary value problems with suitable assumptions on g have been investigated for $d_1 > 0$ and $d_2 \geqslant 0$. When $d_1 > 0$ and $d_2 > 0$, two threshold parameters Θ_m and Θ_M were introduced such that the epidemic eventually tends to extinction for $0 < \Theta_m < 1$ and tends to a spatially inhomogeneous stationary endemic state for $\Theta_M > 1$; while for $d_1 > 0$ and $d_2 = 0$, it has only one threshold parameter as in (1.1). Moreover, the travelling waves of (1.2) have been studied. More precisely, when $d_1 > 0$ and $d_2 > 0$, Hsu and Yang in [19] established the existence, uniqueness, monotonicity and asymptotic behaviour of travelling waves for (1.2) with the term av replaced by a more general function $h(v)$. For the partially degenerate case, Xu and Zhao [39] established the existence, uniqueness (up to translation) and global exponential stability with phase shift of bistable travelling waves; Zhao and Wang [43] proved the existence of Fisher type monotone travelling waves and determined the minimal wave speed.

In view of the latent period of infection, maturation time of population or other factors, time delay is introduced in various biological models. For the general model in [19], Wu and Hsu [38] recently incorporated two discrete time delays into the model as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \dfrac{\partial^2 u}{\partial x^2} - b_1 u + h(v(t-\tau_1, x)), \\ \dfrac{\partial v}{\partial t} = d_2 \dfrac{\partial^2 v}{\partial x^2} - b_2 v + g(u(t-\tau_2, x)), \end{cases} \tag{1.3}$$

where diffusion coefficients (d_1, d_2) satisfy $d_1 > 0$ and $d_2 \geqslant 0$, and time delays (τ_1, τ_2) satisfy $\tau_1 \geqslant 0$ and $\tau_2 > 0$. It was shown in [38] that the system (1.3) with and without the quasi-monotone condition admit entire solution, which is defined in the whole time-space and behaves like a combination of travelling waves as t tends to $-\infty$. When $d_2 = 0$, $\tau_1 = 0$ and $h(v) = av$, [31] investigated the existence of spreading speed and minimal wave speed.

However, the fixed boundary problems (including the bounded domain or the whole space) considered above are not suitable to be used to understand how the bacteria spread spatially to larger area from the initial infected habitat, which motivates us to consider the corresponding free boundary problems. In recent years, the free boundary problems for biological models have been studied extensively. For species models, Du and Lin [15] first studied the free boundary problem for diffusive logistic equation in homogeneous environment. They proved that the species either spreads successfully or vanishes eventually, and determined the spreading speed of free boundary. Based on the work [15], free boundary problems for single species model with periodic coefficients [11–13, 32], nonlocal dispersal [3, 14], seasonal succession [26], general nonlinear term [16] and advection term [29, 36] have been investigated. For epidemic models, free boundary problems for partially degenerate epidemic model [1, 40], SIS [5, 18, 20], SIRS [4], SEIR [23], West Nile virus [24, 35] models were also studied recently. Moreover, the dynamics of biological models with time delay have been studied extensively, but the corresponding free boundary problems were rarely considered. For example, [44] considered the free boundary problem for tumour model with time delay. [23] investigated SEIR model with free boundary and distributed time delay. [28] studied the free boundary problem for the delayed Fisher-KPP equation. [10] considered the partially degenerate epidemic model with free boundary and time delay.

In this paper, we consider the free boundary problem for (1.3) as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \dfrac{\partial^2 u}{\partial x^2} - b_1 u + h(v(t-\tau_1, x)), & t > 0, s_1(t) < x < s_2(t), \\ \dfrac{\partial v}{\partial t} = d_2 \dfrac{\partial^2 v}{\partial x^2} - b_2 v + g(u(t-\tau_2, x)), & t > 0, s_1(t) < x < s_2(t), \\ u(t,x) = v(t,x) = 0, \quad t > 0, x \geqslant s_2(t) \quad \text{or} \quad x \leqslant s_1(t), \\ s_1(0) = -s_0, s_1'(t) = -\mu \dfrac{\partial u}{\partial x}(t, s_1(t)), \quad t > 0, \\ s_2(0) = s_0, s_2'(t) = -\mu \dfrac{\partial u}{\partial x}(t, s_2(t)), \quad t > 0, \\ u(\theta, x) = u_0(\theta, x), \quad -\tau_2 \leqslant \theta \leqslant 0, s_1(\theta) \leqslant x \leqslant s_2(\theta), \\ v(\theta, x) = v_0(\theta, x), \quad -\tau_1 \leqslant \theta \leqslant 0, s_1(\theta) \leqslant x \leqslant s_2(\theta). \end{cases} \tag{1.4}$$

As introduced above, u and v represent the concentration of bacteria in the environment and the population density of infective human, respectively; the diffusion coefficients d_i $(i = 1, 2)$ and the intrinsic decay rates b_i $(i = 1, 2)$ are positive constants. $s_1(t)$ and $s_2(t)$ $(t > 0)$ are free boundaries, which represent the boundary fronts of infected area $(s_1(t), s_2(t))$ at time t. Since the spread of epidemic discussed here is mainly due to the growth of bacteria which results from the infective human population, it is reasonable to assume that the movements of boundary fronts $s_1(t)$ and $s_2(t)$ are fully driven by the bacteria. We assume that the front $s_1(t)$ expands at a rate proportional to the gradient of bacterial concentration at $x = s_1(t)$, which gives rise to the Stefan condition $s_1'(t) = -\mu \frac{\partial u}{\partial x}(t, s_1(t))$. Similarly, the right front $s_2(t)$ satisfies $s_2'(t) = -\mu \frac{\partial u}{\partial x}(t, s_2(t))$.

We assume that the initial functions satisfy

$$\begin{cases} u_0(\theta, x) \in C^{1,2}([-\tau_2, 0] \times [s_1(\theta), s_2(\theta)]), v_0(\theta, x) \in C^{1,2}([-\tau_1, 0] \times [s_1(\theta), s_2(\theta)]), \\ u_0(\theta, x) \begin{cases} > 0 & \text{for } \theta \in [-\tau_2, 0], x \in (s_1(\theta), s_2(\theta)), \\ \equiv 0 & \text{for } \theta \in [-\tau_2, 0], x \notin (s_1(\theta), s_2(\theta)), \end{cases} \\ v_0(\theta, x) \begin{cases} > 0 & \text{for } \theta \in [-\tau_1, 0], x \in (s_1(\theta), s_2(\theta)), \\ \equiv 0 & \text{for } \theta \in [-\tau_1, 0], x \notin (s_1(\theta), s_2(\theta)) \end{cases} \end{cases} \tag{1.5}$$

as well as the compatible condition

$$[s_1(\theta), s_2(\theta)] \subset [-s_0, s_0] \quad \text{for} \quad \theta \in [-\max\{\tau_1, \tau_2\}, 0]. \tag{1.6}$$

The nonlinearities g and h satisfy the following conditions:

$$(\mathrm{M}) \qquad \begin{cases} h \in C^2([0, +\infty)), g \in (C^2 \cap L^\infty)([0, +\infty)), \\ h(0) = 0 = g(0), \text{ and } h'(z), g'(z) > 0 \text{ for any } z \in [0, +\infty), \\ h''(z) \leqslant 0, g''(z) < 0 \text{ for all } z > 0. \end{cases}$$

For example, $h(v) = av$, $g(u) = \frac{pu}{1+qu}$ (the Holling-II type) with $a, p, q > 0$. Some special cases of (1.4) have been investigated recently. More precisely, when $d_2 = 0$, $\tau_1 = 0$ and $\tau_2 = 0$, [1] established the spreading-vanishing dichotomy of the partially degenerate free boundary problem, and [40] determined the spreading speed; when $\tau_1 = 0$ and $\tau_2 = 0$, [34] determined the long-time dynamical behaviour; when $d_2 = 0$ and $\tau_1 = 0$, similar results have been obtained in [10].

The main subject of this paper is to investigate the long-time behaviour of solution and determine the asymptotic spreading speed of free boundaries for the model (1.4) under the assumptions (1.5)–(1.6) and (M). Compared to the partially degenerate case in [1, 10, 40], our model (1.4) in this paper is essentially a two-dimensional problem which cannot be reduced to a nonlocal single-equation problem by solving v from the second equation as in [1, 10, 40]. Moreover, there are two arbitrary positive time delays in

(1.4), but only one time delay was considered in the previous works [10, 28]. These two differences bring more difficulties to us, especially in establishing the existence of monotone solutions to a delayed semi-wave problem (see (3.1)), which plays an important role in determining the asymptotic spreading speed. Indeed, for the models in [10, 28], the authors discussed the distribution of roots of a transcendental equation, which is the sum of an exponential function and a polynomial of degree two or three, and constructed a lower solution of the delayed semi-wave problem by complex root with imaginary part $\mathrm{Im}\lambda \in (0, \frac{\pi}{c\tau})$. However, we cannot derive a suitable upper bound of $\mathrm{Im}\lambda$ for the transcendental equation containing a polynomial of degree four in this paper. Thus, the approach in [10, 28] does not work here. To overcome the difficulty, we first consider the corresponding perturbed semi-wave problem with small parameter δ (see (3.14)) and establish the existence and uniqueness of monotone solutions by combining the lower-upper solutions technique and the Schauder fixed point theorem. By taking the limit $\delta \to 0$, we prove that for any $c > 0$, the system (1.3) has either a monotone travelling wave with speed c or a monotone semi-wave with speed c. Since it admits no monotone travelling wave solution for any $c \in (0, c^*_\tau)$, we can get the desired result on the existence of semi-wave solution.

The rest of this paper is organised as follows. In section 2, the well-posedness and long-time behaviour of the solution are presented. A sharp criteria for spreading and vanishing is also provided. Section 3 is devoted to the study of spreading speeds of free boundaries when spreading happens.

2. Long-time behaviour of the solutions

In this section, we mainly investigate the spreading and vanishing phenomenon of bacteria. A sharp criteria for spreading and vanishing is also provided by choosing the spreading capability μ as varying parameter.

2.1 Preliminaries: well-posedness and comparison principles

We first present the well-posedness and comparison principles of (1.4).

Theorem 2.1. (i) *Assume that $\alpha \in (0,1)$ and $(u_0(\theta,x), v_0(\theta,x), s_1(\theta), s_2(\theta))$ satisfies (1.5) and (1.6). There is a $T_0 > 0$ such that (1.4) admits a unique solution (u, v, s_1, s_2) with $u(t,x)$, $v(t,x) \in C^{(1+\alpha)/2,1+\alpha}(D_{T_0})$, $s_1(t), s_2(t) \in C^{1+\alpha/2}([0,T_0])$, where $D_{T_0} = \{(t,x) \in \mathbb{R}^2 : t \in [0,T_0], x \in [s_1(t), s_2(t)]\}$.*

(ii) *For the solution (u, v, s_1, s_2) established in (i), there exist positive constants K_1, K_2 and K_3 independent of T_0 such that the solution satisfies $0 < u(t,x) \leqslant K_1$, $0 < v(t,x) \leqslant K_2$ and $0 < -s_1'(t), s_2'(t) \leqslant K_3$ for $0 < t \leqslant T_0$ and $s_1(t) < x < s_2(t)$.*

(iii) *The solution (u, v, s_1, s_2) of (1.4) exists and is unique for all $t \in (0, +\infty)$.*

Proof. (i) We introduce the coordinate transformation $(t,x) \to (t,y) = (t, y(t,x))$ as follows

$$y = \frac{2x - s_1(t) - s_2(t)}{s_2(t) - s_1(t)} s_0 \quad \text{for} \quad t > 0, \quad y = x \quad \text{for} \quad -\max\{\tau_1, \tau_2\} \leqslant t \leqslant 0, \tag{2.1}$$

and define

$$\begin{aligned}
&w(t,y) := u\Big(t, \frac{s_2(t) - s_1(t)}{2s_0} y + \frac{s_1(t) + s_2(t)}{2}\Big) = u(t,x) \quad \text{for } t > 0,\\
&w(\theta, y) := u_0(\theta, y) \quad \text{for } -\tau_2 \leqslant \theta \leqslant 0,\\
&z(t,y) := v\Big(t, \frac{s_2(t) - s_1(t)}{2s_0} y + \frac{s_1(t) + s_2(t)}{2}\Big) = v(t,x) \quad \text{for } t > 0,\\
&z(\theta, y) := v_0(\theta, y) \quad \text{for } -\tau_1 \leqslant \theta \leqslant 0.
\end{aligned}$$

https://doi.org/10.1017/S0956792523000220 Published online by Cambridge University Press

Denote

$$A = A(s_1, s_2) = \frac{4s_0^2}{(s_2(t) - s_1(t))^2},$$
$$B = B(s_1, s_2, y) = \frac{s_2'(t) - s_1'(t)}{s_2(t) - s_1(t)} y + \frac{s_2'(t) + s_1'(t)}{s_2(t) - s_1(t)} s_0,$$
$$h(z(t - \tau_1, y)) = h(v(t - \tau_1, x)), \quad g(w(t - \tau_2, y)) = g(u(t - \tau_2, x)).$$

Then, (1.4) can be transformed into a fixed boundary problem

$$\begin{cases} \dfrac{\partial w}{\partial t} = d_1 A \dfrac{\partial^2 w}{\partial y^2} + d_1 B \dfrac{\partial w}{\partial y} - b_1 w + h(z(t - \tau_1, y)), & t > 0, -s_0 < y < s_0, \\ \dfrac{\partial z}{\partial t} = d_2 A \dfrac{\partial^2 z}{\partial y^2} + d_2 B \dfrac{\partial z}{\partial y} - b_2 z + g(w(t - \tau_2, y)), & t > 0, -s_0 < y < s_0, \\ w(t, y) = z(t, y) = 0, \quad t > 0, y \geqslant s_0 \text{ or } y \leqslant -s_0, \\ w(\theta, y) = u_0(\theta, y), \quad -\tau_2 \leqslant \theta \leqslant 0, -s_0 \leqslant y \leqslant s_0, \\ z(\theta, y) = v_0(\theta, y), \quad -\tau_1 \leqslant \theta \leqslant 0, -s_0 \leqslant y \leqslant s_0 \end{cases} \tag{2.2}$$

and

$$s_1'(t) = -\frac{2s_0\mu}{s_2(t) - s_1(t)} \frac{\partial w}{\partial y}(t, -s_0), \quad s_2'(t) = -\frac{2s_0\mu}{s_2(t) - s_1(t)} \frac{\partial w}{\partial y}(t, s_0). \tag{2.3}$$

Let $k_1 = -\mu \frac{\partial u_0}{\partial y}(0, -s_0)$ and $k_2 = -\mu \frac{\partial u_0}{\partial y}(0, s_0)$. For $0 < T_0 \leqslant \min\{\frac{s_0}{2+k_1+k_2}, \tau_1, \tau_2\}$, we define

$$D_{T_0}^{s_1} = \{s_1 \in C^1([0, T_0]) : s_1(0) = -s_0, s_1'(0) = k_1, \|s_1' - k_1\|_{C([0,T_0])} \leqslant 1\},$$
$$D_{T_0}^{s_2} = \{s_2 \in C^1([0, T_0]) : s_2(0) = s_0, s_2'(0) = k_2, \|s_2' - k_2\|_{C([0,T_0])} \leqslant 1\}.$$

Taking any fixed $(s_1, s_2) \in D_{T_0}^{s_1} \times D_{T_0}^{s_2}$, we have

$$\begin{aligned} |s_2(t) - s_1(t) - 2s_0| &= |(s_2(t) - s_0) - (s_1(t) + s_0)| \\ &\leqslant \|s_2'\|_{C([0,T_0])} T_0 + \|s_1'\|_{C([0,T_0])} T_0 \\ &\leqslant (2 + k_1 + k_2) T_0 \leqslant s_0, \end{aligned}$$

which implies $s_2(t) - s_1(t) \geqslant s_0$. Then, the above coordinate transformation $(t, x) \to (t, y)$ is a diffeomorphism from $[0, T_0] \times [s_1(t), s_2(t)]$ to $[0, T_0] \times [-s_0, s_0]$. Note that the parabolic equations in (2.2) are linear, since $z(t - \tau_1, y)$ and $w(t - \tau_2, y)$ contained in the right-hand sides of equations are given initial functions when $t \in [0, T_0]$. By the L^p theory of parabolic equations and the Sobolev embedding theorem, we can prove that (2.2) admits a unique $(w(t, y), z(t, y)) \in [C^{(1+\alpha)/2, 1+\alpha}([0, T_0] \times [-s_0, s_0])]^2$.

Denote

$$\hat{s}_1(t) = -s_0 - \int_0^t \frac{2s_0\mu}{s_2(\zeta) - s_1(\zeta)} \frac{\partial w}{\partial y}(\zeta, -s_0) d\zeta$$

and

$$\hat{s}_2(t) = s_0 - \int_0^t \frac{2s_0\mu}{s_2(\zeta) - s_1(\zeta)} \frac{\partial w}{\partial y}(\zeta, s_0) d\zeta.$$

We define an operator $\mathcal{L}$ by

$$\mathcal{L}(s_1, s_2) = (\hat{s}_1, \hat{s}_2).$$

Similar to the proof of Theorem 2.1 in [15], we can show that for $T_0 > 0$ sufficiently small, $\mathcal{L}$ maps $D_{T_0}^{s_1} \times D_{T_0}^{s_2}$ into itself and $\mathcal{L}$ is a contraction mapping on $D_{T_0}^{s_1} \times D_{T_0}^{s_2}$. The contraction mapping theorem gives that $\mathcal{L}$ has a unique fixed point in $D_{T_0}^{s_1} \times D_{T_0}^{s_2}$. Then, (2.2) and (2.3) have a unique local classical

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solution $(w(t,y), z(t,y), s_1(t), s_2(t))$. We should mention that the local well-posedness can also be established similarly as the proof of Theorem 1.1 in [33], where the existence was proved by the Schauder fixed point theorem.

(*ii*) We only prove $u(t,x) \leqslant K_1$ and $v(t,x) \leqslant K_2$, the remaining part can be obtained by similar arguments as in the proof of Lemma 2.3 in [1].

For any $z > 0$, by Taylor's formula and the concavity of h we have

$$h(z) = h(z) - h(0) = h'(0)z + \frac{1}{2}h''(\xi)z^2 \leqslant h'(0)z$$

with some $\xi \in (0, z)$. Since g is bounded, we can choose K_i $(i = 1, 2)$ sufficiently large such that

$$\frac{\|g\|_{L^\infty}}{K_2} \leqslant b_2, \quad h'(0)\frac{K_2}{K_1} \leqslant b_1,$$

which imply

$$\frac{g(K_1)}{K_2} \leqslant b_2, \quad \frac{h(K_2)}{K_1} = \frac{h(K_2)}{K_2} \cdot \frac{K_2}{K_1} \leqslant h'(0)\frac{K_2}{K_1} \leqslant b_1.$$

We may assume that

$$u_0(\theta, x) \leqslant K_1 \quad \text{for } (\theta, x) \in [-\tau_2, 0] \times [-s_0, s_0],$$
$$v_0(\theta, x) \leqslant K_2 \quad \text{for } (\theta, x) \in [-\tau_1, 0] \times [-s_0, s_0].$$

Let

$$(U(t,x), V(t,x)) := (K_1 - u(t,x), K_2 - v(t,x))e^{-kt}$$

with some constant k to be determined later, then (U, V) satisfies

$$\begin{cases} \dfrac{\partial U}{\partial t} = d_1\dfrac{\partial^2 U}{\partial x^2} - (b_1 + k)U + e^{-kt}[b_1K_1 - h(v(t-\tau_1, x))] \\ \geqslant d_1\dfrac{\partial^2 U}{\partial x^2} - (b_1 + k)U + e^{-k\tau_1}h'(\xi)V(t-\tau_1, x), \quad t > 0, s_1(t) < x < s_2(t), \\ \dfrac{\partial V}{\partial t} = d_2\dfrac{\partial^2 V}{\partial x^2} - (b_2 + k)V + e^{-kt}[b_2K_2 - g(u(t-\tau_2, x))] \\ \geqslant d_2\dfrac{\partial^2 V}{\partial x^2} - (b_2 + k)V + e^{-k\tau_2}g'(\eta)U(t-\tau_2, x), \quad t > 0, s_1(t) < x < s_2(t), \\ U(t,x) = K_1e^{-kt}, \quad t > 0, x \geqslant s_2(t) \text{ or } x \leqslant s_1(t), \\ V(t,x) = K_2e^{-kt}, \quad t > 0, x \geqslant s_2(t) \text{ or } x \leqslant s_1(t), \\ U(\theta, x) \geqslant 0, \quad -\tau_2 \leqslant \theta \leqslant 0, s_1(\theta) \leqslant x \leqslant s_2(\theta), \\ V(\theta, x) \geqslant 0, \quad -\tau_1 \leqslant \theta \leqslant 0, s_1(\theta) \leqslant x \leqslant s_2(\theta), \end{cases} \tag{2.4}$$

where ξ lies between K_2 and $v(t-\tau_1, x)$, η lies between K_1 and $u(t-\tau_2, x)$.

We claim that $U(t,x), V(t,x) \geqslant 0$ in $(0, +\infty) \times (s_1(t), s_2(t))$. Assume by contraction that there exist some T_0 and $(t_0, x_0) \in (0, T_0] \times (s_1(t), s_2(t))$ such that

$$\min\{U(t_0, x_0), V(t_0, x_0)\} = \min_{(t,x)\in[0,T_0]\times[s_1(t),s_2(t)]} \min\{U(t,x), V(t,x)\} < 0.$$

If $U(t_0, x_0) = \min\{U(t_0, x_0), V(t_0, x_0)\} < 0$, then $U(t_0, x_0)$ is the minimum of $U(t,x)$ in $[0, T_0] \times [s_1(t), s_2(t)]$. It follows that $\frac{\partial U}{\partial t}(t_0, x_0) \leqslant 0$ and $\frac{\partial^2 U}{\partial x^2}(t_0, x_0) \geqslant 0$. On the other hand,

$$\begin{aligned} -(b_1 + k)U(t_0, x_0) + e^{-k\tau_1}h'(\xi)V(t_0 - \tau_1, x_0) &\geqslant -(b_1 + k)U(t_0, x_0) + e^{-k\tau_1}h'(\xi)U(t_0, x_0) \\ &\geqslant (-b_1 - k + h'(\xi))U(t_0, x_0). \end{aligned}$$

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Choose

$$k = \max\left\{\|h'\|_{L^\infty([0,\max\{K_2,M_2\}])}, \|g'\|_{L^\infty([0,\max\{K_1,M_1\}])}\right\}$$

with

$$M_1 = \|u\|_{L^\infty([-\tau_2, T_0-\tau_2]\times[s_1(t),s_2(t)])},$$
$$M_2 = \|v\|_{L^\infty([-\tau_1, T_0-\tau_1]\times[s_1(t),s_2(t)])}.$$

Thus,

$$-(b_1+k)U(t_0,x_0) + e^{-k\tau_1}h'(\xi)V(t_0-\tau_1,x_0) > 0,$$

which contradicts with the first equation in (2.4). If $V(t_0,x_0) = \min\{U(t_0,x_0), V(t_0,x_0)\} < 0$, we can similarly prove the claim. This completes the proof of (*ii*).

(*iii*) Since u, v and $s_1'(t)$, $s_2'(t)$ are bounded in $(0,T_0]\times(s_1(t),s_2(t))$ by constants independent of T_0, the global solution is guaranteed. □

Next, we provide two comparison principles for the free boundary problem (1.4). The first one is used for comparing the solution $(u(t,x), v(t,x), s_1(t), s_2(t))$ with a upper solution $(\bar{u}(t,x), \bar{v}(t,x), \bar{s}_1(t), \bar{s}_2(t))$ in the spatial domain $(s_1(t), s_2(t))$, and the second one is used for comparing $(u(t,x), v(t,x), s_2(t))$ with $(\bar{u}(t,x), \bar{v}(t,x), \bar{s}_2(t))$ in one-side interval $(0, s_2(t))$. The proofs are similar as that of Lemma 2.5 in [1], here we omit the details. Moreover, we can also obtain the corresponding conclusions for lower solution by minor modification.

Lemma 2.1. *Suppose that* $T\in(0,\infty)$, $\bar{s}_1, \bar{s}_2 \in C([-\max\{\tau_1,\tau_2\}, T])\cap C^1((0,T])$, $\bar{u}_0(\theta,x)\in C^{1,2}([-\tau_2,0]\times[\bar{s}_1(\theta),\bar{s}_2(\theta)])$, $\bar{v}_0(\theta,x)\in C^{1,2}([-\tau_1,0]\times[\bar{s}_1(\theta),\bar{s}_2(\theta)])$, $\bar{u}(t,x), \bar{v}(t,x)\in C^{1,2}((0,T]\times(\bar{s}_1(t),\bar{s}_2(t)))$, *and*

$$\begin{cases}
\dfrac{\partial \bar{u}}{\partial t} \geqslant d_1\dfrac{\partial^2\bar{u}}{\partial x^2} - b_1\bar{u} + h(\bar{v}(t-\tau_1,x)), & 0<t\leqslant T, \bar{s}_1(t)<x<\bar{s}_2(t),\\
\dfrac{\partial \bar{v}}{\partial t} \geqslant d_2\dfrac{\partial^2\bar{v}}{\partial x^2} - b_2\bar{v} + g(\bar{u}(t-\tau_2,x)), & 0<t\leqslant T, \bar{s}_1(t)<x<\bar{s}_2(t),\\
\bar{u}(t,x)=\bar{v}(t,x)=0, & 0<t\leqslant T, x\geqslant\bar{s}_2(t) \text{ or } x\leqslant\bar{s}_1(t),\\
\bar{s}_2'(t)\geqslant -\mu\dfrac{\partial\bar{u}}{\partial x}(t,\bar{s}_2(t)), & 0<t\leqslant T,\\
\bar{s}_1'(t)\leqslant -\mu\dfrac{\partial\bar{u}}{\partial x}(t,\bar{s}_1(t)), & 0<t\leqslant T,\\
\bar{u}(\theta,x)=\bar{u}_0(\theta,x), & -\tau_2\leqslant\theta\leqslant 0, \bar{s}_1(\theta)\leqslant x\leqslant\bar{s}_2(\theta),\\
\bar{v}(\theta,x)=\bar{v}_0(\theta,x), & -\tau_1\leqslant\theta\leqslant 0, \bar{s}_1(\theta)\leqslant x\leqslant\bar{s}_2(\theta).
\end{cases}$$

If $\bar{u}_0(\theta,x)\geqslant u_0(\theta,x)$ *for* $(\theta,x)\in[-\tau_2,0]\times[s_1(\theta),s_2(\theta)]$, $\bar{v}_0(\theta,x)\geqslant v_0(\theta,x)$ *for* $(\theta,x)\in[-\tau_1,0]\times[s_1(\theta),s_2(\theta)]$ *and* $[\bar{s}_1(\theta),\bar{s}_2(\theta)]\supseteq[s_1(\theta),s_2(\theta)]$ *for* $\theta\in[-\max\{\tau_1,\tau_2\},0]$, *then we have* $\bar{s}_1(t)\geqslant s_1(t)$, $\bar{s}_2(t)\geqslant s_2(t)$ *for* $t\in(0,T]$ *and* $(\bar{u}(t,x),\bar{v}(t,x))\geqslant(u(t,x),v(t,x))$ *for* $(t,x)\in(0,T]\times(s_1(t),s_2(t))$.

Lemma 2.2. *Suppose that* $T\in(0,\infty)$, $\bar{s}_2\in C([-\max\{\tau_1,\tau_2\},T])\cap C^1((0,T])$, $\bar{u}_0(\theta,x)\in C^{1,2}([-\tau_2,0]\times[0,\bar{s}_2(\theta)])$, $\bar{v}_0(\theta,x)\in C^{1,2}([-\tau_1,0]\times[0,\bar{s}_2(\theta)])$, $\bar{u}(t,x),\bar{v}(t,x)\in C^{1,2}((0,T]\times(0,\bar{s}_2(t)))$, *and*

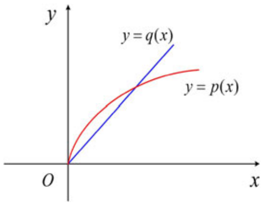

Figure 1. *Two curves $y = p(x)$ and $y = q(x)$.*

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} \geqslant d_1 \dfrac{\partial^2 \bar{u}}{\partial x^2} - b_1 \bar{u} + h(\bar{v}(t-\tau_1, x)), & 0 < t \leqslant T, 0 < x < \bar{s}_2(t), \\ \dfrac{\partial \bar{v}}{\partial t} \geqslant d_2 \dfrac{\partial^2 \bar{v}}{\partial x^2} - b_2 \bar{v} + g(\bar{u}(t-\tau_2, x)), & 0 < t \leqslant T, 0 < x < \bar{s}_2(t), \\ \bar{u}(t,x) = \bar{v}(t,x) = 0, \quad 0 < t \leqslant T, x \geqslant \bar{s}_2(t), \\ \bar{u}(t,0) \geqslant u(t,0), \bar{v}(t,0) \geqslant v(t,0), \quad 0 < t \leqslant T, \\ \bar{s}_2'(t) \geqslant -\mu \dfrac{\partial \bar{u}}{\partial x}(t, \bar{s}_2(t)), \quad 0 < t \leqslant T, \\ \bar{u}(\theta, x) = \bar{u}_0(\theta, x), \quad -\tau_2 \leqslant \theta \leqslant 0, 0 \leqslant x \leqslant \bar{s}_2(\theta), \\ \bar{v}(\theta, x) = \bar{v}_0(\theta, x), \quad -\tau_1 \leqslant \theta \leqslant 0, 0 \leqslant x \leqslant \bar{s}_2(\theta). \end{cases}$$

If $\bar{u}_0(\theta, x) \geqslant u_0(\theta, x)$ for $(\theta, x) \in [-\tau_2, 0] \times [0, s_2(\theta)]$, $\bar{v}_0(\theta, x) \geqslant v_0(\theta, x)$ for $(\theta, x) \in [-\tau_1, 0] \times [0, s_2(\theta)]$ and $[0, \bar{s}_2(\theta)] \supseteq [0, s_2(\theta)]$ for $\theta \in [-\max\{\tau_1, \tau_2\}, 0]$, then we have $\bar{s}_2(t) \geqslant s_2(t)$ for $t \in (0, T]$ and $(\bar{u}(t,x), \bar{v}(t,x)) \geqslant (u(t,x), v(t,x))$ for $(t,x) \in (0, T] \times (0, s_2(t))$.

2.2 Spreading and vanishing

In this subsection, we investigate the long-time behaviour of solution.

Let

$$\mathcal{R}_0 := \sqrt{\frac{h'(0)g'(0)}{b_1 b_2}}.$$

We claim that (1.3) admits a unique positive equilibrium for $\mathcal{R}_0 > 1$ and has no positive equilibrium for $0 < \mathcal{R}_0 \leqslant 1$. Indeed, let

$$p(x) = g\left(\frac{h(x)}{b_1}\right) \quad \text{and} \quad q(x) = b_2 x,$$

we conclude that the curves $y = p(x)$ and $y = q(x)$ have at most one intersection point in the interior of the first quadrant $\mathbb{R}^2_+$. Otherwise, the mean value theorem yields that there exists a $\xi > 0$ such that

$$p''(\xi) = g''\left(\frac{h(\xi)}{b_1}\right)\left(\frac{h'(\xi)}{b_1}\right)^2 + g'\left(\frac{h(\xi)}{b_1}\right)\frac{h''(\xi)}{b_1} = 0,$$

which contradicts with the condition (M). When $\mathcal{R}_0 > 1$, we have $p'(0) = \frac{h'(0)g'(0)}{b_1} > b_2 = q'(0)$. Then, $p(x) > q(x)$ for small $x > 0$ due to $p(0) = q(0) = 0$. Moreover, since $g \in L^\infty$, we know that $p(x) < q(x)$ for large $x > 0$. Thus, as showed in Figure 1 the curves $y = p(x)$ and $y = q(x)$ have exactly one intersection point $(x^*, y^*) > (0,0)$ for $\mathcal{R}_0 > 1$, which implies that $(\frac{h(x^*)}{b_1}, x^*)$ is the unique positive equilibrium of (1.3). For $0 < \mathcal{R}_0 \leqslant 1$, we have $p'(0) \leqslant q'(0)$, then the claim can be easily proved by the concavity of $p(x)$.

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Theorem 2.2. *If* $0<\mathcal{R}_0\leqslant 1$, *then the solution* (u,v,s_1,s_2) *of* (1.4) *satisfies*

$$\lim_{t\to+\infty}(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])}+\|v(t,\cdot)\|_{C([s_1(t),s_2(t)])})=0.$$

Proof. Let $(w_1(t),w_2(t))$ be the unique solution of

$$\begin{cases} w_1'=-b_1w_1+h(w_2(t-\tau_1)), & t>0,\\ w_2'=-b_2w_2+g(w_1(t-\tau_2)), & t>0,\\ w_1(\theta)=\|u_0\|_{C([-\tau_2,0]\times[s_1(\theta),s_2(\theta)])}, & \theta\in[-\tau_2,0],\\ w_2(\theta)=\|v_0\|_{C([-\tau_1,0]\times[s_1(\theta),s_2(\theta)])}, & \theta\in[-\tau_1,0]. \end{cases} \tag{2.5}$$

From the comparison principle, we know that $(u(t,x),v(t,x))\leqslant(w_1(t),w_2(t))$ in $[0,+\infty)\times[s_1(t),s_2(t)]$.

Denote $\bar{\mathcal{C}}:=C([-\tau_2,0],\mathbb{R})\times C([-\tau_1,0],\mathbb{R})$, then $\bar{\mathcal{C}}$ is a Banach space with the usual norm. For any given $w=(w_1,w_2)\in\bar{\mathcal{C}}$ defined on $[-\tau_2,\sigma)\times[-\tau_1,\sigma)$ with $\sigma>0$, we define $w_t:=(w_t^1,w_t^2)\in\bar{\mathcal{C}}$ for $0\leqslant t<\sigma$, where $w_t^1(\theta)=w_1(t+\theta)$ for $\theta\in[-\tau_2,0]$ and $w_t^2(\theta)=w_2(t+\theta)$ for $\theta\in[-\tau_1,0]$. Let

$$\bar{\mathcal{C}}_+=\Big\{(\varphi_1,\varphi_2)\in\bar{\mathcal{C}};\varphi_1(\theta)\geqslant 0 \text{ on } [-\tau_2,0],\varphi_2(\theta)\geqslant 0 \text{ on } [-\tau_1,0]\Big\}$$

and define $f:\bar{\mathcal{C}}_+\longrightarrow\mathbb{R}^2$ by

$$\begin{aligned} f(\varphi_1,\varphi_2) &=(f_1(\varphi_1,\varphi_2),f_2(\varphi_1,\varphi_2))\\ &=\Big(-b_1\varphi_1(0)+h(\varphi_2(-\tau_1)),-b_2\varphi_2(0)+g(\varphi_1(-\tau_2))\Big), \end{aligned}$$

then the equations in (2.5) can be rewritten as

$$w'(t)=f(w_t). \tag{2.6}$$

For any $(y_1,y_2)\in\mathbb{R}^2$, we write $(\hat{y}_1,\hat{y}_2)$ for the element of $\bar{\mathcal{C}}$ satisfying $\hat{y}_1(\theta)\equiv y_1$ for $\theta\in[-\tau_2,0]$ and $\hat{y}_2(\theta)\equiv y_2$ for $\theta\in[-\tau_1,0]$, and define $\hat{f}:\mathbb{R}^2_+\to\mathbb{R}^2$ by $\hat{f}(y_1,y_2)=f(\hat{y}_1,\hat{y}_2)$. Since $f(\hat{0},\hat{0})=(0,0)$ and $df(\hat{0},\hat{0})(\varphi_1,\varphi_2)=(-b_1\varphi_1(0)+h'(0)\varphi_2(-\tau_1),-b_2\varphi_2(0)+g'(0)\varphi_1(-\tau_2))$ for any $(\varphi_1,\varphi_2)\in\bar{\mathcal{C}}$, we can check that $df(\hat{0},\hat{0})$ satisfies the condition (R) in [42].

Note that

$$\hat{f}(y_1,y_2)=f(\hat{y}_1,\hat{y}_2)=\Big(-b_1y_1+h(y_2),-b_2y_2+g(y_1)\Big)$$

and the Fréchet derivative is

$$D\hat{f}(y_1,y_2)=\begin{pmatrix}-b_1 & h'(y_2)\\ g'(y_1) & -b_2\end{pmatrix},\quad D\hat{f}(0,0)=\begin{pmatrix}-b_1 & h'(0)\\ g'(0) & -b_2\end{pmatrix}. \tag{2.7}$$

From $0<\mathcal{R}_0\leqslant 1$, we know that the stability modulus

$$s(D\hat{f}(0,0))=\max\Big\{\operatorname{Re}\lambda;\ \det(\lambda I-D\hat{f}(0,0))=0\Big\}\leqslant 0.$$

Moreover, since $h''(z)\leqslant 0$, $g''(z)<0$ for all $z>0$, we have

$$\frac{h(\lambda y_2)}{\lambda y_2}=\frac{h(\lambda y_2)-h(0)}{\lambda y_2}\geqslant\frac{h(y_2)-h(0)}{y_2}=\frac{h(y_2)}{y_2}$$

and

$$\frac{g(\lambda y_1)}{\lambda y_1}=\frac{g(\lambda y_1)-g(0)}{\lambda y_1}>\frac{g(y_1)-g(0)}{y_1}=\frac{g(y_1)}{y_1}$$

for $y_1,y_2>0$ and $\lambda\in(0,1)$. It follows that $\hat{f}:\mathbb{R}^2_+\to\mathbb{R}^2$ is strictly subhomogeneous. Similarly, we can check that $f:\bar{\mathcal{C}}_+\to\mathbb{R}^2$ is subhomogeneous. From Theorem 3.2 in [42], we know that $(0,0)$ is

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globally asymptotically stable for (2.6) with respect to $\bar{\mathcal{C}}_+$. That is, the solution (w_1, w_2) of (2.5) satisfies $\lim_{t\to+\infty}(w_1(t), w_2(t)) = (0, 0)$, which implies $\lim_{t\to+\infty}(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])}) = 0$. This completes the proof. □

Due to Theorem 2.1, there exist $s_{1,\infty}, s_{2,\infty} \in (0, +\infty]$ such that $\lim_{t\to+\infty} s_1(t) = s_{1,\infty}$ and $\lim_{t\to+\infty} s_2(t) = s_{2,\infty}$. We call that the bacteria are *spreading* if $s_{2,\infty} - s_{1,\infty} = +\infty$ and $\limsup_{t\to+\infty}(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])}) > 0$; the bacteria are *vanishing* if $s_{2,\infty} - s_{1,\infty} < +\infty$ and $\lim_{t\to+\infty}(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])}) = 0$.

Next, we discuss the spreading and vanishing of bacteria for $\mathcal{R}_0 > 1$. For epidemic models governed by differential equations, the threshold dynamics of bacteria or virus are usually established in terms of the basic reproduction number $\mathcal{R}_0$. However, for (1.4), the infected area is changing with time t, and therefore, the basic reproduction number is not a constant and should be a function of t. As in [24], we introduce the spatial-temporal risk index, which is expressed by

$$\mathcal{R}_0^F(s_1(t), s_2(t)) = \sqrt{\frac{h'(0)g'(0)}{[d_1(\frac{\pi}{s_2(t)-s_1(t)})^2 + b_1][d_2(\frac{\pi}{s_2(t)-s_1(t)})^2 + b_2]}}.$$

According to Lemma 4.1 in [24], there exist λ_1 and $\kappa > 0$ such that $\operatorname{sign}(1 - \mathcal{R}_0^F(s_1(t), s_2(t))) = \operatorname{sign}\lambda_1$ and $(\phi, \psi) := (\kappa\psi^*, \psi^*)$ solves the following problem

$$\begin{cases} -d_1\phi_{xx} = h'(0)\psi - b_1\phi + \lambda_1\phi, & x \in (s_1(t), s_2(t)), \\ -d_2\psi_{xx} = g'(0)\phi - b_2\psi + \lambda_1\psi, & x \in (s_1(t), s_2(t)), \\ (\phi(x), \psi(x)) = (0, 0), & x = s_1(t) \text{ or } x = s_2(t), \end{cases} \tag{2.8}$$

where ψ^* is the principal eigenfunction of $-\Delta$ in $(s_1(t), s_2(t))$ with null Dirichlet boundary condition. It is easy to check that $\mathcal{R}_0^F(s_1(t), s_2(t))$ is increasing in t and satisfies $\mathcal{R}_0^F(s_1(t), s_2(t)) \to \mathcal{R}_0$ as $s_2(t) - s_1(t) \to +\infty$.

Theorem 2.3. *(i) If $s_{2,\infty} - s_{1,\infty} < +\infty$, then $s_0 < s_{2,\infty}, -s_{1,\infty} < +\infty$.*

(ii) If $s_{2,\infty} - s_{1,\infty} = +\infty$, then $s_{2,\infty} = -s_{1,\infty} = +\infty$.

(iii) If $s_{2,\infty} - s_{1,\infty} < +\infty$, then $\lim_{t\to+\infty}(\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} + \|v(t,\cdot)\|_{C([s_1(t),s_2(t)])}) = 0$.

(iv) Assume that $\mathcal{R}_0 > 1$. If $s_{2,\infty} - s_{1,\infty} = +\infty$, then $\lim_{t\to+\infty}(u(t,x), v(t,x)) = (u^*, v^*)$ locally uniformly for $x \in \mathbb{R}$, where (u^*, v^*) is the unique positive equilibrium of (1.3).

Proof. Similar as the proof of Lemma 3.1 in [1], we can show that $-2s_0 < s_1(t) + s_2(t) < 2s_0$ for $t \geqslant 0$, which implies (i) and (ii).

For (iii), we only prove $\lim_{t\to+\infty}\|u(t,\cdot)\|_{C([s_1(t),s_2(t)])} = 0$, since the result for v can be obtained similarly. Let $w(t,y)$ and $z(t,y)$ be the functions transformed from $u(t,x)$ and $v(t,x)$ by (2.1). Then, it is sufficient to show $\lim_{t\to+\infty}\|w(t,\cdot)\|_{C([-s_0,s_0])} = 0$. We assume by contradiction that

$$\limsup_{t\to+\infty}\|w(t,\cdot)\|_{C([-s_0,s_0])} = \delta > 0.$$

It follows that there exists a sequence $\{(t_k, y_k)\}_{k=1}^{\infty}$ in $(0, +\infty) \times (-s_0, s_0)$ such that $t_k \to +\infty$ as $k \to \infty$ and $w(t_k, y_k) \geqslant \frac{\delta}{2}$ for all $k \in \mathbb{N}$. Since $\{y_k\}_{k=1}^{\infty}$ is bounded, we may assume that $y_k \to y_0$ as $k \to \infty$. Similar as the proof of Lemma 3.2 in [1], we can obtain $y_0 \neq \pm s_0$.

Denote $w_k(t,y) = w(t + t_k, y)$ and $z_k(t,y) = z(t + t_k, y)$ for any $(t,y) \in (t_0 - t_k, +\infty) \times [-s_0, s_0]$. Note that $s_1'(t), s_2'(t) \to 0$ as $t \to +\infty$. It follows from Theorem 2.1 and the parabolic regularity theory that $\{(w_k, z_k)\}$ has a subsequence, denoted by itself, such that $(w_k, z_k) \to (\tilde{w}, \tilde{z})$ in $C_{loc}^{1,2}(\mathbb{R} \times [-s_0, s_0])$ as $k \to \infty$ and $(\tilde{w}, \tilde{z})$ satisfies

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$$\begin{cases} \dfrac{\partial \tilde{w}}{\partial t} = d_1 \tilde{A} \dfrac{\partial^2 \tilde{w}}{\partial y^2} - b_1 \tilde{w} + h(\tilde{z}(t-\tau_1, y)), & -\infty < t < +\infty, -s_0 < y < s_0, \\ \dfrac{\partial \tilde{z}}{\partial t} = d_2 \tilde{A} \dfrac{\partial^2 \tilde{z}}{\partial y^2} - b_2 \tilde{z} + g(\tilde{w}(t-\tau_2, y)), & -\infty < t < +\infty, -s_0 < y < s_0, \\ \tilde{w}(t,y) = \tilde{z}(t,y) = 0, & -\infty < t < +\infty, y \geqslant s_0 \text{ or } y \leqslant -s_0, \\ \tilde{w}(t,y) \geqslant 0, \tilde{z}(t,y) \geqslant 0, & -\infty < t < +\infty, -s_0 < y < s_0, \end{cases} \tag{2.9}$$

where $\tilde{A} = \frac{4s_0^2}{(s_{2,\infty}-s_{1,\infty})^2}$. Since $w_k(0, y_k) = w(t_k, y_k) \geqslant \frac{\delta}{2}$, we have $\tilde{w}(0, y_0) \geqslant \frac{\delta}{2}$. By the strong maximum principle, we can deduce that $\tilde{w}(t, y) > 0$ for $(t, y) \in \mathbb{R} \times (-s_0, s_0)$. Applying the Hopf boundary lemma, we have $\frac{\partial \tilde{w}}{\partial y}(t, s_0) < 0$.

On the other hand, by the Stefan condition $s_2'(t_k) = -\mu \frac{\partial u}{\partial x}(t_k, s_2(t_k))$ and the fact that $s_2'(t) \to 0$ as $t \to +\infty$, we have $\frac{\partial u}{\partial x}(t_k, s_2(t_k)) \to 0$ as $k \to \infty$. Note that $\frac{\partial u}{\partial x}(t_k, s_2(t_k)) = \sqrt{A} \frac{\partial w_k}{\partial y}(0, s_0) \to \sqrt{\tilde{A}} \frac{\partial \tilde{w}}{\partial y}(0, s_0)$. It follows that $\frac{\partial \tilde{w}}{\partial y}(0, s_0) = 0$, which is a contradiction. Thus, (iii) holds true.

Next, we prove (iv). Since $\mathcal{R}_0 > 1$, we have

$$s(D\hat{f}(0,0)) = \max\{\text{Re } \lambda; \det(\lambda I - D\hat{f}(0,0)) = 0\} > 0,$$

where $D\hat{f}(0, 0)$ is defined in (2.7). By Theorem 3.2 in [42], we can deduce that the solution of (2.5) satisfies

$$\lim_{t \to +\infty} (w_1(t), w_2(t)) = (u^*, v^*). \tag{2.10}$$

Moreover, by the comparison principle, we know that

$$\begin{aligned} u(t,x) &\leqslant w_1(t) \quad \text{for } (t,x) \in [-\tau_2, +\infty) \times [s_1(t), s_2(t)], \\ v(t,x) &\leqslant w_2(t) \quad \text{for } (t,x) \in [-\tau_1, +\infty) \times [s_1(t), s_2(t)]. \end{aligned} \tag{2.11}$$

Hence,

$$\limsup_{t \to +\infty} (u(t,x), v(t,x)) \leqslant (u^*, v^*)$$

uniformly for $x \in [s_1(t), s_2(t)]$.

Next, we prove

$$\liminf_{t \to +\infty} (u(t,x), v(t,x)) \geqslant (u^*, v^*)$$

uniformly in any compact subset of $\mathbb{R}$.

Since $\mathcal{R}_0 > 1$, we can choose a sufficiently large $L_0 > 0$ such that $\mathcal{R}_0^F(-L_0, L_0) > 1$. Then, there exist $\lambda_1 < 0$ and $\kappa > 0$ such that $(\phi, \psi) := (\kappa \psi^*, \psi^*)$ solves the problem (2.8) with $(s_1(t), s_2(t))$ replaced by $(-L_0, L_0)$, where ψ^* is the principal eigenfunction of $-\Delta$ in $(-L_0, L_0)$ with null Dirichlet boundary condition. Due to $s_{2,\infty} - s_{1,\infty} = +\infty$, we have $s_{2,\infty} = -s_{1,\infty} = +\infty$ by (ii). Thus, for any $L \geqslant L_0$, there exists $t_L > 0$ such that $s_2(t) \geqslant L$ and $s_1(t) \leqslant -L$ for all $t > t_L$.

Let $\underline{U}(t, x) = \delta \kappa \psi^*(x)$ and $\underline{V}(t, x) = \delta \psi^*(x)$. We can choose a sufficiently small $\delta > 0$ such that $(\underline{U}, \underline{V})$ satisfies

$$\begin{cases} \dfrac{\partial \underline{U}}{\partial t} \leqslant d_1 \dfrac{\partial^2 \underline{U}}{\partial x^2} - b_1 \underline{U} + h(\underline{V}(t-\tau_1, x)), & t > t_{L_0}, -L_0 < x < L_0, \\ \dfrac{\partial \underline{V}}{\partial t} \leqslant d_2 \dfrac{\partial^2 \underline{V}}{\partial x^2} - b_2 \underline{V} + g(\underline{U}(t-\tau_2, x)), & t > t_{L_0}, -L_0 < x < L_0, \\ \underline{U}(t,x) = \underline{V}(t,x) = 0, & t > t_{L_0}, x = \pm L_0, \\ \underline{U}(\theta, x) \leqslant u(\theta, x), & \theta \in [t_{L_0} - \tau_2, t_{L_0}], x \in [-L_0, L_0], \\ \underline{V}(\theta, x) \leqslant v(\theta, x), & \theta \in [t_{L_0} - \tau_1, t_{L_0}], x \in [-L_0, L_0]. \end{cases}$$

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For any $L \geqslant L_0$, we extend $(\underline{U}, \underline{V})$ by defining $(\underline{U}(t,x), \underline{V}(t,x)) = (0,0)$ for $(t,x) \in \mathbb{R} \times ([-L, L] \setminus [-L_0, L_0])$. Consider the following problem

$$\begin{cases} \dfrac{\partial \underline{u}}{\partial t} = d_1 \dfrac{\partial^2 \underline{u}}{\partial x^2} - b_1 \underline{u} + h(\underline{v}(t-\tau_1, x)), & t > t_L, -L < x < L, \\ \dfrac{\partial \underline{v}}{\partial t} = d_2 \dfrac{\partial^2 \underline{v}}{\partial x^2} - b_2 \underline{v} + g(\underline{u}(t-\tau_2, x)), & t > t_L, -L < x < L, \\ \underline{u}(t,x) = \underline{v}(t,x) = 0, \quad t > t_L, x = \pm L, \\ \underline{u}(\theta, x) = \underline{U}(\theta, x) = \delta\kappa\psi^*(x), \quad \theta \in [t_L - \tau_2, t_L], x \in [-L, L], \\ \underline{v}(\theta, x) = \underline{V}(\theta, x) = \delta\psi^*(x), \quad \theta \in [t_L - \tau_1, t_L], x \in [-L, L]. \end{cases} \tag{2.12}$$

By the comparison principle, we have

$$\underline{u}(t,x) \leqslant u(t,x) \quad \text{for } (t,x) \in [t_L - \tau_2, +\infty) \times [-L, L],$$
$$\underline{v}(t,x) \leqslant v(t,x) \quad \text{for } (t,x) \in [t_L - \tau_1, +\infty) \times [-L, L].$$

Moreover, since $(\underline{U}(t,x), \underline{V}(t,x))$ is a lower solution of (2.12), the solution of (2.12) is increasing in t for $x \in [-L, L]$. Thus, $(\underline{u}(t,x), \underline{v}(t,x)) \to (\underline{u}^L, \underline{v}^L)$ in $C^2([-L, L])$ as $t \to +\infty$, where $(\underline{u}^L, \underline{v}^L)$ solves

$$\begin{cases} -d_1 \underline{u}^L_{xx} = -b_1 \underline{u}^L + h(\underline{v}^L(x)), & -L < x < L, \\ -d_2 \underline{v}^L_{xx} = -b_2 \underline{v}^L + g(\underline{u}^L(x)), & -L < x < L, \\ \underline{u}^L(\pm L) = \underline{v}^L(\pm L) = 0. \end{cases}$$

Note that $(\underline{u}^L, \underline{v}^L)$ is increasing in L. By classical elliptic regularity theory and a diagonal procedure, we obtain that as $L \to +\infty$, $(\underline{u}^L, \underline{v}^L)$ converges to $(\underline{u}^\infty, \underline{v}^\infty)$ in $[C^2_{loc}(\mathbb{R})]^2$, and $(\underline{u}^\infty, \underline{v}^\infty)$ solves

$$\begin{cases} -d_1 \underline{u}^\infty_{xx} = -b_1 \underline{u}^\infty + h(\underline{v}^\infty(x)), & -\infty < x < +\infty, \\ -d_2 \underline{v}^\infty_{xx} = -b_2 \underline{v}^\infty + g(\underline{u}^\infty(x)), & -\infty < x < +\infty. \end{cases} \tag{2.13}$$

Similarly as the proof of Theorem 4.5 in [1] and Lemma 3.5 in [34], we can prove $(\underline{u}^\infty, \underline{v}^\infty) = (u^*, v^*)$ and then get the desired result. □

Remark 2.1. From Theorem 2.3 (*iii*) and (*iv*), we know that the spreading-vanishing dichotomy holds for $\mathcal{R}_0 > 1$, that is, either the bacteria are spreading ($s_{2,\infty} - s_{1,\infty} = +\infty$) or vanishing ($s_{2,\infty} - s_{1,\infty} < +\infty$).

Next, we exhibit some sufficient conditions to determine whether the bacteria are vanishing or spreading.

Theorem 2.4. *If $\mathcal{R}_0^F(-s_0, s_0) < 1$ and $\mu > 0$ is sufficiently small, then $s_{2,\infty} - s_{1,\infty} < +\infty$.*

Proof. Since $\mathcal{R}_0^F(-s_0, s_0) < 1$, we know that there exist $\lambda_1 > 0$ and $\kappa > 0$ such that $(\phi, \psi) := (\kappa\psi^*, \psi^*)$ satisfies the problem (2.8) with $(s_1(t), s_2(t))$ replaced by $(-s_0, s_0)$, where ψ^* is the principal eigenfunction of $-\Delta$ in $(-s_0, s_0)$ with null Dirichlet boundary condition. Moreover, we have $\psi^*(x) > 0$ in $(-s_0, s_0)$, $(\psi^*)'(x) < 0$ in $(0, s_0]$ and $(\psi^*)'(x) > 0$ in $[-s_0, 0)$. Since $\lambda_1 > 0$, there exists a small $\delta > 0$ such that

$$-\delta\kappa + \frac{1}{(1+\delta)^2}(h'(0) - b_1\kappa + \lambda_1\kappa) + b_1\kappa - h'(0)e^{\delta\tau_1} \geqslant 0 \tag{2.14}$$

and

$$-\delta + \frac{1}{(1+\delta)^2}(\kappa g'(0) - b_2 + \lambda_1) + b_2 - \kappa g'(0)e^{\delta\tau_2} \geqslant 0. \tag{2.15}$$

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Define

$$\sigma(t)=\begin{cases} s_0\left(1+\delta-\frac{\delta}{2}e^{-\delta t}\right), & t\in[0,+\infty),\\ s_0\left(1+\frac{\delta}{2}\right), & t\in[-\max\{\tau_1,\tau_2\},0],\end{cases}$$

$$\bar{u}(t,x)=\begin{cases} \kappa Ce^{-\delta t}\psi^*\left(\frac{xs_0}{\sigma(t)}\right), & (t,x)\in[0,+\infty)\times[-\sigma(t),\sigma(t)],\\ \kappa C\psi^*\left(\frac{2x}{2+\delta}\right), & (t,x)\in[-\tau_2,0]\times[-\sigma(t),\sigma(t)],\end{cases}$$

$$\bar{v}(t,x)=\begin{cases} Ce^{-\delta t}\psi^*\left(\frac{xs_0}{\sigma(t)}\right), & (t,x)\in[0,+\infty)\times[-\sigma(t),\sigma(t)],\\ C\psi^*\left(\frac{2x}{2+\delta}\right), & (t,x)\in[-\tau_1,0]\times[-\sigma(t),\sigma(t)].\end{cases}$$

For $(t,x)\in(0,+\infty)\times(-\sigma(t),\sigma(t))$, we can deduce

$$\begin{aligned}
&\frac{\partial\bar{u}}{\partial t}-d_1\frac{\partial^2\bar{u}}{\partial x^2}+b_1\bar{u}-h(\bar{v}(t-\tau_1,x))\\
&=-\delta\bar{u}-\kappa Ce^{-\delta t}\frac{xs_0\sigma'(t)}{\sigma^2(t)}(\psi^*)'\left(\frac{xs_0}{\sigma(t)}\right)-d_1\kappa Ce^{-\delta t}\frac{s_0^2}{\sigma^2(t)}(\psi^*)''\left(\frac{xs_0}{\sigma(t)}\right)+b_1\bar{u}-h(\bar{v}(t-\tau_1,x))\\
&\geqslant-\delta\bar{u}-\kappa Ce^{-\delta t}\frac{xs_0\sigma'(t)}{\sigma^2(t)}(\psi^*)'\left(\frac{xs_0}{\sigma(t)}\right)+Ce^{-\delta t}\frac{s_0^2}{\sigma^2(t)}(h'(0)-b_1\kappa+\lambda_1\kappa)\psi^*\left(\frac{xs_0}{\sigma(t)}\right)+b_1\bar{u}\\
&\quad-h'(0)\bar{v}(t-\tau_1,x)\\
&=-\kappa Ce^{-\delta t}\frac{xs_0\sigma'(t)}{\sigma^2(t)}(\psi^*)'\left(\frac{xs_0}{\sigma(t)}\right)\\
&\quad+\left[-\delta\kappa+\frac{s_0^2}{\sigma^2(t)}(h'(0)-b_1\kappa+\lambda_1\kappa)+b_1\kappa-h'(0)\frac{\bar{v}(t-\tau_1,x)}{\bar{v}(t,x)}\right]\bar{v}(t,x).
\end{aligned}$$

Case 1. If $(t,x)\in[\tau_1,+\infty)\times(-\sigma(t),\sigma(t))$, then

$$\frac{\bar{v}(t-\tau_1,x)}{\bar{v}(t,x)}=e^{\delta\tau_1}\frac{\psi^*\left(\frac{xs_0}{\sigma(t-\tau_1)}\right)}{\psi^*\left(\frac{xs_0}{\sigma(t)}\right)}.$$

When $x\in[0,\sigma(t))$, we have $\frac{xs_0}{\sigma(t-\tau_1)}>\frac{xs_0}{\sigma(t)}>0$. Since $(\psi^*)'(x)<0$ in $(0,s_0]$, we have

$$\frac{\psi^*\left(\frac{xs_0}{\sigma(t-\tau_1)}\right)}{\psi^*\left(\frac{xs_0}{\sigma(t)}\right)}\leqslant 1,\quad -\kappa Ce^{-\delta t}\frac{xs_0\sigma'(t)}{\sigma^2(t)}(\psi^*)'\left(\frac{xs_0}{\sigma(t)}\right)\geqslant 0.$$

When $x\in(-\sigma(t),0]$, we can get the same inequalities by the fact $(\psi^*)'(x)>0$. Consequently,

$$\begin{aligned}
&\frac{\partial\bar{u}}{\partial t}-d_1\frac{\partial^2\bar{u}}{\partial x^2}+b_1\bar{u}-h(\bar{v}(t-\tau_1,x))\\
&\geqslant\left[-\delta\kappa+\frac{s_0^2}{\sigma^2(t)}(h'(0)-b_1\kappa+\lambda_1\kappa)+b_1\kappa-h'(0)e^{\delta\tau_1}\right]\bar{v}(t,x)\\
&\geqslant\left[-\delta\kappa+\frac{1}{(1+\delta)^2}(h'(0)-b_1\kappa+\lambda_1\kappa)+b_1\kappa-h'(0)e^{\delta\tau_1}\right]\bar{v}(t,x)\\
&\geqslant 0,
\end{aligned}\tag{2.16}$$

where the last inequality uses (2.14).

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Case 2. If $(t,x)\in(0,\tau_1)\times(-\sigma(t),\sigma(t))$, by using similar arguments as above, we can deduce

$$\frac{\bar v(t-\tau_1,x)}{\bar v(t,x)}=e^{\delta\tau_1}\frac{\psi^*\left(\frac{2x}{2+\delta}\right)}{\psi^*\left(\frac{xs_0}{\sigma(t)}\right)}\leqslant e^{\delta\tau_1},\quad -\kappa Ce^{-\delta t}\frac{xs_0\sigma'(t)}{\sigma^2(t)}(\psi^*)'\left(\frac{xs_0}{\sigma(t)}\right)\geqslant 0,$$

which implies that (2.16) also holds.

Similarly, by (2.15), we have

$$\frac{\partial\bar v}{\partial t}-d_2\frac{\partial^2\bar v}{\partial x^2}+b_2\bar v-g(\bar u(t-\tau_2,x))\geqslant 0\quad\text{for }(t,x)\in(0,+\infty)\times(-\sigma(t),\sigma(t)).$$

Besides, choose C large enough such that

$$\bar u_0(\theta,x)=\kappa C\psi^*\left(\frac{2x}{2+\delta}\right)\geqslant\kappa C\psi^*\left(\frac{2s_0}{2+\delta}\right)\geqslant\|u_0\|_{L^\infty([-\tau_2,0]\times[s_1(\theta),s_2(\theta)])}\geqslant u_0(\theta,x)$$

for $(\theta,x)\in[-\tau_2,0]\times[s_1(\theta),s_2(\theta)]$ and

$$\bar v_0(\theta,x)=C\psi^*\left(\frac{2x}{2+\delta}\right)\geqslant C\psi^*\left(\frac{2s_0}{2+\delta}\right)\geqslant\|v_0\|_{L^\infty([-\tau_1,0]\times[s_1(\theta),s_2(\theta)])}\geqslant v_0(\theta,x)$$

for $(\theta,x)\in[-\tau_1,0]\times[s_1(\theta),s_2(\theta)]$. Then, take $\mu>0$ sufficiently small such that

$$\sigma'(t)=s_0\frac{\delta^2}{2}e^{-\delta t}\geqslant-\mu\bar u_x(t,\sigma(t)),-\sigma'(t)=-s_0\frac{\delta^2}{2}e^{-\delta t}\leqslant-\mu\bar u_x(t,-\sigma(t))\quad\text{for }t>0.$$

Since

$$[-\sigma(\theta),\sigma(\theta)]=\left[-s_0(1+\frac{\delta}{2}),s_0(1+\frac{\delta}{2})\right]\supseteq[-s_0,s_0]\supseteq[s_1(\theta),s_2(\theta)]$$

for $\theta\in[-\max\{\tau_1,\tau_2\},0]$, we can apply Lemma 2.1 to conclude that $s_1(t)\geqslant-\sigma(t)$ and $s_2(t)\leqslant\sigma(t)$ for $t>0$. Thus, $s_{2,\infty}-s_{1,\infty}\leqslant s_0(2+2\delta)$, which completes the proof. □

Theorem 2.5. *If $\mathcal{R}_0^F(-s_0,s_0)\geqslant 1$, then $s_{2,\infty}-s_{1,\infty}=+\infty$.*

Proof. Since $\mathcal{R}_0^F(s_1(t_0),s_2(t_0))>\mathcal{R}_0^F(-s_0,s_0)\geqslant 1$ holds for any $t_0>0$, we can choose t_0 as initial time when $\mathcal{R}_0^F(-s_0,s_0)=1$, so it is sufficient to consider the case $\mathcal{R}_0^F(-s_0,s_0)>1$. In such case, we know that there exist $\lambda_1<0$ and $\kappa>0$ such that $(\phi,\psi):=(\kappa\psi^*,\psi^*)$ satisfies the problem (2.8) with $(s_1(t),s_2(t))$ replaced by $(-s_0,s_0)$, where ψ^* is the principal eigenfunction of $-\Delta$ in $(-s_0,s_0)$ with null Dirichlet boundary condition.

Let $\varepsilon>0$ be a sufficiently small constant and define

$$\underline u(t,x)=\varepsilon\kappa\psi^*(x),\underline v(t,x)=\varepsilon\psi^*(x)\quad\text{for }(t,x)\in[\max\{\tau_1,\tau_2\},+\infty)\times[-s_0,s_0].$$

By direct calculations, we have

$$\begin{aligned}\frac{\partial\underline u}{\partial t}-d_1\frac{\partial^2\underline u}{\partial x^2}-h(\underline v(t-\tau_1,x))+b_1\underline u&=\varepsilon\psi^*(x)(h'(0)+\lambda_1\kappa)-h(\varepsilon\psi^*(x))\\&=\varepsilon\psi^*(x)(h'(0)+\lambda_1\kappa)-\varepsilon\psi^*(x)h'(\xi_1(x))\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\underline v}{\partial t}-d_2\frac{\partial^2\underline v}{\partial x^2}-g(\underline u(t-\tau_2,x))+b_2\underline v&=\varepsilon\psi^*(x)(g'(0)\kappa+\lambda_1)-g(\varepsilon\kappa\psi^*(x))\\&=\varepsilon\psi^*(x)(g'(0)\kappa+\lambda_1)-\varepsilon\kappa\psi^*(x)g'(\xi_2(x)),\end{aligned}$$

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where $\xi_1(x) \in (0, \varepsilon\psi^*(x))$ and $\xi_2(x) \in (0, \varepsilon\kappa\psi^*(x))$. We may choose ε sufficiently small such that

$$\begin{cases} \dfrac{\partial \underline{u}}{\partial t} - d_1 \dfrac{\partial^2 \underline{u}}{\partial x^2} - h(\underline{v}(t-\tau_1, x)) + b_1 \underline{u} \leqslant 0, & t > \max\{\tau_1, \tau_2\}, -s_0 < x < s_0, \\ \dfrac{\partial \underline{v}}{\partial t} - d_2 \dfrac{\partial^2 \underline{v}}{\partial x^2} - g(\underline{u}(t-\tau_2, x)) + b_2 \underline{v} \leqslant 0, & t > \max\{\tau_1, \tau_2\}, -s_0 < x < s_0, \\ \underline{u}(\theta, x) \leqslant u(\theta, x), & \max\{\tau_1, \tau_2\} - \tau_2 \leqslant \theta \leqslant \max\{\tau_1, \tau_2\}, -s_0 \leqslant x \leqslant s_0, \\ \underline{v}(\theta, x) \leqslant v(\theta, x), & \max\{\tau_1, \tau_2\} - \tau_1 \leqslant \theta \leqslant \max\{\tau_1, \tau_2\}, -s_0 \leqslant x \leqslant s_0. \end{cases}$$

Moreover, it is easy to check that

$$\underline{u}(t, x) = \underline{v}(t, x) = 0, \quad t \geqslant \max\{\tau_1, \tau_2\}, x \geqslant s_0 \text{ or } x \leqslant -s_0,$$

$$0 = s_0' \leqslant -\mu \frac{\partial \underline{u}}{\partial x}(t, s_0), \quad t > \max\{\tau_1, \tau_2\},$$

$$0 = -s_0' \geqslant -\mu \frac{\partial \underline{u}}{\partial x}(t, -s_0), \quad t > \max\{\tau_1, \tau_2\},$$

$$[-s_0, s_0] \subseteq [s_1(\theta), s_2(\theta)], \quad t > \max\{\tau_1, \tau_2\}.$$

It follows from the comparison principle that $(u(t, x), v(t, x)) \geqslant (\underline{u}(t, x), \underline{v}(t, x))$ for $(t, x) \in [\max\{\tau_1, \tau_2\}, +\infty) \times [-s_0, s_0]$, thus

$$\liminf_{t\to\infty} (\|u(t, \cdot)\|_{C([s_1(t), s_2(t)])} + \|v(t, \cdot)\|_{C([s_1(t), s_2(t)])}) \geqslant \varepsilon(\kappa + 1)\psi^*(0) > 0.$$

By Theorem 2.3 (*iii*), we have $s_{2,\infty} - s_{1,\infty} = +\infty$. □

Theorem 2.6. *If $\mathcal{R}_0^F(-s_0, s_0) < 1 < \mathcal{R}_0$ and $\mu > 0$ is sufficiently large, then $s_{2,\infty} - s_{1,\infty} = +\infty$.*

Proof. Since $\mathcal{R}_0^F(-L, L) \to \mathcal{R}_0 > 1$ as $L \to +\infty$, there exits $L_0 > 0$ such that $\mathcal{R}_0^F(-L_0, L_0) > 1$. Note that

$$\frac{\partial u}{\partial t} - d_1 \frac{\partial^2 u}{\partial x^2} \geqslant -b_1 u \quad \text{for } (t, x) \in (0, +\infty) \times (s_1(t), s_2(t)).$$

It follows from Lemma 4.3 in [30] that there exists $\mu_{L_0} > 0$ such that for any $\mu > \mu_{L_0}$, the corresponding solution $(u(t, x), v(t, x), s_1(t), s_2(t))$ of (1.1) satisfies

$$\limsup_{t\to+\infty} s_1(t) < -L_0, \quad \liminf_{t\to+\infty} s_2(t) > L_0.$$

Since $-s_1(t)$ and $s_2(t)$ are strictly increasing in t, there exists $T_0 > 0$ such that $s_1(T_0) < -L_0$ and $s_2(T_0) > L_0$. Hence, $\mathcal{R}_0^F(s_1(T_0), s_2(T_0)) > \mathcal{R}_0^F(-L_0, L_0) > 1$. By Theorem 2.5, we have $s_{2,\infty} - s_{1,\infty} = +\infty$ for any $\mu > \mu_{L_0}$. □

By choosing μ as varying parameter, we can obtain the following sharp criteria from Theorems 2.4–2.6. We shall use the notations u^μ, v^μ, s_i^μ, $s_{i,\infty}^\mu$ $(i = 1, 2)$ to emphasise the dependence of u, v, s_i, $s_{i,\infty}$ on μ in the following theorem.

Theorem 2.7. (Sharp criteria) *Assume that $\mathcal{R}_0 > 1$. For any given $(u_0(\theta, x), v_0(\theta, x), s_1(\theta), s_2(\theta))$ satisfying (1.5) and (1.6), there exists $\mu^* \in [0, +\infty)$ such that $s_{2,\infty} - s_{1,\infty} = +\infty$ for $\mu > \mu^*$, and $s_{2,\infty} - s_{1,\infty} < +\infty$ for $0 < \mu \leqslant \mu^*$.*

Proof. For $\mathcal{R}_0^F(-s_0, s_0) \geqslant 1$, Theorem 2.5 implies $\mu^* = 0$.

For $\mathcal{R}_0^F(-s_0, s_0) < 1 < \mathcal{R}_0$, we define

$$\Sigma = \{\mu > 0 : s_{2,\infty}^\mu - s_{1,\infty}^\mu < +\infty\}$$

and

$$\mu^* = \sup \Sigma.$$

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By Theorems 2.4 and 2.6, we have $\Sigma \neq \emptyset$ and $0 < \mu^* < +\infty$. It follows that $s^{\mu}_{2,\infty} - s^{\mu}_{1,\infty} = +\infty$ for $\mu > \mu^*$.

By Lemma 2.1, if $\mu_1 > \mu_2$ and $s^{\mu_1}_{2,\infty} - s^{\mu_1}_{1,\infty} < +\infty$, then $s^{\mu_2}_{2,\infty} - s^{\mu_2}_{1,\infty} < +\infty$. Thus, to complete the proof it is sufficient to show $s^{\mu^*}_{2,\infty} - s^{\mu^*}_{1,\infty} < +\infty$. Assume, by contradiction, that $s^{\mu^*}_{2,\infty} - s^{\mu^*}_{1,\infty} = +\infty$, we have $\lim_{t\to+\infty} \mathcal{R}^F_0(s^{\mu^*}_1(t), s^{\mu^*}_2(t)) = \mathcal{R}_0 > 1$. Then, there exists a sufficiently large $T^* > 0$ such that $\mathcal{R}^F_0(s^{\mu^*}_1(T^*), s^{\mu^*}_2(T^*)) > 1$. By the continuous dependence of $(u^{\mu}, v^{\mu}, s^{\mu}_1, s^{\mu}_2)$ on μ, we can find a sufficiently small $\delta > 0$ such that $\mathcal{R}^F_0(s^{\mu}_1(T^*), s^{\mu}_2(T^*)) > 1$ for all $\mu \in [\mu^* - \delta, \mu^* + \delta]$. Choosing T^* as initial time, similarly as the proof of Theorem 2.5, we can deduce that $s^{\mu}_{2,\infty} - s^{\mu}_{1,\infty} = +\infty$ for all $\mu \in [\mu^* - \delta, \mu^* + \delta]$. This contradicts with the definition of μ^*. □

3. Asymptotic spreading speeds

In this section, we mainly determine the spreading speeds of free boundaries when spreading happens. To achieve it, we first consider a semi-wave problem with time delays, whose monotone increasing solutions provide a pair of upper and lower solutions in handling spreading speeds.

3.1 Semi-wave problem with time delays

Consider the following nonlinear semi-wave problem with time delays

$$\begin{cases} c\phi'(s) - d_1\phi''(s) = h(\psi(s - c\tau_1)) - b_1\phi(s), & s > 0, \\ c\psi'(s) - d_2\psi''(s) = g(\phi(s - c\tau_2)) - b_2\psi(s), & s > 0, \\ (\phi(s), \psi(s)) = (0, 0), & s \leqslant 0, \\ (\phi(+\infty), \psi(+\infty)) = (u^*, v^*), \end{cases} \tag{3.1}$$

where $c > 0$ and $\tau_1, \tau_2 > 0$. The condition $\mathcal{R}_0 > 1$ is always assumed in this section, which ensures that (3.1) admits a positive equilibrium (u^*, v^*). In this subsection, we aim to establish the existence, uniqueness and properties of monotone increasing solutions to (3.1).

For the existence of monotone increasing solutions to (3.1), we first characterise the distribution of roots of the following transcendental equation containing a polynomial of degree four:

$$\Delta^c(\lambda, \tau) := p^c_1(\lambda)p^c_2(\lambda) - h'(0)g'(0)e^{-\lambda c\tau} = 0 \tag{3.2}$$

with

$$p^c_1(\lambda) = d_1\lambda^2 - c\lambda - b_1, \quad p^c_2(\lambda) = d_2\lambda^2 - c\lambda - b_2, \quad \tau = \tau_1 + \tau_2.$$

Obviously, for $i = 1, 2$, $p^c_i(\lambda) = 0$ has two real roots

$$\lambda^{\pm}_i = \frac{c \pm \sqrt{c^2 + 4d_ib_i}}{2d_i}.$$

Denote

$$\lambda^c_m = \min\{\lambda^+_1, \lambda^+_2\}, \quad \lambda^c_M = \max\{\lambda^+_1, \lambda^+_2\}$$

and define

$$c^*_0 = \inf\left\{c_0 > 0 \,\middle|\, \text{all roots of } \Delta^c(\lambda, 0) = 0 \text{ are real for } c \geqslant c_0\right\}.$$

Next we establish the distribution of roots of $\Delta^c(\lambda, \tau) = 0$ according to the value of c in the following lemma.

Lemma 3.1. *Assume that $c > 0$ and $\tau_1, \tau_2 > 0$. Then, the following conclusions hold:*

(i) $\Delta^c(\lambda, \tau) = 0$ has only one positive root $\tilde{\lambda}^c_\tau$ in $[\lambda^c_m, +\infty)$. Moreover, $\tilde{\lambda}^c_\tau$ satisfies $\tilde{\lambda}^c_\tau > \lambda^c_M$ and $\frac{d}{d\lambda}\Delta^c(\lambda, \tau)|_{\lambda=\tilde{\lambda}^c_\tau} > 0$;

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(ii) there exists $c_\tau^* \in (0, c_0^*)$ such that

(a) for $c > c_\tau^*$, $\Delta^c(\lambda, \tau) = 0$ has exactly two positive roots $\check{\lambda}_\tau^c$, $\hat{\lambda}_\tau^c$ in $(0, \lambda_m^c)$ and satisfies $\Delta^c(\lambda, \tau) > 0$ for $\lambda \in (\check{\lambda}_\tau^c, \hat{\lambda}_\tau^c)$;

(b) for $0 < c < c_\tau^*$, $\Delta^c(\lambda, \tau) = 0$ has no positive root in $(0, \lambda_m^c)$ and satisfies $\Delta^c(\lambda, \tau) < 0$ in $(0, \lambda_m^c)$. In such case, $\Delta^c(\lambda, \tau) = 0$ admits a pair of conjugate complex roots with one lying in the domain $D := \{\lambda \in \mathbb{C} : Re\lambda > 0, Im\lambda > 0\}$;

(c) for $c = c_\tau^*$, $\Delta^c(\lambda, \tau) = 0$ has only one positive root in $(0, \lambda_m^c)$.

Proof. (i) For any given $c > 0$, if $\lambda \in [\lambda_m^c, \lambda_M^c)$ then we have $p_1^c(\lambda)p_2^c(\lambda) \leqslant 0$. It follows that

$$\Delta^c(\lambda, \tau) = p_1^c(\lambda)p_2^c(\lambda) - h'(0)g'(0)e^{-\lambda c\tau} < 0 \quad \text{for } \lambda \in [\lambda_m^c, \lambda_M^c).$$

If $\lambda \in [\lambda_M^c, +\infty)$, then $p_1^c(\lambda)p_2^c(\lambda)$ is strictly increasing in λ, which implies that $\Delta^c(\lambda, \tau) = p_1^c(\lambda)p_2^c(\lambda) - h'(0)g'(0)e^{-\lambda c\tau}$ is also strictly increasing with respect to λ in $[\lambda_M^c, +\infty)$. Note that

$$\Delta^c(\lambda_M^c, \tau) = p_1^c(\lambda_M^c)p_2^c(\lambda_M^c) - h'(0)g'(0)e^{-\lambda_M^c c\tau} = -h'(0)g'(0)e^{-\lambda_M^c c\tau} < 0$$

and

$$\Delta^c(+\infty, \tau) = +\infty.$$

Then, $\Delta^c(\lambda, \tau) = 0$ has only one positive root $\tilde{\lambda}_\tau^c$ in $[\lambda_m^c, +\infty)$ and $\tilde{\lambda}_\tau^c > \lambda_M^c$. Moreover, for any fixed $c > 0$ and $\tau > 0$,

$$\begin{aligned}\frac{d}{d\lambda}\Delta^c(\lambda, \tau)\Big|_{\lambda=\tilde{\lambda}_\tau^c} &= (2d_1\tilde{\lambda}_\tau^c - c)\Big[d_2(\tilde{\lambda}_\tau^c)^2 - c\tilde{\lambda}_\tau^c - b_2\Big] + (2d_2\tilde{\lambda}_\tau^c - c)\Big[d_1(\tilde{\lambda}_\tau^c)^2 - c\tilde{\lambda}_\tau^c - b_1\Big] \\ &\quad + 2c\tau h'(0)g'(0)e^{-\tilde{\lambda}_\tau^c c\tau} \\ &> 0.\end{aligned}$$

(ii) By the proof of Lemma 2.1 in [38], $\Delta^c(\lambda, \tau) = 0$ has at most three distinct positive roots in $(0, +\infty)$ for any given $c > 0$ and $\tau > 0$. Thus, from (i) we know that $\Delta^c(\lambda, \tau) = 0$ has at most two distinct positive roots in $(0, \lambda_m^c)$. Moreover,

$$\Delta^c\left(\frac{1}{\sqrt{c}}, \tau\right) = \left(\frac{d_1}{c} - \sqrt{c} - b_1\right)\left(\frac{d_2}{c} - \sqrt{c} - b_2\right) - h'(0)g'(0)e^{-\sqrt{c}\tau} \quad \text{and} \quad \lim_{c\to+\infty}\Delta^c\left(\frac{1}{\sqrt{c}}, \tau\right) = +\infty,$$

which imply that $\Delta^c(\frac{1}{\sqrt{c}}, \tau) > 0$ and $0 < \frac{1}{\sqrt{c}} < \lambda_m^c$ for large c. Note that

$$\Delta^c(0, \tau) = b_1b_2 - h'(0)g'(0) < 0 \quad \text{and} \quad \Delta^c(\lambda_m^c, \tau) = -h'(0)g'(0)e^{-\lambda_m^c c\tau} < 0.$$

Therefore, for all large c, $\Delta^c(\lambda, \tau) = 0$ has exactly two distinct positive roots in $(0, \lambda_m^c)$ and then has three distinct positive roots in $(0, +\infty)$. It follows that the set

$$S(\tau) := \Big\{\mathfrak{C} > 0 \Big| \Delta^c(\lambda, \tau) \text{ has three distinct positive zeros in } (0, +\infty) \text{ for all } c \geqslant \mathfrak{C}\Big\}$$

is not empty. Thus,

$$c_\tau^* := \inf S(\tau) \geqslant 0$$

is well-defined, and we know that $\Delta^c(\lambda, \tau) = 0$ has three distinct positive roots in $(0, +\infty)$ for $c > c_\tau^*$. From (i), we deduce that $\Delta^c(\lambda, \tau) = 0$ has two distinct positive roots in $(0, \lambda_m^c)$ for $c > c_\tau^*$. We denote by $\check{\lambda}_\tau^c$, $\hat{\lambda}_\tau^c$ the two positive roots and assume $\check{\lambda}_\tau^c < \hat{\lambda}_\tau^c$, then $\Delta^c(\lambda, \tau) > 0$ for $\lambda \in (\check{\lambda}_\tau^c, \hat{\lambda}_\tau^c)$, and $\Delta^c(\lambda, \tau) < 0$ for $\lambda \in (0, \check{\lambda}_\tau^c) \cup (\hat{\lambda}_\tau^c, \lambda_m^c)$. This completes the proof of (ii)-(a).

We claim that $c_\tau^* > 0$. Otherwise, for any sequence $\{c_n\}$ satisfying $0 < c_n < 1$ and $\lim_{n\to\infty} c_n = 0$, $\Delta^{c_n}(\lambda, \tau) = 0$ has two distinct positive roots $\check{\lambda}_\tau^{c_n}$, $\hat{\lambda}_\tau^{c_n}$ in $(0, \lambda_m^{c_n})$ and one positive root $\tilde{\lambda}_\tau^{c_n}$ in $(\lambda_M^{c_n}, +\infty)$. Since

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$$0<\lambda_m^{c_n} = \min\left\{\frac{c_n+\sqrt{c_n^2+4d_1b_1}}{2d_1}, \frac{c_n+\sqrt{c_n^2+4d_2b_2}}{2d_2}\right\}$$
$$< \min\left\{\frac{1+\sqrt{1+4d_1b_1}}{2d_1}, \frac{1+\sqrt{1+4d_2b_2}}{2d_2}\right\}$$

and $\tilde{\lambda}_\tau^{c_n}$ satisfies

$$\left[d_1(\tilde{\lambda}_\tau^{c_n})^2-c_n\tilde{\lambda}_\tau^{c_n}-b_1\right]\left[d_2(\tilde{\lambda}_\tau^{c_n})^2-c_n\tilde{\lambda}_\tau^{c_n}-b_2\right]\leqslant h'(0)g'(0),$$

the above sequences $\{\check{\lambda}_\tau^{c_n}\}$, $\{\hat{\lambda}_\tau^{c_n}\}$, $\{\tilde{\lambda}_\tau^{c_n}\}$ are uniformly bounded with respect to n. By extracting convergence subsequences and taking $n\to\infty$, we can deduce $(\check{\lambda}_\tau^{c_n}, \hat{\lambda}_\tau^{c_n}, \tilde{\lambda}_\tau^{c_n})\to(\check{\lambda}_\tau^{0}, \hat{\lambda}_\tau^{0}, \tilde{\lambda}_\tau^{0})$ with some $\check{\lambda}_\tau^{0}, \hat{\lambda}_\tau^{0}, \tilde{\lambda}_\tau^{0}\geqslant 0$. Note that for any fixed $\tau>0$, $\Delta^{c_n}(\lambda,\tau)\to\Delta^0(\lambda,\tau)$ in $C^2_{loc}(\mathbb{R})$ as $n\to\infty$. It follows that $\check{\lambda}_\tau^{0}, \hat{\lambda}_\tau^{0}, \tilde{\lambda}_\tau^{0}$ are nonnegative roots of $\Delta^0(\lambda,\tau)=0$. Assume that one of three roots is 0, a contradiction occurs since $\Delta^0(0,\tau)<0$, and the claim is proved. If all the three roots are positive, then there are two possible cases: (1) at least two of these three roots are not equal; (2) all three roots are equal. On the other hand, we can directly check that $\Delta^0(\lambda,\tau)=0$ has only one positive real root, denoted by λ_τ^0, and

$$\frac{d}{d\lambda}\Delta^0(\lambda,\tau)|_{\lambda=\lambda_\tau^0}>0. \tag{3.3}$$

Thus, the first case cannot happen. Next, we consider the second case. Since $\check{\lambda}_\tau^{c_n}$, $\hat{\lambda}_\tau^{c_n}$ are two roots of $\Delta^{c_n}(\lambda,\tau)=0$, by the mean value theorem, there exists a ξ_n between $\check{\lambda}_\tau^{c_n}$ and $\hat{\lambda}_\tau^{c_n}$ such that $\frac{d}{d\lambda}\Delta^{c_n}(\lambda,\tau)|_{\lambda=\xi_n}=0$. Since $\lim_{n\to\infty}\check{\lambda}_\tau^{c_n}=\lim_{n\to\infty}\hat{\lambda}_\tau^{c_n}=\lambda_\tau^0$, we also have $\lim_{n\to\infty}\xi_n=\lambda_\tau^0$. By taking $n\to\infty$ in $\frac{d}{d\lambda}\Delta^{c_n}(\lambda,\tau)|_{\lambda=\xi_n}=0$, we get $\frac{d}{d\lambda}\Delta^{0}(\lambda,\tau)|_{\lambda=\lambda_\tau^0}=0$, which contradicts with (3.3). Then, the claim holds true.

Now we prove the first part of *(ii)*-*(b)*. Assume by contradiction that $\Delta^{c_1}(\lambda,\tau)$ has at least one positive root in $(0,\lambda_m^{c_1})$ for some $0<c_1<c_\tau^*$, we will show that

$$c_1\geqslant\inf S(\tau)=c_\tau^*, \tag{3.4}$$

which leads to a contradiction. To prove (3.4), it is sufficient to show that $\Delta^{c_2}(\lambda,\tau)$ has two distinct positive roots in $(0,\lambda_m^{c_2})$ for any $c_2>c_1$. In fact, since $\lambda_m^{c_2}>\lambda_m^{c_1}$, we know that

$$p_1^{c_1}(\lambda), p_2^{c_1}(\lambda), p_1^{c_2}(\lambda), p_2^{c_2}(\lambda)<0 \quad \text{for } \lambda\in(0,\lambda_m^{c_1}).$$

Note that $p_1^c(\lambda), p_2^c(\lambda)$ are decreasing in c. We conclude that $p_1^c(\lambda)p_2^c(\lambda)$ and then $\Delta^c(\lambda,\tau)$ are increasing with respect to c for $\lambda\in(0,\lambda_m^{c_1})$. Denote $\tilde{\lambda}_1$ by one positive root of $\Delta^{c_1}(\lambda,\tau)=0$ in $(0,\lambda_m^{c_1})$, that is, $\Delta^{c_1}(\tilde{\lambda}_1,\tau)=0$. Then,

$$\Delta^{c_2}(\tilde{\lambda}_1,\tau)>\Delta^{c_1}(\tilde{\lambda}_1,\tau)=0.$$

It follows that $\Delta^{c_2}(\lambda,\tau)=0$ has exactly two distinct positive root in $(0,\lambda_m^{c_2})$ for any $c_2>c_1$. Hence, (3.4) holds and we can get the desired result. Clearly, $\Delta^c(\lambda,\tau)<0$ in $(0,\lambda_m^c)$ for any $0<c<c_\tau^*$.

Based on the above discussions, it is easy to know that $\Delta^c(\lambda,\tau)=0$ has only one positive root in $(0,\lambda_m^c)$ for $c=c_\tau^*$, i.e. *(ii)*-*(c)* holds. We claim that $c_\tau^*<c_0^*$. Indeed, by Lemma 3.3 *(iii)*–*(iv)* in [35] and the fact $\Delta^c(\lambda,\tau)>\Delta^c(\lambda,0)$, we know that $\Delta^c(\lambda,\tau)=0$ has three distinct positive roots in $(0,+\infty)$ for all $c\geqslant c_0^*$. Thus, $c_\tau^*<c_0^*$.

Next, we prove the second part of *(ii)*-*(b)*. We will prove the result by taking τ as a parameter which continuously increases from 0 to $+\infty$. Since the range $(0,c_\tau^*)$ of c is dependent on τ, we cannot simply fix a c as τ varies. However, we find that c_τ^* is decreasing with respect to τ. In fact, we have showed that $\Delta^c(\lambda,\tau)<0$ in $(0,\lambda_m^c)$ for $0<c<c_\tau^*$. If $\tilde{\tau}>\tau$, then we can check that

$$\Delta^c(\lambda,\tilde{\tau})>\Delta^c(\lambda,\tau).$$

It follows that $\Delta^c(\lambda,\tau)<\Delta^c(\lambda,\tilde{\tau})\leqslant 0$ in $(0,\lambda_m^c)$ for $0<c\leqslant c_{\tilde{\tau}}^*$, which implies that

$$c_{\tilde{\tau}}^*<c_\tau^*. \tag{3.5}$$

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Otherwise, $\Delta^c(\lambda, \tau) < 0$ in $(0, \lambda_m^c)$ for $c = c_\tau^*$, which contradicts with (*ii*)-(*c*). Therefore, for any fixed $T > 0$, we have $c_T^* < c_\tau^*$ for all $\tau \in (0, T)$. In what follows, by choosing any fixed $c \in (0, c_T^*)$ and varying τ continuously from 0 to T, we prove the existence of complex roots with positive real parts.

Note that the zeros of $\Delta^c(\lambda, \tau)$ are continuous in $\tau \in (0, T)$ for fixed $c \in (0, c_T^*)$. Define $\lambda = \alpha(\tau) + i\beta(\tau)$, where $\alpha(\tau)$ and $\beta(\tau)$ are continuous in $\tau \in (0, T)$. Separating the real and imaginary parts of $\Delta^c(\lambda, \tau) = 0$, we derive

$$\begin{cases} F_1(\alpha, \beta, \tau) = \Big[d_1(\alpha^2 - \beta^2) - c\alpha - b_1\Big]\Big[d_2(\alpha^2 - \beta^2) - c\alpha - b_2\Big] \\ \qquad\qquad -(2d_1\alpha\beta - c\beta)(2d_2\alpha\beta - c\beta) - h'(0)g'(0)e^{-c\tau\alpha}\cos c\tau\beta = 0, \\ F_2(\alpha, \beta, \tau) = (2d_1\alpha\beta - c\beta)\Big[d_2(\alpha^2 - \beta^2) - c\alpha - b_2\Big] \\ \qquad\qquad +(2d_2\alpha\beta - c\beta)\Big[d_1(\alpha^2 - \beta^2) - c\alpha - b_1\Big] + h'(0)g'(0)e^{-c\tau\alpha}\sin c\tau\beta = 0. \end{cases} \tag{3.6}$$

We divide the rest of proof into four steps.

Step 1. There is a complex root in D provided that τ is small enough.

For $\tau = 0$, $\Delta^c(\lambda, 0) = p_1^c(\lambda)p_2^c(\lambda) - h'(0)g'(0)$, then it follows from Lemma 3.3 in [35] that $\Delta^c(\lambda, 0) = 0$ has a pair of conjugate complex roots for $c \in (0, c_T^*) \subset (0, c_0^*)$. We denote one of the complex roots in D by $\lambda = \alpha + i\beta$. By direct calculations,

$$\begin{aligned} &\det\begin{pmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{pmatrix}\Bigg|_{\tau=0} \\ &= \Big\{(2d_1\alpha - c)\Big[d_2(\alpha^2 - \beta^2) - c\alpha - b_2\Big] + (2d_2\alpha - c)\Big[d_1(\alpha^2 - \beta^2) - c\alpha - b_1\Big] \\ &\quad - 2d_1\beta^2(2d_2\alpha - c) - 2d_2\beta^2(2d_1\alpha - c)\Big\}^2 \\ &\quad + \Big\{2d_1\beta\Big[d_2(\alpha^2 - \beta^2) - c\alpha - b_2\Big] + 2d_2\beta\Big[d_1(\alpha^2 - \beta^2) - c\alpha - b_1\Big] \\ &\quad + (2d_1\alpha\beta - c\beta)(2d_2\alpha - c) + (2d_2\alpha\beta - c\beta)(2d_1\alpha - c)\Big\}^2 \geqslant 0. \end{aligned}$$

Since $\tau = 0$, $\lambda = \alpha + i\beta$ satisfies (3.6), it follows from the second equation of (3.6) that

$$(2d_1\alpha - c)\Big[d_2(\alpha^2 - \beta^2) - c\alpha - b_2\Big] + (2d_2\alpha - c)\Big[d_1(\alpha^2 - \beta^2) - c\alpha - b_1\Big] = 0.$$

Thus,

$$\begin{aligned} &\det\begin{pmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{pmatrix}\Bigg|_{\tau=0} \\ &= 4\beta^4\Big[d_1(2d_2\alpha - c) + d_2(2d_1\alpha - c)\Big]^2 + 4\beta^2\Big\{d_1\Big[d_2(\alpha^2 - \beta^2) - c\alpha - b_2\Big] \\ &\quad + d_2\Big[d_1(\alpha^2 - \beta^2) - c\alpha - b_1\Big] + (2d_1\alpha - c)(2d_2\alpha - c)\Big\}^2 \geqslant 0, \end{aligned}$$

where the equality holds if and only if

$$\begin{cases} d_1(2d_2\alpha - c) + d_2(2d_1\alpha - c) = 0, \\ d_1\Big[d_2(\alpha^2 - \beta^2) - c\alpha - b_2\Big] + d_2\Big[d_1(\alpha^2 - \beta^2) - c\alpha - b_1\Big] + (2d_1\alpha - c)(2d_2\alpha - c) = 0. \end{cases} \tag{3.7}$$

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In view of the first equation of (3.7), we have $\alpha = \frac{cd_1+cd_2}{4d_1d_2} > 0$. Substituting this α into the second equation of (3.7) gives

$$
\begin{aligned}
&6d_1d_2\alpha^2 + d_1(-d_2\beta^2 - b_2) + d_2(-d_1\beta^2 - b_1) - 3cd_1\alpha - 3cd_2\alpha \\
&= \frac{3}{2}\alpha(cd_1 + cd_2) + d_1(-d_2\beta^2 - b_2) + d_2(-d_1\beta^2 - b_1) - 3cd_1\alpha - 3cd_2\alpha \\
&= -\frac{3}{2}cd_1\alpha - \frac{3}{2}cd_2\alpha + d_1(-d_2\beta^2 - b_2) + d_2(-d_1\beta^2 - b_1) = 0,
\end{aligned}
$$

which is not solvable for β. Therefore, we get

$$
\det\begin{pmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{pmatrix}\Big|_{\tau=0} > 0.
$$

Then, the implicit function theorem indicates that, for small $\tau > 0$, $\Delta^c(\lambda, \tau) = 0$ with $c \in (0, c_T^*)$ admits a pair of complex solutions near $\alpha + i\beta$, and thus in the open domain D.

Step 2. If $\lambda = \alpha(\tau) + i\beta(\tau)$ touches the pure imaginary axis at some $\tau = \tau_0 \in (0, T)$, i.e. $\alpha(\tau_0) = 0$, then $\alpha'(\tau_0) > 0$. Moreover, $\beta_0 := \beta(\tau_0) > 0$.

Since $\alpha(\tau_0)$ and $\beta(\tau_0)$ satisfy (3.6), we get

$$
\begin{cases} (d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2) - c^2\beta_0^2 - h'(0)g'(0)\cos c\tau_0\beta_0 = 0, \\ c\beta_0(d_2\beta_0^2 + b_2) + c\beta_0(d_1\beta_0^2 + b_1) + h'(0)g'(0)\sin c\tau_0\beta_0 = 0. \end{cases} \tag{3.8}
$$

Besides,

$$
\begin{aligned}
&\det\begin{pmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{pmatrix}\Big|_{\tau=\tau_0} \\
&= \Big\{c(d_2\beta_0^2 + b_2) + c(d_1\beta_0^2 + b_1) + 2d_1c\beta_0^2 + 2d_2c\beta_0^2 + 2c\tau h'(0)g'(0)\cos c\tau_0\beta_0\Big\}^2 \\
&\quad + \Big\{2d_1\beta_0(d_2\beta_0^2 + b_2) + 2d_2\beta_0(d_1\beta_0^2 + b_1) - 2c^2\beta_0 + 2c\tau h'(0)g'(0)\sin c\tau_0\beta_0\Big\}^2 \geqslant 0,
\end{aligned}
$$

where the equality holds if and only if

$$
\begin{cases} 3d_1\beta_0^2 + 3d_2\beta_0^2 + b_1 + b_2 + 2c\tau h'(0)g'(0)\cos c\tau_0\beta_0 = 0, \\ 2d_1\beta_0^2(d_2\beta_0^2 + b_2) + 2d_2\beta_0^2(d_1\beta_0^2 + b_1) - 2c^2\beta_0^2 + 2c\beta\tau h'(0)g'(0)\sin c\tau_0\beta_0 = 0. \end{cases} \tag{3.9}
$$

Substituting (3.8) into (3.9), we give

$$
\begin{cases} 3d_1\beta_0^2 + 3d_2\beta_0^2 + b_1 + b_2 - 2\tau c^2\beta_0^2 + 2\tau(d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2) = 0, \\ 2d_1\beta_0^2(d_2\beta_0^2 + b_2) + 2d_2\beta_0^2(d_1\beta_0^2 + b_1) - 2c^2\beta_0^2 - 2\tau c^2\beta_0^2(d_1\beta_0^2 + d_2\beta_0^2 + b_1 + b_2) = 0. \end{cases} \tag{3.10}
$$

Multiplying the first equation in (3.10) by $d_1\beta_0^2 + d_2\beta_0^2 + b_1 + b_2$, and then subtracting it from the second equation in (3.10), we get

$$
\begin{aligned}
&2d_1d_2\beta_0^4 + 3d_2^2\beta_0^4 + 3d_1^2\beta_0^4 + d_2b_1\beta_0^2 + d_1b_2\beta_0^2 + 3d_1^2\beta_0^2 + 3d_2^2\beta_0^2 + 2c^2\beta_0^2 + b_1d_1\beta_0^2 \\
&\quad + b_2d_2\beta_0^2 + (b_1 + b_2)^2 + 2\tau(d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2)(d_1\beta_0^2 + d_2\beta_0^2 + b_1 + b_2) = 0,
\end{aligned}
$$

which is not solvable for $\beta_0 > 0$. Thus,

$$
\det\begin{pmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{pmatrix}\Big|_{\tau=\tau_0} > 0.
$$

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Moreover,

$$\begin{pmatrix} \partial_\tau F_1 \\ \partial_\tau F_2 \end{pmatrix} \Big|_{\tau=\tau_0} = 2h'(0)g'(0)c\beta_0 \begin{pmatrix} \sin c\tau_0\beta_0 \\ \cos c\tau_0\beta_0 \end{pmatrix}.$$

Consequently, the implicit function theorem indicates that

$$\begin{pmatrix} \alpha'(\tau) \\ \beta'(\tau) \end{pmatrix} \Big|_{\tau=\tau_0} = - \begin{pmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{pmatrix}^{-1} \Big|_{\tau=\tau_0} \begin{pmatrix} \partial_\tau F_1 \\ \partial_\tau F_2 \end{pmatrix} \Big|_{\tau=\tau_0}.$$

Direct computation induces that

$$\alpha'(\tau_0) = -\frac{(\partial_\beta F_2 \partial_\tau F_1 - \partial_\beta F_1 \partial_\tau F_2)|_{\tau=\tau_0}}{\det \begin{pmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{pmatrix} \Big|_{\tau=\tau_0}}.$$

Besides, by (3.8), we have

$$\begin{aligned}
&(\partial_\beta F_2 \partial_\tau F_1 - \partial_\beta F_1 \partial_\tau F_2)|_{\tau=\tau_0} \\
&= \Big[c(d_2\beta_0^2 + b_2) + c(d_1\beta_0^2 + b_1) + 2d_1c\beta_0^2 + 2d_2c\beta_0^2 + 2c\tau h'(0)g'(0)\cos c\tau_0\beta_0\Big] \\
&\quad \times \Big(2h'(0)g'(0)c\beta_0 \sin c\tau_0\beta_0\Big) \\
&\quad + \Big[2d_1\beta_0(-d_2\beta_0^2 - b_2) + 2d_2\beta_0(-d_1\beta_0^2 - b_1) + 2c^2\beta_0 - 2c\tau h'(0)g'(0)c\beta_0 \sin c\tau_0\beta_0\Big] \\
&\qquad \times \Big(2h'(0)g'(0)c\beta_0 \cos c\tau_0\beta_0\Big) \\
&= \Big[c(d_2\beta_0^2 + b_2) + c(d_1\beta_0^2 + b_1) + 2d_1c\beta_0^2 + 2d_2c\beta_0^2 + 2c\tau(d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2) \\
&\quad - 2\tau c^3\beta_0^2\Big] \times \Big[-2c^2\beta_0^2(d_2\beta_0^2 + b_2) - 2c^2\beta_0^2(d_1\beta_0^2 + b_1)\Big] \\
&\quad + \Big[2d_1\beta_0(-d_2\beta_0^2 - b_2) + 2d_2\beta_0(-d_1\beta_0^2 - b_1) + 2c^2\beta_0 + 2\tau c^2\beta_0(d_2\beta_0^2 + b_2) \\
&\qquad + 2\tau c^2\beta_0(d_1\beta_0^2 + b_1)\Big] \times \Big[2c\beta_0(d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2) - 2c^3\beta_0^3\Big] \\
&= -2c^3\beta_0^2\Big[(d_2\beta_0^2 + b_2) + (d_1\beta_0^2 + b_1) + 2d_1\beta_0^2 + 2d_2\beta_0^2 + 2\tau(d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2)\Big] \\
&\quad \times \Big[(d_2\beta_0^2 + b_2) + (d_1\beta_0^2 + b_1)\Big] + 4\tau c^5\beta_0^4\Big[(d_2\beta_0^2 + b_2) + (d_1\beta_0^2 + b_1)\Big] \\
&\quad - 4c\beta_0^2\Big[d_1(d_2\beta_0^2 + b_2) + d_2(d_1\beta_0^2 + b_1)\Big] \times \Big[(d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2)\Big] \\
&\quad - 4c^5\beta_0^4\Big[1 + \tau(d_2\beta_0^2 + b_2) + \tau(d_1\beta_0^2 + b_1)\Big] + 4c^3\beta_0^4\Big[d_1(d_2\beta_0^2 + b_2) + d_2(d_1\beta_0^2 + b_1)\Big] \\
&\quad + 4c^3\beta_0^2\Big[1 + \tau(d_2\beta_0^2 + b_2) + \tau(d_1\beta_0^2 + b_1)\Big] \times \Big[(d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2)\Big] \\
&= -\Big[2c^3\beta_0^2(d_2\beta_0^2 + b_2)^2 + 2c^3\beta_0^2(d_1\beta_0^2 + b_1)^2 + 4d_2c^3\beta_0^4(d_2\beta_0^2 + b_2) \\
&\qquad + 4d_1c^3\beta_0^4(d_1\beta_0^2 + b_1) + 4d_1c\beta_0^2(d_1\beta_0^2 + b_1)(d_2\beta_0^2 + b_2)^2 \\
&\qquad + 4d_2c\beta_0^2(d_1\beta_0^2 + b_1)^2(d_2\beta_0^2 + b_2) + 4c^5\beta_0^4\Big] < 0.
\end{aligned}$$

Due to $\det \begin{pmatrix} \partial_\alpha F_1 & \partial_\beta F_1 \\ \partial_\alpha F_2 & \partial_\beta F_2 \end{pmatrix} \Big|_{\tau=\tau_0} > 0$, hence $\alpha'(\tau_0) > 0$.

Step 3. As τ varies from 0 to T, the complex root cannot touch the real axis at some $\tau_1 \in (0, T)$ for $c \in (0, c_T^*)$.

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If not, then the complex root changes into $\tilde{\lambda} > 0$ at $\tau = \tau_1$. Since complex roots appear in conjugate pairs, the multiplicity of $\tilde{\lambda}$ is larger than two. However, due to $\Delta^c(\lambda, \tau_1) < 0$ in $(0, \lambda_m^c)$ for $c \in (0, c_T^*) \subset (0, c_{\tau_1}^*)$, we know that $\tilde{\lambda}$ must satisfies $\tilde{\lambda} \in [\lambda_m^c, +\infty)$. From (i), $\tilde{\lambda}$ is a simple root. This is a contradiction.

Step 4. Since $T > 0$ is arbitrary, by Steps 1–3 we can obtain that as τ continuously increases from 0 to $+\infty$, the complex root cannot escape the first quadrant of the complex plane. Thus, it always stays in D. This completes the proof of this lemma. □

Lemma 3.2. *For any $c \geqslant c_\tau^*$, problem (3.1) admits no monotone solution.*

Proof. Assume on the contrary that (3.1) has a monotone solution (ϕ, ψ) for some $c \geqslant c_\tau^*$. Clearly, $(\phi(s), \psi(s)) \leqslant (u^*, v^*)$ for $s \in \mathbb{R}$. From the first equation in (3.1), we have

$$\phi(s) = \frac{1}{d_1(\lambda_1^+ - \lambda_1^-)}\Big[\int_0^s (e^{\lambda_1^-(s-\xi)} - e^{\lambda_1^- s - \lambda_1^+ \xi}) h(\psi(\xi - c\tau_1))d\xi + \int_s^{+\infty} (e^{\lambda_1^+(s-\xi)} - e^{\lambda_1^- s - \lambda_1^+ \xi}) h(\psi(\xi - c\tau_1))d\xi\Big] \tag{3.11}$$

for all $s > 0$. By direct calculations,

$$\begin{aligned}|\phi'(s)| &\leqslant \frac{1}{d_1(\lambda_1^+ - \lambda_1^-)}\Big|\int_0^s \lambda_1^-(e^{\lambda_1^-(s-\xi)} - e^{\lambda_1^- s - \lambda_1^+ \xi}) h(\psi(\xi - c\tau_1))d\xi \\ &\quad + \int_s^{+\infty} (\lambda_1^+ e^{\lambda_1^+(s-\xi)} - \lambda_1^- e^{\lambda_1^- s - \lambda_1^+ \xi}) h(\psi(\xi - c\tau_1))d\xi\Big| \\ &\leqslant \frac{h(v^*)}{d_1(\lambda_1^+ - \lambda_1^-)}(1 - \frac{\lambda_1^-}{\lambda_1^+})e^{\lambda_1^- s} \leqslant \frac{h(v^*)}{d_1\lambda_1^+} =: C_1\end{aligned}$$

and

$$|\phi''(s)| = \frac{|c\phi' - h(\psi(s - c\tau_1)) + b_1\phi|}{d_1} \leqslant \frac{cC_1 + h(v^*) + b_1 u^*}{d_1} =: C_2.$$

Similarly, there exist $C_3 > 0$ and $C_4 > 0$ such that $|\psi'(s)| \leqslant C_3$ and $|\psi''(s)| \leqslant C_4$. Denote the derivatives of (ϕ, ψ) by $(\phi^{(i)}, \psi^{(i)})$ for $i = 1, 2$, then

$$\begin{gathered}\textstyle\int_0^\infty \phi^{(i)}(s)e^{-\lambda_0 s}ds < \infty, \int_0^\infty \psi^{(i)}(s)e^{-\lambda_0 s}ds < \infty, i = 0, 1, 2, \\ \textstyle\int_0^\infty \psi(s - c\tau_1)e^{-\lambda_0 s}ds < \infty, \int_0^\infty \phi(s - c\tau_2)e^{-\lambda_0 s}ds < \infty,\end{gathered}$$

for each $\lambda > 0$. From Lemma 3.1 (ii), $\Delta^c(\lambda, \tau) = 0$ has a positive root $\lambda_0 \in (0, \lambda_m^c)$ for $c \geqslant c_\tau^*$ (when $c > c_\tau^*$ we only choose one of positive roots). Next, multiplying the equations in (3.1) by $e^{-\lambda_0 s}$ and integrating from 0 to $+\infty$, we obtain

$$\begin{aligned}0 < d_1\phi'(0) &= p_1^c(\lambda_0) \textstyle\int_0^{+\infty} \phi(s)e^{-\lambda_0 s}ds + \int_0^{+\infty} h(\psi(s - c\tau_1))e^{-\lambda_0 s}ds, \\ 0 < d_2\psi'(0) &= p_2^c(\lambda_0) \textstyle\int_0^{+\infty} \psi(s)e^{-\lambda_0 s}ds + \int_0^{+\infty} g(\phi(s - c\tau_2))e^{-\lambda_0 s}ds.\end{aligned}$$

Since $h(z) \leqslant h'(0)z$ and $g(z) < g'(0)z$ for $z > 0$ by Taylor's formula and the concavity of h, g, we have

$$h(\psi(s - c\tau_1)) \leqslant h'(0)\psi(s - c\tau_1), \quad g(\phi(s - c\tau_2)) \leqslant g'(0)\phi(s - c\tau_2),$$

which imply that

$$\begin{aligned}&p_1^c(\lambda_0) \textstyle\int_0^{+\infty} \phi(s)e^{-\lambda_0 s}ds + h'(0) \int_0^{+\infty} \psi(s - c\tau_1)e^{-\lambda_0 s}ds > 0, \\ &p_2^c(\lambda_0) \textstyle\int_0^{+\infty} \psi(s)e^{-\lambda_0 s}ds + g'(0) \int_0^{+\infty} \phi(s - c\tau_2)e^{-\lambda_0 s}ds > 0.\end{aligned} \tag{3.12}$$

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Hence,

$$\begin{aligned}
0 &< p_1^c(\lambda_0)\int_0^{+\infty}\phi(s)e^{-\lambda_0 s}ds + h'(0)\int_0^{+\infty}\psi(s-c\tau_1)e^{-\lambda_0 s}ds\\
&= \frac{p_1^c(\lambda_0)e^{\lambda_0 c\tau}}{g'(0)}\left[g'(0)e^{-\lambda_0 c\tau}\int_0^{+\infty}\phi(s)e^{-\lambda_0 s}ds + \frac{h'(0)g'(0)e^{-\lambda_0 c\tau}}{p_1^c(\lambda_0)}\int_0^{+\infty}\psi(s-c\tau_1)e^{-\lambda_0 s}ds\right]\\
&= \frac{p_1^c(\lambda_0)e^{\lambda_0 c\tau}}{g'(0)}\left[g'(0)e^{-\lambda_0 c\tau_1}\int_{c\tau_2}^{+\infty}\phi(s'-c\tau_2)e^{-\lambda_0 s'}ds' + p_2^c(\lambda_0)e^{-\lambda_0 c\tau_1}\int_{-c\tau_1}^{+\infty}\psi(s')e^{-\lambda_0 s'}ds'\right]\\
&= \frac{p_1^c(\lambda_0)e^{\lambda_0 c\tau_2}}{g'(0)}\left[g'(0)\int_0^{+\infty}\phi(s'-c\tau_2)e^{-\lambda_0 s'}ds' + p_2^c(\lambda_0)\int_0^{+\infty}\psi(s')e^{-\lambda_0 s'}ds'\right].
\end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we have $p_1^c(\lambda_0) > 0$. On the other hand, since $\lambda_0 \in (0, \lambda_m^c)$, we can deduce that $p_1^c(\lambda_0) < 0$, which is a contradiction. □

Next, we establish the existence of monotone solutions to (3.1) for $0 < c < c_\tau^*$.

In [10, 28], to get the monotone increasing solutions of semi-wave problems with one time delay $\tau > 0$, lower solutions were constructed by complex roots with imaginary part $\mathrm{Im}\lambda \in (0, \frac{\pi}{c\tau})$. However, we cannot derive a suitable upper bound of $\mathrm{Im}\lambda$ for the transcendental equation (3.2) in the above Lemma 3.1 (*ii*)-*b*, which together with the influence of two time delays (τ_1, τ_2) bring some difficulties in constructing lower solution of (3.1).

To get the monotone increasing solutions of (3.1), we first consider the corresponding perturbed problem

$$\begin{cases}
c\phi'(s) - d_1\phi''(s) = h(\psi(s-c\tau_1)) - b_1\phi(s), & s>0,\\
c\psi'(s) - d_2\psi''(s) = g(\phi(s-c\tau_2)) - b_2\psi(s), & s>0,\\
(\phi(s), \psi(s)) = (\delta u^*, \delta v^*), & s\leqslant 0,\\
(\phi(+\infty), \psi(+\infty)) = (u^*, v^*),
\end{cases} \tag{3.14}$$

where $\delta \in (0, \frac{1}{2})$ is a small parameter, and then get the desired solution of (3.1) by taking $\delta \to 0$. The idea of approximation is motivated by the works on semi-wave problems with nonlocal diffusion (see [14]).

In the following, we establish sufficient conditions for the existence of monotone solutions to (3.14) by employing the lower-upper solutions technique and the Schauder fixed point theorem, which was proposed in [25].

For a fixed

$$\sigma \in (0, \min_{i=1,2}\{-\lambda_i^-, \lambda_i^+\}),$$

we introduce the Banach space $(\mathbb{X}_\sigma(\mathbb{R}, \mathbb{R}^2), |\cdot|_\sigma)$ with

$$\mathbb{X}_\sigma(\mathbb{R}, \mathbb{R}^2) := \left\{\Phi \in C(\mathbb{R}, \mathbb{R}^2);\, |\Phi|_\sigma := \sup_{s\in\mathbb{R}} |\Phi(s)|e^{-\sigma|s|} < \infty\right\}.$$

For $i = 1, 2$, we define the mapping

$$F_i:\quad \mathbb{X}_\sigma(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R})$$
$$\Phi = (\phi, \psi) \mapsto F_i(\Phi)(s)$$

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by

$$F_1(\Phi)(s)=\begin{cases}\delta u^* e^{\lambda_1^- s}+\dfrac{1}{d_1(\lambda_1^+-\lambda_1^-)}\Big[\int_0^s (e^{\lambda_1^-(s-\xi)}-e^{\lambda_1^- s-\lambda_1^+\xi})h(\psi(\xi-c\tau_1))d\xi \\ \qquad\qquad + \int_s^{+\infty}(e^{\lambda_1^+(s-\xi)}-e^{\lambda_1^- s-\lambda_1^+\xi})h(\psi(\xi-c\tau_1))d\xi\Big], \quad s>0, \\ \delta u^*, \quad s\leqslant 0\end{cases} \tag{3.15}$$

and

$$F_2(\Phi)(s)=\begin{cases}\delta v^* e^{\lambda_1^- s}+\dfrac{1}{d_2(\lambda_2^+-\lambda_2^-)}\Big[\int_0^s (e^{\lambda_2^-(s-\xi)}-e^{\lambda_2^- s-\lambda_2^+\xi})g(\phi(\xi-c\tau_2))d\xi \\ \qquad\qquad + \int_s^{+\infty}(e^{\lambda_2^+(s-\xi)}-e^{\lambda_2^- s-\lambda_2^+\xi})g(\phi(\xi-c\tau_2))d\xi\Big], \quad s>0, \\ \delta v^*, \quad s\leqslant 0.\end{cases} \tag{3.16}$$

By direct calculations, we see that (F_1, F_2) is well-defined and satisfies

$$\begin{cases} c(F_1(\Phi))'(s)-d_1(F_1(\Phi))''(s)=h(\psi(s-c\tau_1))-b_1F_1(\Phi)(s), \quad s>0,\\ c(F_2(\Phi))'(s)-d_2(F_2(\Phi))''(s)=g(\phi(s-c\tau_2))-b_2F_2(\Phi)(s), \quad s>0,\\ (F_1(\Phi)(s),F_2(\Phi)(s))=(\delta u^*,\delta v^*), \quad s\leqslant 0.\end{cases}$$

Clearly, any fixed point of (F_1, F_2) in $\mathbb{X}_\sigma(\mathbb{R},\mathbb{R}^2)$ is a solution of (3.14). Moreover, the mapping (F_1, F_2) satisfies the following monotonicity properties.

Lemma 3.3. *(i) If $\Phi_1=(\phi_1,\psi_1)$, $\Phi_2=(\phi_2,\psi_2)\in\mathbb{X}_\sigma(\mathbb{R},\mathbb{R}^2)$ satisfy $\Phi_1\leqslant\Phi_2$, then $(F_1(\Phi_1)(s),F_2(\Phi_1)(s))\leqslant(F_1(\Phi_2)(s),F_2(\Phi_2)(s))$ for $s\in\mathbb{R}$.*

(ii) If $\Phi=(\phi,\psi)\in\mathbb{X}_\sigma(\mathbb{R},\mathbb{R}^2)$ is monotone increasing, then $(F_1(\Phi)(s),F_2(\Phi)(s))$ is monotone increasing in $s\in\mathbb{R}$ for all $\delta\in(0,1)$.

Proof. Since h, g are strictly increasing functions, we can directly get (i) from (3.15) and (3.16). Now we prove (ii). It is sufficient to consider the case $s>0$. Differentiating (3.15) with respect to s, we have

$$(F_1(\Phi))'(s)=\delta\lambda_1^- u^* e^{\lambda_1^- s}+\frac{1}{d_1(\lambda_1^+-\lambda_1^-)}\left[\int_0^s \lambda_1^-(e^{\lambda_1^-(s-\xi)}-e^{\lambda_1^- s-\lambda_1^+\xi})h(\psi(\xi-c\tau_1))d\xi\right.$$
$$\left.+\int_s^{+\infty}(\lambda_1^+e^{\lambda_1^+(s-\xi)}-\lambda_1^-e^{\lambda_1^- s-\lambda_1^+\xi})h(\psi(\xi-c\tau_1))d\xi\right].$$

Since $h(\psi(s-c\tau_1))$ is monotone increasing in $s\in\mathbb{R}$ and $h(\psi(s-c\tau_1))\geqslant h(\delta v^*)>0$, we have

$$\begin{aligned}(F_1(\Phi))'(s) &\geqslant \delta\lambda_1^- u^* e^{\lambda_1^- s}+\frac{1}{d_1(\lambda_1^+-\lambda_1^-)}\Big[\int_0^s \lambda_1^- e^{\lambda_1^- s}(e^{-\lambda_1^-\xi}-e^{-\lambda_1^+\xi})h(\psi(s-c\tau_1))d\xi \\ &\qquad +\int_s^{+\infty}(\lambda_1^+e^{\lambda_1^+(s-\xi)}-\lambda_1^-e^{\lambda_1^- s-\lambda_1^+\xi})h(\psi(s-c\tau_1))d\xi\Big] \\ &=\delta\lambda_1^- u^* e^{\lambda_1^- s}+\frac{h(\psi(s-c\tau_1))}{d_1(\lambda_1^+-\lambda_1^-)}\Big[\lambda_1^- e^{\lambda_1^- s}\int_0^s (e^{-\lambda_1^-\xi}-e^{-\lambda_1^+\xi})d\xi \\ &\qquad +\int_s^{+\infty}(\lambda_1^+e^{\lambda_1^+(s-\xi)}-\lambda_1^-e^{\lambda_1^- s-\lambda_1^+\xi})d\xi\Big] \\ &=\delta\lambda_1^- u^* e^{\lambda_1^- s}+\frac{h(\psi(s-c\tau_1))}{d_1\lambda_1^+}e^{\lambda_1^- s} \\ &\geqslant \delta\lambda_1^- u^* e^{\lambda_1^- s}+\frac{h(\delta v^*)}{d_1\lambda_1^+}e^{\lambda_1^- s}.\end{aligned}$$

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Since $\delta \in (0, 1)$, we have $h(\delta v^*) \geqslant \delta h(v^*)$, which together with the fact $b_1 u^* = h(v^*)$ imply that

$$\begin{aligned}(F_1(\Phi))'(s) &\geqslant \delta\lambda_1^- u^* e^{\lambda_1^- s} + \delta \frac{h(v^*)}{d_1\lambda_1^+} e^{\lambda_1^- s} = \delta\lambda_1^- u^* e^{\lambda_1^- s} + \delta \frac{b_1 u^*}{d_1 \lambda_1^+} e^{\lambda_1^- s} \\ &= \delta u^* e^{\lambda_1^- s}(\lambda_1^- + \frac{b_1}{d_1\lambda_1^+}) = 0.\end{aligned}$$

This completes the proof. □

Now we introduce the definition of lower and upper solutions to (3.14). If $(\underline{\phi}, \underline{\psi})$ and $(\bar{\phi}, \bar{\psi})$ are continuous from $\mathbb{R}$ into $[\delta u^*, u^*] \times [\delta v^*, v^*]$, twice continuously differentiable on $\overline{\mathbb{R}_+} \backslash \{\xi_i\}_{i=1}^n$, and satisfy

$$\begin{cases} c\underline{\phi}'(s) - d_1\underline{\phi}''(s) \leqslant h(\underline{\psi}(s - c\tau_1)) - b_1\underline{\phi}(s), & s > 0, s \notin \{\xi_i\}_{i=1}^n, \\ c\underline{\psi}'(s) - d_2\underline{\psi}''(s) \leqslant g(\underline{\phi}(s - c\tau_2)) - b_2\underline{\psi}(s), & s > 0, s \notin \{\xi_i\}_{i=1}^n, \\ \underline{\phi}'_+(\xi_i) \geqslant \underline{\phi}'_-(\xi_i), \quad \underline{\psi}'_+(\xi_i) \geqslant \underline{\psi}'_-(\xi_i), & 1 \leqslant i \leqslant n, \\ (\underline{\phi}(s), \underline{\psi}(s)) = (\delta u^*, \delta v^*), & s \leqslant 0, \\ (\underline{\phi}(+\infty), \underline{\psi}(+\infty)) \leqslant (u^*, v^*) \end{cases}$$

and

$$\begin{cases} c\bar{\phi}'(s) - d_1\bar{\phi}''(s) \geqslant h(\bar{\psi}(s - c\tau_1)) - b_1\bar{\phi}(s), & s > 0, s \notin \{\xi_i\}_{i=1}^n, \\ c\bar{\psi}'(s) - d_2\bar{\psi}''(s) \geqslant g(\bar{\phi}(s - c\tau_2)) - b_2\bar{\psi}(s), & s > 0, s \notin \{\xi_i\}_{i=1}^n, \\ \bar{\phi}'_+(\xi_i) \leqslant \bar{\phi}'_-(\xi_i), \quad \bar{\psi}'_+(\xi_i) \leqslant \bar{\psi}'_-(\xi_i), & 1 \leqslant i \leqslant n, \\ (\bar{\phi}(s), \bar{\psi}(s)) = (\delta u^*, \delta v^*), & s \leqslant 0, \\ (\bar{\phi}(+\infty), \bar{\psi}(+\infty)) = (u^*, v^*), \end{cases}$$

then $(\underline{\phi}, \underline{\psi})$ and $(\bar{\phi}, \bar{\psi})$ are called a *lower solution* and an *upper solution* of (3.14), respectively.

Given a pair of lower and upper solutions $(\underline{\phi}, \underline{\psi})$ and $(\bar{\phi}, \bar{\psi})$ satisfying

$$\sup_{t\leqslant s} (\underline{\phi}(t), \underline{\psi}(t)) \leqslant (\bar{\phi}(s), \bar{\psi}(s)), \quad \forall s \in \mathbb{R},$$

then the set

$$\begin{aligned}\Gamma &:= \left\{(\phi, \psi) \in \mathbb{X}_\sigma(\mathbb{R}, \mathbb{R}^2) : (\underline{\phi}, \underline{\psi}) \leqslant (\phi, \psi) \leqslant (\bar{\phi}, \bar{\psi}) \text{ and } (\phi, \psi) \text{ is monotone increasing on } \mathbb{R}_+\right\} \\ &\neq \emptyset\end{aligned}$$

since $(\phi(s), \psi(s)) = \sup_{t\leqslant s} (\underline{\phi}(t), \underline{\psi}(t)) \in \Gamma$. Next, we prove that (F_1, F_2) maps Γ into itself and is completely continuous.

Lemma 3.4. *$(F_1, F_2)(\Gamma) \subset \Gamma$ and (F_1, F_2) is a continuous compact mapping on Γ.*

Proof. We first prove $(F_1, F_2)(\Gamma) \subset \Gamma$. Let $\underline{\Phi}(s) = (\underline{\phi}(s), \underline{\psi}(s))$ and $\bar{\Phi}(s) = (\bar{\phi}(s), \bar{\psi}(s))$. By Lemma 3.3, it is sufficient to prove that $\underline{\Phi}(s) \leqslant (F_1(\underline{\Phi})(s), F_2(\underline{\Phi})(s)) \leqslant (F_1(\bar{\Phi})(s), F_2(\bar{\Phi})(s)) \leqslant \bar{\Phi}(s)$ for all $s \in \mathbb{R}_+$. Let $\xi_0 = 0$ and $\xi_{m+1} = \infty$. Assume that $s \in (\xi_i, \xi_{i+1})$ for some i, then we have

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$$\begin{aligned}
F_1(\underline{\Phi})(s) &= \delta u^* e^{\lambda_1^- s} + \frac{1}{d_1(\lambda_1^+ - \lambda_1^-)}\Big[\int_0^s (e^{\lambda_1^-(s-\xi)} - e^{\lambda_1^- s - \lambda_1^+ \xi}) h(\underline{\psi}(\xi - c\tau_1))d\xi \\
&\qquad + \int_s^{+\infty} (e^{\lambda_1^+(s-\xi)} - e^{\lambda_1^- s - \lambda_1^+ \xi}) h(\underline{\psi}(\xi - c\tau_1))d\xi\Big] \\
&\geqslant \delta u^* e^{\lambda_1^- s} + \frac{1}{d_1(\lambda_1^+ - \lambda_1^-)}\Big[\int_0^s (e^{\lambda_1^-(s-\xi)} - e^{\lambda_1^- s - \lambda_1^+ \xi})(b_1\underline{\phi}(\xi) - d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi))d\xi \\
&\qquad + \int_s^{+\infty} (e^{\lambda_1^+(s-\xi)} - e^{\lambda_1^- s - \lambda_1^+ \xi})(b_1\underline{\phi}(\xi) - d_1\underline{\phi}''(\xi) + c\underline{\phi}'(\xi))d\xi\Big] \\
&= \delta u^* e^{\lambda_1^- s} - e^{\lambda_1^- s}\underline{\phi}(0) + \underline{\phi}(s) + \frac{1}{(\lambda_1^+ - \lambda_1^-)}\Big[\sum_{k=1}^{i} (e^{\lambda_1^-(s-\xi_k)} - e^{\lambda_1^- s - \lambda_1^+ \xi_k})(\underline{\phi}'_+(\xi_k) - \underline{\phi}'_-(\xi_k)) \\
&\qquad + \sum_{k=i+1}^{m} (e^{\lambda_1^+(s-\xi_k)} - e^{\lambda_1^- s - \lambda_1^+ \xi_k})(\underline{\phi}'_+(\xi_k) - \underline{\phi}'_-(\xi_k))\Big] \\
&\geqslant \underline{\phi}(s).
\end{aligned}$$

Similarly, we can prove that $F_2(\underline{\Phi})(s) \geqslant \underline{\psi}(s)$ and $(F_1(\bar{\Phi})(s), F_2(\bar{\Phi})(s)) \leqslant (\bar{\phi}(s), \bar{\psi}(s))$ for all $s \in (\xi_i, \xi_{i+1})$. By the continuity, we can get the same results for the endpoints ξ_i $(i = 1, \cdots, m)$. Thus, $(F_1, F_2)(\Gamma) \subset \Gamma$.

The continuity and compactness of (F_1, F_2) can be obtained by similar arguments in Lemmas 2.4–2.5 of [35], here we omit the details. □

Lemma 3.5. *Suppose that (3.14) admits a pair of lower and upper solutions* $(\underline{\phi}(s), \underline{\psi}(s))$ *and* $(\bar{\phi}(s), \bar{\psi}(s))$ *satisfying* $\sup_{t \leqslant s}(\underline{\phi}(t), \underline{\psi}(t)) \leqslant (\bar{\phi}(s), \bar{\psi}(s))$, $\forall s \in \mathbb{R}$. *Then, (3.14) has a monotone increasing solution.*

Proof. From Lemma 3.4, (F_1, F_2) has a fixed point $\Phi = (\phi, \psi) \in \Gamma$ by the Schauder's fixed point theorem. Note that $(\bar{\phi}(s), \bar{\psi}(s)) = (\underline{\phi}(s), \underline{\psi}(s)) = (\delta u^*, \delta v^*)$ for $s \leqslant 0$. We have $(\phi(s), \psi(s)) = (\delta u^*, \delta v^*)$ for $s \leqslant 0$. Since (ϕ, ψ) is monotone increasing and bounded, $\lim_{s\to+\infty}(\phi(s), \psi(s))$ is well-defined. By Lemma 2.3 in [37], we can deduce that $\lim_{s\to+\infty}(\phi(s), \psi(s)) = (u^*, v^*)$, which implies $\Phi = (\phi, \psi) \in \Gamma$ is a monotone increasing solution of (3.14). □

Theorem 3.1. *For any fixed* $c > 0$*, the perturbed problem (3.14) has a monotone increasing solution* $(\phi^\delta(s), \psi^\delta(s))$*. Moreover,* $(\phi^\delta(s), \psi^\delta(s))$ *obtained in this way is strictly increasing in* $\delta \in (0, \frac{1}{2})$.

Proof. By Lemma 3.5, it is sufficient to construct a pair of lower and upper solutions of (3.14).

Take

$$(\underline{\phi}(s), \underline{\psi}(s)) = (\delta u^*, \delta v^*), \quad s \in \mathbb{R}.$$

Since

$$h(\underline{\psi}(s - c\tau_1)) = h(\delta v^*) \geqslant \delta h(v^*), \quad g(\underline{\phi}(s - c\tau_1)) = g(\delta u^*) \geqslant \delta g(u^*),$$

we can check that $(\underline{\phi}(s), \underline{\psi}(s))$ satisfies

$$\begin{cases}
c\underline{\phi}'(s) - d_1\underline{\phi}''(s) = 0 \leqslant h(\underline{\psi}(s - c\tau_1)) - b_1\underline{\phi}(s), & s > 0, \\
c\underline{\psi}'(s) - d_2\underline{\psi}''(s) = 0 \leqslant g(\underline{\phi}(s - c\tau_2)) - b_2\underline{\psi}(s), & s > 0, \\
(\underline{\phi}(s), \underline{\psi}(s)) = (\delta u^*, \delta v^*), & s \leqslant 0, \\
(\underline{\phi}(+\infty), \underline{\psi}(+\infty)) \leqslant (u^*, v^*).
\end{cases}$$

Thus, $(\underline{\phi}(s), \underline{\psi}(s))$ is a lower solution of (3.14).

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Now, we construct an upper solution of (3.14). Choose $k>0$ large enough such that

$$0<\frac{1}{k}<\min\left\{c\tau_1, c\tau_2, \frac{c}{b_1}, \frac{c}{b_2}\right\}.$$

Define

$$\bar{\phi}(s)=\begin{cases}\delta u^*, & s\leqslant 0,\\ u^*\left[-k^2(1-\delta)\left(s-\frac{1}{k}\right)^2+1\right], & 0<s\leqslant\frac{1}{k},\\ u^*, & s>\frac{1}{k}\end{cases}$$

and

$$\bar{\psi}(s)=\begin{cases}\delta v^*, & s\leqslant 0,\\ v^*\left[-k^2(1-\delta)\left(s-\frac{1}{k}\right)^2+1\right], & 0<s\leqslant\frac{1}{k},\\ v^*, & s>\frac{1}{k}.\end{cases}$$

Obviously, $(\bar{\phi}(s), \bar{\psi}(s))$ satisfies $(\bar{\phi}(s), \bar{\psi}(s))=(\delta u^*, \delta v^*)$ for $s\leqslant 0$ and $(\bar{\phi}(+\infty), \bar{\psi}(+\infty))=(u^*, v^*)$. We claim that $(\bar{\phi}, \bar{\psi})$ is an upper solution of (3.14). In fact, for $0<s<\frac{1}{k}$, we have $\bar{\psi}(s-c\tau_1)=\delta v^*$ and

$$\begin{aligned}&c\bar{\phi}'(s)-d_1\bar{\phi}''(s)-h(\bar{\psi}(s-c\tau_1))+b_1\bar{\phi}(s)\\ &=u^*k^2(1-\delta)\left[2d_1-2c\left(s-\frac{1}{k}\right)-b_1\left(s-\frac{1}{k}\right)^2\right]+b_1u^*-h(\delta v^*)\\ &=u^*k^2(1-\delta)\left[2d_1-2c\left(s-\frac{1}{k}\right)-b_1\left(s-\frac{1}{k}\right)^2\right]+h(v^*)-h(\delta v^*).\end{aligned}$$

Since $-\frac{c}{b_1}<-\frac{1}{k}\leqslant s-\frac{1}{k}<0$, we can deduce $2d_1-2c(s-\frac{1}{k})-b_1(s-\frac{1}{k})^2>0$ and then

$$c\bar{\phi}'(s)-d_1\bar{\phi}''(s)-h(\bar{\psi}(s-c\tau_1))+b_1\bar{\phi}(s)>h(v^*)-h(\delta v^*)\geqslant 0.$$

For $s>\frac{1}{k}$, we have

$$\begin{aligned}c\bar{\phi}'(s)-d_1\bar{\phi}''(s)-h(\bar{\psi}(s-c\tau_1))+b_1\bar{\phi}(s)&=-h(\bar{\psi}(s-c\tau_1))+b_1u^*\\ &=-h(\bar{\psi}(s-c\tau_1))+h(v^*)\geqslant 0.\end{aligned}$$

Thus,

$$c\bar{\phi}'(s)-d_1\bar{\phi}''(s)-h(\bar{\psi}(s-c\tau_1))+b_1\bar{\phi}(s)\geqslant 0\quad \text{for } s\in\mathbb{R}_+\setminus\left\{\frac{1}{k}\right\}.$$

We can similarly obtain

$$c\bar{\psi}'(s)-d_2\bar{\psi}''(s)-g(\bar{\phi}(s-c\tau_2))+b_2\bar{\psi}(s)\geqslant 0\quad \text{for } s\in\mathbb{R}_+\setminus\left\{\frac{1}{k}\right\}.$$

Moreover, $\bar{\phi}'_+(\frac{1}{k})=\bar{\phi}'_-(\frac{1}{k})=0$ and $\bar{\psi}'_+(\frac{1}{k})=\bar{\psi}'_-(\frac{1}{k})=0$. Therefore, $(\bar{\phi}, \bar{\psi})$ is an upper solution of (3.14). It follows that the perturbed problem (3.14) admits a monotone increasing solution $(\phi^\delta(s), \psi^\delta(s))$.

Next, we prove that if $\delta_1, \delta_2\in(0, \frac{1}{2})$ and $\delta_1>\delta_2$, then the solutions obtained in the above way satisfy $(\phi^{\delta_1}(s), \psi^{\delta_1}(s))>(\phi^{\delta_2}(s), \psi^{\delta_2}(s))$ for all $s\in\mathbb{R}$. Let $\tilde{\phi}(s)=\phi^{\delta_1}(s)-\phi^{\delta_2}(s)$ and $\tilde{\psi}(s)=\psi^{\delta_1}(s)-\psi^{\delta_2}(s)$. Without loss of generality, we assume $\tau_1<\tau_2$.

First, we consider $s\in[0, c\tau_1]$. For this case, $h(\psi^{\delta_i}(s-c\tau_1))=h(\delta_i v^*)$ and $g(\phi^{\delta_i}(s-c\tau_2))=g(\delta_i u^*)$. Then, $(\tilde{\phi}, \tilde{\psi})$ satisfies

$$\begin{cases}c\tilde{\phi}'(s)-d_1\tilde{\phi}''(s)=h(\delta_1v^*)-h(\delta_2v^*)-b_1\tilde{\phi}(s)>-b_1\tilde{\phi}(s), & 0<s\leqslant c\tau_1,\\ c\tilde{\psi}'(s)-d_2\tilde{\psi}''(s)=g(\delta_1u^*)-g(\delta_2u^*)-b_2\tilde{\psi}(s)>-b_2\tilde{\psi}(s), & 0<s\leqslant c\tau_1,\\ (\tilde{\phi}(0), \tilde{\psi}(0))=(\delta_1u^*-\delta_2u^*, \delta_1v^*-\delta_2v^*)>(0,0).\end{cases}$$

https://doi.org/10.1017/S0956792523000220 Published online by Cambridge University Press

By the maximum principle, we have $\tilde{\phi}(s) > 0$ and $\tilde{\psi}(s) > 0$ for $s \in [0, c\tau_1]$. Thus, $\phi^{\delta_1}(s) > \phi^{\delta_2}(s)$ and $\psi^{\delta_1}(s) > \psi^{\delta_2}(s)$ for $s \in (-\infty, c\tau_1]$.

For $s \in [c\tau_1, 2c\tau_1]$, we have $s - c\tau_1, s - c\tau_2 \in (-\infty, c\tau_1]$. By the above result, $h(\psi^{\delta_1}(s - c\tau_1)) > h(\psi^{\delta_2}(s - c\tau_1))$ and $g(\phi^{\delta_1}(s - c\tau_2)) > g(\phi^{\delta_2}(s - c\tau_2))$. It follows that $(\tilde{\phi}, \tilde{\psi})$ satisfies

$$\begin{cases} c\tilde{\phi}'(s) - d_1\tilde{\phi}''(s) > -b_1\tilde{\phi}(s), & c\tau_1 < s \leqslant 2c\tau_1, \\ c\tilde{\psi}'(s) - d_2\tilde{\psi}''(s) > -b_2\tilde{\psi}(s), & c\tau_1 < s \leqslant 2c\tau_1, \\ (\tilde{\phi}(c\tau_1), \tilde{\psi}(c\tau_1)) > (0, 0). \end{cases}$$

By the maximum principle, we have $\tilde{\phi}(s) > 0$ and $\tilde{\psi}(s) > 0$ for $s \in [c\tau_1, 2c\tau_1]$, which imply that $\phi^{\delta_1}(s) > \phi^{\delta_2}(s)$ and $\psi^{\delta_1}(s) > \psi^{\delta_2}(s)$ for $s \in (-\infty, 2c\tau_1]$.

By repeating the above steps, we conclude that $(\phi^{\delta_1}(s), \psi^{\delta_1}(s)) > (\phi^{\delta_2}(s), \psi^{\delta_2}(s))$ for $s \in \mathbb{R}$. □

In what follows, we exhibit the existence and uniqueness of monotone increasing solutions to (3.1).

Lemma 3.6. *For any $c \in (0, c_\tau^*)$, (3.1) has a unique monotone increasing solution, which is strictly increasing on $\mathbb{R}_+$.*

Proof. (*i*) (*Uniqueness of Monotone Increasing Solutions*) Suppose that (3.1) has two monotone increasing solutions (ϕ_1, ψ_1) and (ϕ_2, ψ_2). Then $(0, 0) < (\phi_i, \psi_i) < (u^*, v^*)$ in $(0, +\infty)$ and $(\phi_i(+\infty), \psi_i(+\infty)) = (u^*, v^*)$ for $i = 1, 2$. The Hopf boundary lemma implies that $((\phi_i)'_+(0), (\psi_i)'_+(0)) > (0, 0)$ for $i = 1, 2$.

Define

$$\rho_1 := \inf\left\{\frac{\phi_1(s)}{\phi_2(s)}\bigg| s > 0\right\}, \quad \rho_2 := \inf\left\{\frac{\psi_1(s)}{\psi_2(s)}\bigg| s > 0\right\}$$

and

$$\rho^* := \min\{\rho_1, \rho_2\}.$$

By the L'Hôpital's rule, it is easy to see that

$$\lim_{s \to 0^+}\left(\frac{\phi_1(s)}{\phi_2(s)}, \frac{\psi_1(s)}{\psi_2(s)}\right) = \left(\frac{(\phi_1)'_+(0)}{(\phi_2)'_+(0)}, \frac{(\psi_1)'_+(0)}{(\psi_2)'_+(0)}\right) > (0, 0).$$

Moreover, $\lim_{s \to +\infty}\left(\frac{\phi_1(s)}{\phi_2(s)}, \frac{\psi_1(s)}{\psi_2(s)}\right) = (1, 1)$. Thus, ρ_1, ρ_2 are well-defined and $\rho^* \in (0, 1]$.

We claim that $\rho^* \equiv 1$. Assume on the contrary that $\rho^* \in (0, 1)$. Let

$$\tilde{\phi}(s) = \phi_1(s) - \rho^*\phi_2(s), \quad \tilde{\psi}(s) = \psi_1(s) - \rho^*\psi_2(s).$$

Obviously, $(\tilde{\phi}(s), \tilde{\psi}(s)) \geqslant (0, 0)$ for $s > 0$, $(\tilde{\phi}(0), \tilde{\psi}(0)) = (0, 0)$, $(\tilde{\phi}(+\infty), \tilde{\psi}(+\infty)) = ((1 - \rho^*)u^*, (1 - \rho^*)v^*) > (0, 0)$, and

$$\begin{aligned} &c\tilde{\phi}'(s) - d_1\tilde{\phi}''(s) + b_1\tilde{\phi}(s) \\ &\quad = h(\psi_1(s - c\tau_1)) - \rho^* h(\psi_2(s - c\tau_1)) \\ &\quad \geqslant h(\psi_1(s - c\tau_1)) - h(\rho^*\psi_2(s - c\tau_1)) \geqslant 0 \quad \text{for } s > 0. \end{aligned}$$

Similarly, we have

$$c\tilde{\psi}'(s) - d_2\tilde{\psi}''(s) + b_2\tilde{\psi}(s) \geqslant g(\phi_1(s - c\tau_2)) - g(\rho^*\phi_2(s - c\tau_2)) \geqslant 0 \quad \text{for } s > 0.$$

From the Hopf boundary lemma, we can deduce that $(0, 0) < (\tilde{\phi}'_+(0), \tilde{\psi}'_+(0)) = ((\phi_1)'_+(0) - \rho^*(\phi_2)'_+(0), (\psi_1)'_+(0) - \rho^*(\psi_2)'_+(0))$, which implies

$$\lim_{s \to 0^+}\left(\frac{\phi_1(s)}{\phi_2(s)}, \frac{\psi_1(s)}{\psi_2(s)}\right) > (\rho^*, \rho^*).$$

Therefore, by the definition of ρ^*, there exists $s_0 \in (0, +\infty)$ such that $\tilde{\phi}(s_0) = 0$ or $\tilde{\psi}(s_0) = 0$. By the maximum principle, we know $\tilde{\phi}(s) \equiv 0$ or $\tilde{\psi}(s) \equiv 0$ for $s \geqslant 0$, which is a contradiction since $(\tilde{\phi}(+\infty), \tilde{\psi}(+$

$\infty)) = ((1-\rho^*)u^*, (1-\rho^*)v^*) > (0,0)$. Thus, $\rho^* \equiv 1$. It follows that $(\phi_1(s), \psi_1(s)) \geqslant (\phi_2(s), \psi_2(s))$ for $s > 0$.

Clearly, the same method can be employed to prove $(\phi_1(s), \psi_1(s)) \leqslant (\phi_2(s), \psi_2(s))$ for $s > 0$. Hence, the uniqueness follows directly.

(*ii*) (*Strict Monotonicity of Solution on* $\mathbb{R}_+$) Assume that (ϕ, ψ) is a monotone increasing solution of (3.1). For any $\vartheta > 0$, by (3.11), we get

$$\begin{aligned}
\phi(s+\vartheta) &= \frac{1}{d_1(\lambda_1^+ - \lambda_1^-)}\Big[\int_0^{s+\vartheta} (e^{\lambda_1^-(s+\vartheta-\xi)} - e^{\lambda_1^-(s+\vartheta)-\lambda_1^+\xi})h(\psi(\xi - c\tau_1))d\xi \\
&\qquad + \int_{s+\vartheta}^{+\infty} (e^{\lambda_1^+(s+\vartheta-\xi)} - e^{\lambda_1^-(s+\vartheta)-\lambda_1^+\xi})h(\psi(\xi - c\tau_1))d\xi\Big] \\
&= \frac{1}{d_1(\lambda_1^+ - \lambda_1^-)}\Big[\int_{-\vartheta}^{s} (e^{\lambda_1^-(s-\xi')} - e^{\lambda_1^-(s+\vartheta)-\lambda_1^+(\xi'+\vartheta)})h(\psi(\xi' + \vartheta - c\tau_1))d\xi' \\
&\qquad + \int_{s}^{+\infty} (e^{\lambda_1^+(s-\xi')} - e^{\lambda_1^-(s+\vartheta)-\lambda_1^+(\xi'+\vartheta)})h(\psi(\xi' + \vartheta - c\tau_1))d\xi'\Big] \\
&> \frac{1}{d_1(\lambda_1^+ - \lambda_1^-)}\Big[\int_0^{s} (e^{\lambda_1^-(s-\xi')} - e^{\lambda_1^- s-\lambda_1^+\xi'})h(\psi(\xi' - c\tau_1))d\xi' \\
&\qquad + \int_{s}^{+\infty} (e^{\lambda_1^+(s-\xi')} - e^{\lambda_1^- s-\lambda_1^+\xi'})h(\psi(\xi' - c\tau_1))d\xi'\Big] \\
&= \phi(s), \quad \forall s \in \mathbb{R}_+.
\end{aligned}$$

Similarly, we can prove $\psi(s+\vartheta) > \psi(s)$ on $\mathbb{R}_+$ for any $\vartheta > 0$.

(*iii*) (*Existence of Monotone Increasing Solution*) For the existence, we divide the proof into the following steps.

Step 1. For any fixed $c > 0$, we prove that the equations

$$\begin{cases} c\phi'(s) - d_1\phi''(s) = h(\psi(s - c\tau_1)) - b_1\phi(s), & s \in \mathbb{R}, \\ c\psi'(s) - d_2\psi''(s) = g(\phi(s - c\tau_2)) - b_2\psi(s), & s \in \mathbb{R} \end{cases}$$

have a monotone increasing solution (ϕ, ψ) satisfying either

$$(a) \qquad (\phi(-\infty), \psi(-\infty)) = (0,0), \quad (\phi(+\infty), \psi(+\infty)) = (u^*, v^*),$$

or

$$(b) \quad (\phi(s), \psi(s)) = (0,0) \quad \text{for } s \leqslant 0, \quad (\phi(+\infty), \psi(+\infty)) = (u^*, v^*).$$

For the case (a), $(u(t,x), v(t,x)) := (\phi(x+ct), \psi(x+ct))$ is a travelling wave solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\dfrac{\partial^2 u}{\partial x^2} - b_1 u + h(v(t-\tau_1, x)), & t > 0, x \in \mathbb{R}, \\ \dfrac{\partial v}{\partial t} = d_2\dfrac{\partial^2 v}{\partial x^2} - b_2 v + g(u(t-\tau_2, x)), & t > 0, x \in \mathbb{R}. \end{cases} \tag{3.17}$$

For the case (b), $(\phi(s), \psi(s))$ is a monotone increasing solution of (3.1), and $(u(t,x), v(t,x)) := (\phi(x+ct), \psi(x+ct))$ is called a semi-wave solution of (3.17). Thus, the conclusion of this step implies that (3.17) has either a monotone travelling wave or a monotone semi-wave with speed c for any fixed $c > 0$.

Take a sequence $\{\delta_n\}_{n=1}^{\infty}$ satisfying $\delta_n \in (0, \frac{1}{2})$ and $\delta_n \searrow 0$ as $n \to \infty$, and let (ϕ_n, ψ_n) be the monotone increasing solution of (3.14) with δ replaced by δ_n, which is obtained in Theorem 3.1. Then, $s_n := \max\{s : \phi_n(s) = \frac{1}{2}u^*\}$ is well-defined. By Theorem 3.1, we know that s_n is monotone increasing in n.

Define $(\tilde{\phi}_n(s), \tilde{\psi}_n(s)) := (\phi_n(s+s_n), \psi_n(s+s_n))$ for $s \in \mathbb{R}$. Then, $\tilde{\phi}_n(0) = \frac{1}{2}u^*$ and $(\tilde{\phi}_n(s), \tilde{\psi}_n(s))$ satisfies

$$\begin{cases} c\tilde{\phi}_n'(s) - d_1\tilde{\phi}_n''(s) = h(\tilde{\psi}_n(s - c\tau_1)) - b_1\tilde{\phi}_n(s), \quad s > -s_n, \\ c\tilde{\psi}_n'(s) - d_2\tilde{\psi}_n''(s) = g(\tilde{\phi}_n(s - c\tau_2)) - b_2\tilde{\psi}_n(s), \quad s > -s_n, \\ (\tilde{\phi}_n(s), \tilde{\psi}_n(s)) = (\delta_n u^*, \delta_n v^*), \quad s \leqslant -s_n, \\ (\tilde{\phi}_n(+\infty), \tilde{\psi}_n(+\infty)) = (u^*, v^*). \end{cases}$$

Since s_n is monotone increasing in n, we have $s_0 := \lim_{n\to\infty} s_n \in (0, +\infty]$.

From the proof of Lemma 3.2, we know that there exists a positive constant C independent of n such that

$$|\phi_n(s)|, |\phi_n'(s)|, |\phi_n''(s)|, |\psi_n(s)|, |\psi_n'(s)|, |\psi_n''(s)| \leqslant C$$

for all $s \in \mathbb{R}_+$. That is, ϕ_n and ψ_n are uniformly bounded in $C^2(\mathbb{R}_+)$, from which we can deduce that $\phi_n(s)$, $\phi_n'(s)$, $\psi_n(s)$ and $\psi_n'(s)$ are equicontinuous on $\mathbb{R}_+$. Applying the equations in (3.14), we know that $\phi_n''(s)$ and $\psi_n''(s)$ are also equicontinuous on $\mathbb{R}_+$. By the Arzela–Ascoli theorem, there is a subsequence, still denoted by $(\tilde{\phi}_n, \tilde{\psi}_n)$, converges to $(\tilde{\phi}, \tilde{\psi})$ in $C^2_{loc}(\mathbb{R})$. It is easy to know that $(\tilde{\phi}(s), \tilde{\psi}(s))$ is monotone increasing in s and satisfies $\tilde{\phi}(0) = \frac{1}{2}u^*$.

(a) If $s_0 = +\infty$, then $(\tilde{\phi}(s), \tilde{\psi}(s))$ satisfies

$$\begin{cases} c\tilde{\phi}'(s) - d_1\tilde{\phi}''(s) = h(\tilde{\psi}(s - c\tau_1)) - b_1\tilde{\phi}(s), \quad s \in \mathbb{R}, \\ c\tilde{\psi}'(s) - d_2\tilde{\psi}''(s) = g(\tilde{\phi}(s - c\tau_2)) - b_2\tilde{\psi}(s), \quad s \in \mathbb{R}. \end{cases}$$

Since $(\tilde{\phi}(s), \tilde{\psi}(s))$ is monotone increasing and uniformly continuous on $\mathbb{R}_+$, by Lemma 2.3 in [37] we can deduce that $\lim_{s\to\infty} \tilde{\phi}'(s) = \lim_{s\to\infty} \tilde{\phi}''(s) = 0$ and $\lim_{s\to\infty} \tilde{\psi}'(s) = \lim_{s\to\infty} \tilde{\psi}''(s) = 0$, which imply $(\tilde{\phi}(\pm\infty), \tilde{\psi}(\pm\infty)) = (0, 0)$ or (u^*, v^*). In view of $\tilde{\phi}(0) = \frac{1}{2}u^*$, we know that $(\tilde{\phi}(-\infty), \tilde{\psi}(-\infty)) = (0, 0)$ and $(\tilde{\phi}(+\infty), \tilde{\psi}(+\infty)) = (u^*, v^*)$.

(b) If $s_0 \in (0, +\infty)$, then $(\tilde{\phi}(s), \tilde{\psi}(s))$ satisfies

$$\begin{cases} c\tilde{\phi}'(s) - d_1\tilde{\phi}''(s) = h(\tilde{\psi}(s - c\tau_1)) - b_1\tilde{\phi}(s), \quad s > -s_0, \\ c\tilde{\psi}'(s) - d_2\tilde{\psi}''(s) = g(\tilde{\phi}(s - c\tau_2)) - b_2\tilde{\psi}(s), \quad s > -s_0, \\ (\tilde{\phi}(s), \tilde{\psi}(s)) = (0, 0), \quad s \leqslant -s_0. \end{cases}$$

Let $(\phi(s), \psi(s)) = (\tilde{\phi}(s - s_0), \tilde{\psi}(s - s_0))$, we can also prove $(\phi(+\infty), \psi(+\infty)) = (u^*, v^*)$. Obviously, $(\phi(s), \psi(s)) = (0, 0)$ for $s \leqslant 0$. This completes the proof of Step 1.

Step 2. For any $c \in (0, c_\tau^*)$, we prove that (3.17) has no travelling wave solution.

We claim that c_τ^* is the minimal wave speed for travelling wave solutions of (3.17), that is, (3.17) admits a monotone travelling wave solution for $c \geqslant c_\tau^*$ and has no monotone travelling wave solution for $c \in (0, c_\tau^*)$.

Indeed, for the quasi-monotone system (3.17), applying the theory of monotone semiflows developed by Liang and Zhao [22], we can prove that c_τ^* is the asymptotic spreading speed and coincides with the minimal wave speed. The proof is similar as that of Theorem 5.1 in [22], Theorems 2.1–2.2 in [17] and Theorem 3.1 in [21]. The key step in the proof is verifying the conditions in [22].

We first introduce some notations. Let $\mathcal{C} = C([-\tau_2, 0] \times \mathbb{R}, \mathbb{R}) \times C([-\tau_1, 0] \times \mathbb{R}, \mathbb{R})$, $\bar{\mathcal{C}} = C([-\tau_2, 0], \mathbb{R}) \times C([-\tau_1, 0], \mathbb{R})$, $X = (C \cap L^\infty)(\mathbb{R}, \mathbb{R}^2)$,

$$\mathcal{C}_{(u^*,v^*)} = \left\{(\varphi_1, \varphi_2) \in \mathcal{C}; 0 \leqslant \varphi_1(\theta, x) \leqslant u^* \text{on } [-\tau_2, 0] \times \mathbb{R}, 0 \leqslant \varphi_2(\theta, x) \leqslant v^* \text{on } [-\tau_1, 0] \times \mathbb{R}\right\}$$

and

$$\bar{\mathcal{C}}_{(u^*,v^*)} = \left\{(\varphi_1, \varphi_2) \in \bar{\mathcal{C}}; 0 \leqslant \varphi_1(\theta) \leqslant u^* \text{on } [-\tau_2, 0], 0 \leqslant \varphi_2(\theta) \leqslant v^* \text{on } [-\tau_1, 0]\right\}.$$

https://doi.org/10.1017/S0956792523000220 Published online by Cambridge University Press

Clearly, any element in $\bar{\mathcal{C}}$ or X can be regarded as a function in $\mathcal{C}$. We equip $\mathcal{C}$ with the compact open topology and define the metric function $d(\cdot,\cdot)$ in $\mathcal{C}$ with respect to this topology by

$$d((\varphi_1,\varphi_2),(\tilde{\varphi}_1,\tilde{\varphi}_2)) = \sum_{k=1}^{\infty} \frac{\max_{\theta\in[-\tau_2,0],|x|\leqslant k} |\varphi_1(\theta,x)-\tilde{\varphi}_1(\theta,x)| + \max_{\theta\in[-\tau_1,0],|x|\leqslant k} |\varphi_2(\theta,x)-\tilde{\varphi}_2(\theta,x)|}{2^k}$$

for any $(\varphi_1,\varphi_2),(\tilde{\varphi}_1,\tilde{\varphi}_2)\in\mathcal{C}$, so that $(\mathcal{C},d)$ is a metric space.

Define $f:\mathcal{C}\longrightarrow X$ by

$$\begin{aligned} f(\varphi_1,\varphi_2)(x) &= \Big(f_1(\varphi_1,\varphi_2)(x), f_2(\varphi_1,\varphi_2)(x)\Big) \\ &= \Big(-b_1\varphi_1(0,x)+h(\varphi_2(-\tau_1,x)), -b_2\varphi_2(0,x)+g(\varphi_1(-\tau_2,x))\Big). \end{aligned}$$

We can then rewrite (3.17) as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\dfrac{\partial^2 u}{\partial x^2} + f_1(u_t,v_t)(x), \\ \dfrac{\partial v}{\partial t} = d_2\dfrac{\partial^2 v}{\partial x^2} + f_2(u_t,v_t)(x), \end{cases} \tag{3.18}$$

where $(u_t,v_t)\in\mathcal{C}$ with $u_t(\theta,x)=u(t+\theta,x)$ for $\theta\in[-\tau_2,0]$ and $v_t(\theta,x)=v(t+\theta,x)$ for $\theta\in[-\tau_1,0]$. The condition $(A1)$ in [22] is satisfied by the property that both $(u(t,-x),v(t,-x))$ and $(u(t,x+y),v(t,x+y))$, $\forall y\in\mathbb{R}$, are also solutions when $(u(t,x),v(t,x))$ is a solution.

By Lemma 2.9 in [38], there exists a solution map $Q_t=(Q_t^u,Q_t^v):\mathcal{C}_{(u^*,v^*)}\to\mathcal{C}_{(u^*,v^*)}$ for $t>0$:

$$Q_t^u[(\varphi_1,\varphi_2)](\theta,x) = u_t(\theta,x;\varphi_1,\varphi_2), \quad \forall(\theta,x)\in[-\tau_2,0]\times\mathbb{R},$$
$$Q_t^v[(\varphi_1,\varphi_2)](\theta,x) = v_t(\theta,x;\varphi_1,\varphi_2), \quad \forall(\theta,x)\in[-\tau_1,0]\times\mathbb{R},$$

and $Q_t:\mathcal{C}_{(u^*,v^*)}\to\mathcal{C}_{(u^*,v^*)}$ is monotone. Thus, the condition $(A4)$ in [22] holds. Similar as the proof of Lemma 2.2 in [17] and Lemma 3.2 in [21], we can show that Q_t is continuous in (φ_1,φ_2) with respect to the compact open topology uniformly for $t\in[0,t_0]$ with any $t_0>0$. Since (3.18) is an autonomous system, $\{Q_t\}_{t\geqslant 0}$ is a semiflow on $\mathcal{C}_{(u^*,v^*)}$. Thus, the condition $(A2)$ in [22] holds.

Let $\hat{Q}_t$ be the restriction of Q_t to $\bar{\mathcal{C}}_{(u^*,v^*)}$. It is easy to see that $\hat{Q}_t:\bar{\mathcal{C}}_{(u^*,v^*)}\to\bar{\mathcal{C}}_{(u^*,v^*)}$ is the solution semiflow generated by

$$\begin{cases} u'(t)=f_1(u_t,v_t), \\ v'(t)=f_2(u_t,v_t) \end{cases} \tag{3.19}$$

with the initial data $(\varphi_1,\varphi_2)\in\bar{\mathcal{C}}_{(u^*,v^*)}$. By Corollary 5.3.5 in [27], $\hat{Q}_t$ is eventually strongly monotone on $\bar{\mathcal{C}}_{(u^*,v^*)}$. In the proof of Theorem 2.3 (iv), we have known that $s(D\hat{f}(0,0))=\max\{\mathrm{Re}\lambda;\det(\lambda I - D\hat{f}(0,0))=0\}>0$. By Corollary 5.5.2 in [27], $(\hat{0},\hat{0})$ is an unstable equilibrium of (3.19). By the Dancer-Hess connecting orbit lemma (see, e.g. Section 2.1 in [41]), the semiflow $\hat{Q}_t$ admits a strongly monotone full orbit connecting $(0,0)$ to (u^*,v^*). Thus, the condition $(A5)$ holds for each Q_t, $t>0$.

Similar as the proof of Lemma 2.3 in [17] and Lemma 3.3 in [21], we can prove that the following two statements hold:

(a) for $(t_1,t_2)\in(\tau_2,+\infty)\times(\tau_1,+\infty)$, $(Q_{t_1}^u[\mathcal{C}_{(u^*,v^*)}], Q_{t_2}^v[\mathcal{C}_{(u^*,v^*)}])$ is precompact in $\mathcal{C}_{(u^*,v^*)}$;

(b) for $(t_1,t_2)\in[0,\tau_2]\times[0,\tau_1]$, $(Q_{t_1}^u[\mathcal{C}_{(u^*,v^*)}](0,\cdot), Q_{t_2}^v[\mathcal{C}_{(u^*,v^*)}](0,\cdot))$ is precompact in X, and there are positive numbers $(\zeta,\vartheta)\leqslant(\tau_2,\tau_1)$ such that $Q_{t_1}^u[(u,v)](\theta,\cdot)=u(\theta+\zeta,x)$ for $-\tau_2\leqslant\theta\leqslant-\zeta$, $Q_{t_2}^v[(u,v)](\theta,\cdot)=v(\theta+\vartheta,x)$ for $-\tau_1\leqslant\theta\leqslant-\vartheta$, and the operators

$$S_1[(u,v)](\theta,x) = \begin{cases} u(0,x), & \theta \in [-\tau_2, -\zeta), \\ Q^u_{t_1}[(u,v)](\theta,x), & \theta \in [-\zeta, 0], \end{cases}$$

$$S_2[(u,v)](\theta,x) = \begin{cases} v(0,x), & \theta \in [-\tau_1, -\vartheta), \\ Q^v_{t_2}[(u,v)](\theta,x), & \theta \in [-\vartheta, 0] \end{cases}$$

have the property that $S_1[D], S_2[D]$ are precompact in $\mathcal{C}_{(u^*,v^*)}$ for any T-invariant set $D \subset \mathcal{C}_{(u^*,v^*)}$ with $D(0,\cdot)$ precompact in X. Thus, the condition $(A6)(a), (b')$ in [22] holds. By the theory of monotone semiflows developed in [22], there exists $c^* > 0$ such that c^* is the asymptotic spreading speed, which coincides with the minimal wave speed.

Next, we prove $c^* = c^*_\tau$. Let $M_t = (M^u_t, M^v_t) : \mathcal{C} \to \mathcal{C}$ be the solution map at time t of the following linear equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \dfrac{\partial^2 u}{\partial x^2} - b_1 u + h'(0) v_t, \\ \dfrac{\partial v}{\partial t} = d_2 \dfrac{\partial^2 v}{\partial x^2} - b_2 v + g'(0) u_t. \end{cases} \tag{3.20}$$

For $\lambda \geqslant 0$, we define the linear map $B_t = (B^u_t, B^v_t) : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ by

$$B^u_t[(\varphi_1,\varphi_2)](\theta) = M^u_t[(\varphi_1,\varphi_2)e^{-\lambda x}](\theta,0), \quad \forall \theta \in [-\tau_2, 0],$$

$$B^v_t[(\varphi_1,\varphi_2)](\theta) = M^v_t[(\varphi_1,\varphi_2)e^{-\lambda x}](\theta,0), \quad \forall \theta \in [-\tau_1, 0].$$

Then, $B_t = (B^u_t, B^v_t) : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ is the solution map of the following equations

$$\begin{cases} u'(t) = d_1 \lambda^2 u(t) - b_1 u(t) + h'(0) v_t, \\ v'(t) = d_2 \lambda^2 v(t) - b_2 v(t) + g'(0) u_t. \end{cases} \tag{3.21}$$

Let

$$A(\chi) = \begin{pmatrix} d_1\lambda^2 - b_1 & h'(0)e^{-\chi\tau_1} \\ g'(0)e^{-\chi\tau_2} & d_2\lambda^2 - b_2 \end{pmatrix}.$$

Since (3.21) is a cooperative and irreducible delay equations, it follows that

$$\det(\chi I - A(\chi)) = 0,$$

i.e.

$$\chi^2 - [(d_1\lambda^2 - b_1) + (d_2\lambda^2 - b_2)]\chi + (d_1\lambda^2 - b_1)(d_2\lambda^2 - b_2) - h'(0)g'(0)e^{-\chi(\tau_1+\tau_2)} = 0,$$

admits a real root $\chi(\lambda)$ which is greater than the real parts of all other ones (see Theorem 5.5.1 in [27]).

Define $\Phi(\lambda) = \frac{\chi(\lambda)}{\lambda}$. By Theorem 3.10 in [22], we know $c^* = \inf_{\lambda>0} \Phi(\lambda) = \inf_{\lambda>0} \frac{\chi(\lambda)}{\lambda}$. Let $c = \frac{\chi(\lambda)}{\lambda}$, then c^* satisfies $c^* = \Phi(\lambda^*)$ and $\frac{dc}{d\lambda}|_{\lambda=\lambda^*}=0$. Then, (c^*, λ^*) can be determined as the positive solution to the system

$$\Delta^c(\lambda,\tau) = 0, \quad \frac{\partial \Delta^c(\lambda,\tau)}{\partial \lambda} = 0, \tag{3.22}$$

where $\Delta^c(\lambda,\tau)$, as a function of c and λ, is defined in (3.2). From the conclusions of Lemma 3.1, $c^* = c^*_\tau$. This completes the proof of the claim.

We should mention that the existence of monotone travelling wave solution to (3.17) for $c \geqslant c^*_\tau$ has also been proved in [38] by the method of lower and upper solutions.

Step 3. For any $c \in (0, c^*_\tau)$, we prove that (3.17) has a monotone semi-wave with speed c, or equivalently, (3.1) has a monotone increasing solution.

By the conclusions in Steps 1–2, we can immediately get the desired result. □

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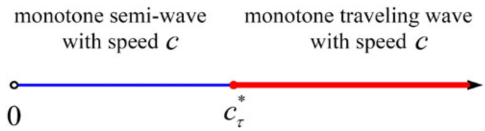

Figure 2. *Dichotomy between semi-waves and travelling waves.*

Remark 3.1. From Lemma 3.2, Lemma 3.6 and its proof (Step 2), we conclude that there is a dichotomy between monotone semi-waves and monotone travelling waves for (3.17). More precisely, for each $c>0$, (3.17) has either a monotone semi-wave solution with speed c or a monotone travelling wave solution with speed c, but not both. c_τ^* is the critical value. The ranges of c for both cases are shown in Figure 2.

Lemma 3.7. *Let (ϕ^c,ψ^c) be the unique strictly increasing solution of (3.1) with any fixed $c\in(0,c_\tau^*)$. Then, the following conclusions hold.*

(i) If $0<c_1<c_2<c_\tau^*$, then $((\phi^{c_1})'_+(0),(\psi^{c_1})'_+(0))>((\phi^{c_2})'_+(0),(\psi^{c_2})'_+(0))$ and $(\phi^{c_1}(s),\psi^{c_1}(s))>(\phi^{c_2}(s),\psi^{c_2}(s))$ for $s>0$.

(ii) For any fixed $\mu>0$, there exists a unique $c_\mu(\tau)\in(0,c_\tau^*)$ such that $\mu(\phi^{c_\mu(\tau)})'_+(0)=c_\mu(\tau)$. Moreover, $c_\mu(\tau)$ is strictly increasing in μ and $\lim_{\mu\to+\infty}c_\mu(\tau)=c_\tau^*$.

(iii) If $(\tilde\tau_1,\tilde\tau_2)\geqslant(\tau_1,\tau_2)$, then $c_\mu(\tilde\tau)\leqslant c_\mu(\tau)$ with $\tilde\tau=\tilde\tau_1+\tilde\tau_2$ and $\tau=\tau_1+\tau_2$.

Proof. (i) Since $0<c_1<c_2$ and $((\phi^{c_2})'(s),(\psi^{c_2})'(s))>(0,0)$ for $s\in\mathbb{R}_+$, we have

$$\begin{cases} c_1(\phi^{c_2})'(s)-d_1(\phi^{c_2})''(s)<h(\psi^{c_2}(s-c\tau_1))-b_1\phi^{c_2}(s), & s>0,\\ c_1(\psi^{c_2})'(s)-d_2(\psi^{c_2})''(s)<g(\phi^{c_2}(s-c\tau_2))-b_2\psi^{c_2}(s), & s>0\end{cases}$$

with $(\phi^{c_i}(s),\psi^{c_i}(s))=(0,0)$ for $s\leqslant 0$ and $(\phi^{c_i}(+\infty),\psi^{c_i}(+\infty))=(u^*,v^*)$ $(i=1,2)$. The comparison principle implies $(\phi^{c_1}(s),\psi^{c_1}(s))>(\phi^{c_2}(s),\psi^{c_2}(s))$ for $s>0$. Let $\tilde\phi(s)=\phi^{c_1}(s)-\phi^{c_2}(s)$ and $\tilde\psi(s)=\psi^{c_1}(s)-\psi^{c_2}(s)$, then

$$\begin{cases} c_1\tilde\phi'(s)-d_1\tilde\phi''(s)+b_1\tilde\phi(s)=h(\psi^{c_1}(s-c\tau_1))-h(\psi^{c_2}(s-c\tau_1))>0, & s>0,\\ c_1\tilde\psi'(s)-d_2\tilde\psi''(s)+b_2\tilde\psi(s)=g(\phi^{c_1}(s-c\tau_2))-g(\phi^{c_2}(s-c\tau_2))>0, & s>0,\\ (\tilde\phi(0),\tilde\psi(0))=(0,0).\end{cases}$$

By the Hopf boundary lemma, we obtain $(\tilde\phi'_+(0),\tilde\psi'_+(0))>(0,0)$ and then

$$((\phi^{c_1})'_+(0),(\psi^{c_1})'_+(0))>((\phi^{c_2})'_+(0),(\psi^{c_2})'_+(0)).$$

(ii) Similar as the proof of Lemma 3.11 in [35], we can show that the mapping $c\mapsto(\phi^c,\psi^c)$ is continuous from $(0,c_\tau^*)$ to $[C^2_{loc}([0,+\infty)]^2$ and satisfies $\lim_{c\nearrow c_\tau^*}(\phi^c,\psi^c)=(0,0)$ in $[C^2_{loc}([0,+\infty)]^2$. It follows that $\xi_\mu(c;\tau):=(\phi^c)'_+(0)-\frac{c}{\mu}$ is continuous on $(0,c_\tau^*)$ and $\lim_{c\nearrow c_\tau^*}\xi_\mu(c;\tau)=-\frac{c_\tau^*}{\mu}<0$. From (i), we see that $\xi_\mu(c;\tau)$ is strictly decreasing in $c\in(0,c_\tau^*)$. Note that $\xi_\mu(0;\tau)>0$. Therefore, there exists a unique $c_\mu(\tau)\in(0,c_\tau^*)$ such that $\xi_\mu(c_\mu(\tau);\tau)=0$, i.e. $\mu(\phi^{c_\mu(\tau)})'_+(0)=c_\mu(\tau)$.

Since $\xi_\mu(c;\tau)$ is strictly increasing in μ, we know that $c_\mu(\tau)$ is also strictly increasing in μ for $c\in(0,c_\tau^*)$. Thus, $\bar c:=\lim_{\mu\to+\infty}c_\mu(\tau)$ is well-defined and $\bar c\in[0,c_\tau^*]$. We claim that $\bar c=c_*^\tau$. In fact, assume by contradiction that $\bar c<c_*^\tau$, then $\lim_{\mu\to+\infty}(\phi^{c_\mu(\tau)})'_+(0)=\lim_{\mu\to+\infty}\frac{c_\mu(\tau)}{\mu}=0$, which yields $(\phi^{\bar c})'_+(0)=\lim_{c\nearrow\bar c}(\phi^c)'_+(0)=0$. By (i), we have $\lim_{c\nearrow c_\tau^*}(\phi^c)'_+(0)<(\phi^{\bar c})'_+(0)=0$, which is a contradiction. Thus, $\lim_{\mu\to+\infty}c_\mu(\tau)=c_\tau^*$.

(iii) To stress the dependence of the solution to (3.1) on time delays, we denote it by $(\phi_\tau^c,\psi_\tau^c)$. Since $\tilde\tau>\tau$, from (3.5) we have $c_{\tilde\tau}^*<c_\tau^*$, which implies that $\xi_\mu(c;\tilde\tau),\xi_\mu(c;\tau)$ are well-defined for $c\in(0,c_{\tilde\tau}^*)$. To get $c_\mu(\tilde\tau)\leqslant c_\mu(\tau)$, it is sufficient to prove that $\xi_\mu(c;\tilde\tau)<\xi_\mu(c;\tau)$ on $(0,c_{\tilde\tau}^*)$, or equivalently, $(\phi_{\tilde\tau}^c)'_+(0)<(\phi_\tau^c)'_+(0)$.

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Since $(\phi^c_{\tilde\tau}(s),\psi^c_{\tilde\tau}(s))$ is increasing on $\mathbb{R}$, we have $(\phi^c_{\tilde\tau}(s-c\tilde\tau_1),\psi^c_{\tilde\tau}(s-c\tilde\tau_2))\leqslant(\phi^c_{\tilde\tau}(s-c\tau_1),\psi^c_{\tilde\tau}(s-c\tau_1))$. It follows that $(\phi^c_{\tilde\tau}(s),\psi^c_{\tilde\tau}(s))$ satisfies

$$\begin{cases} c(\phi^c_{\tilde\tau})'(s)-d_1(\phi^c_{\tilde\tau})''(s)+b_1\phi^c_{\tilde\tau}(s)=h(\psi^c_{\tilde\tau}(s-c\tilde\tau_1))\leqslant h(\psi^c_{\tilde\tau}(s-c\tau_1)), & s>0,\\ c(\psi^c_{\tilde\tau})'(s)-d_2(\psi^c_{\tilde\tau})''(s)+b_2\psi^c_{\tilde\tau}(s)=g(\phi^c_{\tilde\tau}(s-c\tilde\tau_2))\leqslant g(\phi^c_{\tilde\tau}(s-c\tau_2)), & s>0,\\ \phi^c_{\tilde\tau}(s)=\psi^c_{\tilde\tau}(s)=0, \quad s\leqslant 0, \end{cases}$$

which implies that $(\phi^c_{\tilde\tau}(s),\psi^c_{\tilde\tau}(s))$ is a lower solution of the following problem

$$\begin{cases} \Phi_t=d_1\Phi_{ss}-c\Phi_s-b_1\Phi+h(\Psi(t,s-c\tau_1)), & t>0, s>0,\\ \Psi_t=d_2\Psi_{ss}-c\Psi_s-b_2\Psi+g(\Phi(t,s-c\tau_2)), & t>0, s>0,\\ \Phi(t,s)=\Psi(t,s)=0, \quad t>0, s\leqslant 0,\\ (\Phi(0,s),\Psi(0,s))=(\phi^c_{\tilde\tau}(s),\psi^c_{\tilde\tau}(s)). \end{cases} \tag{3.23}$$

By the maximum principle, we know that the solution $(\Phi(t,s),\Psi(t,s))$ of (3.23) is increasing in $t\geqslant 0$ and satisfies $\lim_{t\to+\infty}(\Phi(t,s),\Psi(t,s))=(\phi^*(s),\psi^*(s))$, where $(\phi^*(s),\psi^*(s))$ is a solution of (3.1). Clearly, the uniqueness of the solutions to (3.1) ensures that $(\phi^*(s),\psi^*(s))=(\phi^c_\tau(s),\psi^c_\tau(s))$. Thus, for all $s>0$, we have

$$\begin{aligned}(\phi^c_{\tilde\tau}(s),\psi^c_{\tilde\tau}(s))=(\Phi(0,s),\Psi(0,s)) &\leqslant(\Phi(t,s),\Psi(t,s))\\ &\leqslant(\Phi(+\infty,s),\Psi(+\infty,s))=(\phi^c_\tau(s),\psi^c_\tau(s)).\end{aligned}$$

Let $\hat\phi(s)=\phi^c_\tau(s)-\phi^c_{\tilde\tau}(s)$, then $\hat\phi$ satisfies

$$\begin{cases} c\hat\phi'(s)-d_1\hat\phi''(s)+b_1\hat\phi(s)=h(\psi^c_\tau(s-c\tau_1))-h(\psi^c_{\tilde\tau}(s-c\tilde\tau_1))\\ \qquad\qquad\qquad\geqslant h(\psi^c_\tau(s-c\tilde\tau_1))-h(\psi^c_{\tilde\tau}(s-c\tilde\tau_1))\geqslant 0, \quad s>0,\\ \hat\phi(0)=0. \end{cases}$$

The Hopf boundary lemma yields $\hat\phi'(0)>0$, that is, $(\phi^c_\tau)'_+(0)>(\phi^c_{\tilde\tau})'_+(0)$. This completes the proof. □

3.2 Asymptotic spreading speed when spreading happens

In this subsection, we present the asymptotic spreading speed of (1.4) when spreading happens.

Theorem 3.2. *Assume that $\mathcal{R}_0>1$ and $s_{2,\infty}-s_{1,\infty}=+\infty$. Then $\lim_{t\to\infty}\frac{s_2(t)}{t}=\lim_{t\to\infty}\frac{-s_1(t)}{t}=c_\mu(\tau)$.*

Proof. It is sufficient to show $\lim_{t\to\infty}\frac{s_2(t)}{t}=c_\mu(\tau)$, since the limit of $\frac{-s_1(t)}{t}$ can be similarly proved by considering the free boundary problem for $(u(t,-x),v(t,-x),-s_2(t),-s_1(t))$.

We first prove $\limsup_{t\to\infty}\frac{s_2(t)}{t}\leqslant c_\mu(\tau)$. For any sufficiently small constant $\varepsilon>0$, let $(\phi^c_\varepsilon,\psi^c_\varepsilon)$ be the unique strictly increasing solution of the following semi-wave problem

$$\begin{cases} c\phi'(s)-d_1\phi''(s)=h(\psi(s-c\tau_1))-(b_1-\varepsilon)\phi(s), & s>0,\\ c\psi'(s)-d_2\psi''(s)=g(\phi(s-c\tau_2))-(b_2-\varepsilon)\psi(s), & s>0,\\ (\phi(s),\psi(s))=(0,0), \quad s\leqslant 0,\\ (\phi(+\infty),\psi(+\infty))=(u^*_\varepsilon,v^*_\varepsilon), \end{cases} \tag{3.24}$$

where $(u^*_\varepsilon,v^*_\varepsilon)$ is defined as the positive equilibrium of (3.24). By Lemma 3.7, there exists some $c^\varepsilon_\mu(\tau)>0$ such that

$$\mu(\phi^{c^\varepsilon_\mu(\tau)}_\varepsilon)'_+(0)=c^\varepsilon_\mu(\tau), \quad \lim_{\varepsilon\to 0}c^\varepsilon_\mu(\tau)=c_\mu(\tau).$$

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Let $\varepsilon_0 = \frac{\varepsilon}{2}$, it is easy to check that $(u^*_\varepsilon, v^*_\varepsilon) > (u^*_{\varepsilon_0}, v^*_{\varepsilon_0}) > (u^*, v^*)$, which together with (2.10)–(2.11) imply that there exists some $t_0 > 0$ such that

$$\begin{aligned} u(t,x) &\leqslant u^*_{\varepsilon_0} \quad \text{for } (t,x) \in [t_0 - \tau_2, +\infty) \times [s_1(t), s_2(t)], \\ v(t,x) &\leqslant v^*_{\varepsilon_0} \quad \text{for } (t,x) \in [t_0 - \tau_1, +\infty) \times [s_1(t), s_2(t)]. \end{aligned}$$

In view of $(\phi^c_\varepsilon(+\infty), \psi^c_\varepsilon(+\infty)) = (u^*_\varepsilon, v^*_\varepsilon)$, we have

$$(\phi^c_\varepsilon(l_0 - s_2(t_0)), \psi^c_\varepsilon(l_0 - s_2(t_0))) > (u^*_{\varepsilon_0}, v^*_{\varepsilon_0}) \quad \text{for some } l_0 > s_2(t_0).$$

Define

$$\bar{s}_2(t) = \begin{cases} c^\varepsilon_\mu(\tau)(t - t_0) + l_0, & t \in [t_0, +\infty), \\ l_0, & t \in [t_0 - \max\{\tau_1, \tau_2\}, t_0], \end{cases}$$

$$\bar{u}(t,x) = \begin{cases} \phi^c_\varepsilon(\bar{s}_2(t) - x), & t \in [t_0, +\infty), x \in [0, \bar{s}_2(t)), \\ \phi^c_\varepsilon(l_0 - x), & t \in [t_0 - \tau_2, t_0], x \in [0, \bar{s}_2(t)) = [0, l_0), \end{cases}$$

$$\bar{v}(t,x) = \begin{cases} \psi^c_\varepsilon(\bar{s}_2(t) - x), & t \in [t_0, +\infty), x \in (0, \bar{s}_2(t)), \\ \psi^c_\varepsilon(l_0 - x), & t \in [t_0 - \tau_1, t_0], x \in [0, \bar{s}_2(t)) = [0, l_0). \end{cases}$$

Clearly, straightforward computation yields that $(\bar{u}(t,x), \bar{v}(t,x), \bar{s}_2(t))$ is an upper solution of (1.4) with $(s_1(t), s_2(t))$ replaced by $(0, s_2(t))$. It follows from Lemma 2.2 that $\bar{s}_2(t) \geqslant s_2(t)$ for $t \geqslant t_0$. Therefore,

$$\limsup_{t\to\infty} \frac{s_2(t)}{t} \leqslant \limsup_{t\to\infty} \frac{\bar{s}_2(t)}{t} \leqslant c^\varepsilon_\mu(\tau).$$

Taking $\varepsilon \to 0$, we have $\limsup_{t\to\infty} \frac{s_2(t)}{t} \leqslant c_\mu(\tau)$.

Next, we prove $\liminf_{t\to\infty} \frac{s_2(t)}{t} \geqslant c_\mu(\tau)$. For any sufficiently small constant $\epsilon > 0$, let $(\phi^c_\epsilon, \psi^c_\epsilon)$ be the unique strictly increasing solution of the following semi-wave problem

$$\begin{cases} c\phi'(s) - d_1\phi''(s) = h(\psi(s - c\tau_1)) - (b_1 + \epsilon)\phi(s), & s > 0, \\ c\psi'(s) - d_2\psi''(s) = g(\phi(s - c\tau_2)) - (b_2 + \epsilon)\psi(s), & s > 0, \\ (\phi(s), \psi(s)) = (0, 0), & s \leqslant 0, \\ (\phi(+\infty), \psi(+\infty)) = (u^*_\epsilon, v^*_\epsilon), \end{cases}$$

where $(u^*_\epsilon, v^*_\epsilon)$ is the unique positive equilibrium and satisfies $(u^*_\epsilon, v^*_\epsilon) < (u^*, v^*)$. Since $(\phi^c_\epsilon, \psi^c_\epsilon)$ is strictly increasing, we have $(\phi^c_\epsilon(s), \psi^c_\epsilon(s)) < (u^*_\epsilon, v^*_\epsilon)$ for any $s \geqslant 0$. Note that $\lim_{t\to\infty} (u(t,x), v(t,x)) = (u^*, v^*)$ locally uniformly for $x \in \mathbb{R}$ when spreading occurs. Thus, for any $L_0 > 0$ there exists $T_0 > \max\{\tau_1, \tau_2\}$ such that

$$s_2(T_0 - \max\{\tau_1, \tau_2\}) > L_0,$$

$$(u(t,x), v(t,x)) \geqslant (u^*_\epsilon, v^*_\epsilon) \quad \text{for any } (t,x) \in [T_0 - \max\{\tau_1, \tau_2\}, +\infty) \times [-L_0, L_0].$$

From Lemma 3.7, there exists some $c^\epsilon_\mu(\tau) > 0$ such that $\mu(\phi^{c^\epsilon_\mu(\tau)}_\epsilon)'_+(0) = c^\epsilon_\mu(\tau)$ and $\lim_{\epsilon\to 0} c^\epsilon_\mu(\tau) = c_\mu(\tau)$. Define

$$\underline{s}_2(t) = \begin{cases} c^\epsilon_\mu(\tau)(t - T_0) + L_0, & t \in [T_0, +\infty), \\ L_0, & t \in [T_0 - \max\{\tau_1, \tau_2\}, T_0], \end{cases}$$

$$\underline{u}(t,x) = \begin{cases} \phi^c_\epsilon(\underline{s}_2(t) - x), & t \in [T_0, +\infty), x \in [0, \underline{s}_2(t)), \\ \phi^c_\epsilon(L_0 - x), & t \in [T_0 - \tau_2, T_0], x \in [0, \underline{s}_2(t)) = [0, L_0), \end{cases}$$

$$\underline{v}(t,x) = \begin{cases} \psi^c_\epsilon(\underline{s}_2(t) - x), & t \in [T_0, +\infty), x \in (0, \underline{s}_2(t)), \\ \psi^c_\epsilon(L_0 - x), & t \in [T_0 - \tau_1, T_0], x \in [0, \underline{s}_2(t)) = [0, L_0). \end{cases}$$

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It is easy to check that $(\underline{u}(t,x), \underline{v}(t,x), \underline{s}_2(t))$ is a lower solution of (1.4) with $(s_1(t), s_2(t))$ replaced by $(0, s_2(t))$. Then, the comparison principle implies that $s_2(t) \geqslant \underline{s}_2(t)$ for $t \geqslant T_0$. It follows that $\liminf_{t\to\infty} \frac{s_2(t)}{t} \geqslant \liminf_{t\to\infty} \frac{\underline{s}_2(t)}{t} \geqslant c^{\epsilon}_{\mu}(\tau)$. Taking $\epsilon \to 0$, we have $\liminf_{t\to\infty} \frac{s_2(t)}{t} \geqslant c_\mu(\tau)$. Thus, $\lim_{t\to\infty} \frac{s_2(t)}{t} = c_\mu(\tau)$, which completes the proof. □

Remark 3.2. In Theorem 3.2, when $\mathcal{R}_0 > 1$ and spreading occurs we obtain that $s_1(t) = -c_\mu(\tau)t + o(t)$ and $s_2(t) = c_\mu(\tau)t + o(t)$ as $t \to \infty$. However, finer estimates of $|s_1(t) + c_\mu(\tau)t|$ and $|s_2(t) - c_\mu(\tau)t|$ are still unknown, which will be considered in the future work.

Acknowledgements. The authors would like to thank the referees for their helpful comments and suggestions.

Financial support. Chen was partially supported by the NSFC of China (Grant Nos: 12271421,11801432), China Postdoctoral Science Foundation (Grant No:2019M663610) and the Young Talent fund of University Association for Science and Technology in Shaanxi (Grant No:20200510). Tang was partially supported by the NSFC of China (Grant Nos:12031010,61772017). Teng was partially supported by the NSFC of China (Grant No:11771373). Wang was partially supported by the NSFC of China (Grant No:11801429) and the Natural Science Basic Research Plan in Shaanxi Province of China (Grant No:2019JQ-136).

Competing interests. None.

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Cite this article: Chen Q., Tang S., Teng Z. and Wang F. (2023). Spreading dynamics of a diffusive epidemic model with free boundaries and two delays. *European Journal of Applied Mathematics*, **34**, 1133–1169. https://doi.org/10.1017/S0956792523000220