Can. J. Math., Vol. XXIX, No. 3, 1977, pp. 638–647

ON A TRANSCENDENCE PROBLEM OF K. MAHLER

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K. Mahler [**8**] has proposed the following problem. Let Ω_r for $r \geqq 1$ be a sequence of $n \times n$ non-negative rational integer matrices. Each $\Omega_r = (\omega_{rij})$ defines a map $\Omega_r : \mathbf{C}^n \rightarrow \mathbf{C}^n$ by

$$z = (z_1, \ldots, z_n) \rightarrow \Omega_r z = \left(\prod_{j=1}^{n} z_j^{\omega_{r1j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{rnj}} \right).$$

Let $z_0 = (z_{01}, \ldots, z_{0n})$ be an algebraic point and $f_r(z)$ for $r \geqq 0$ be a sequence of convergent power series satisfying recursive relations of the form

$$f_r(z) = a_r(z) f_{r+1}(\Omega_{r+1} z) + b_r(z)$$

where the $a_r(z)$ and $b_r(z)$ are rational functions. For certain special classes of matrices, find conditions on the z_{0j} and f_i which imply that $f_0(z_0)$ is transcendental.

For example, Mahler [**6**; **7**] has shown that for integers $\rho \geqq 2$, the transcendental function $f(z) = \sum_{k=0}^{\infty} z^{\rho^k}$ which satisfies

$$f(z) = f(z^\rho) + z$$

(corresponding to the case $n = 1$, $\Omega_r = (\rho)$, $f_r = f$, $a_r = 1$, and $b_r = z$) takes on transcendental values at all algebraic points z_0 in the punctured open unit disc. To generalize this situation, one might take a sequence $\rho_1, \rho_2, \ldots$ of 2's and 3's and consider the functions

$$f_r(z) = \sum_{k=r}^{\infty} z^{\rho_{r+1} \cdots \rho_k}$$

which satisfy the functional equations

$$f_r(z) = f_{r+1}(\Omega_{r+1} z) + z$$

where $\Omega_r = (\rho_r)$. Then as a special case of the theorem proved below, it will be shown that the function

$$f_0(z) = \sum_{k=0}^{\infty} z^{\rho^{(k)}}$$

where $\rho^{(k)} = \rho_1 \rho_2 \ldots \rho_k$ takes on transcendental values at all algebraic points in the punctured open unit disc.

Results along these same lines have been obtained independently by Loxton and van der Poorten. Their very strong Theorem 1 in [**4**, III] amply contains

Received July 13, 1976 and in revised form, January 12, 1977. The author is a Sloan Fellow.

most of the examples, including that of the last paragraph, which can be obtained using the theorem given below. However, the approach given here has at least the advantage that it generalizes readily to give an algebraic independence result [**2**]. Further references and details can be found in the survey article [**5**].

1. General functional equations. Let n be a positive integer and $z = (z_1, \ldots, z_n)$ be an n-tuple of indeterminants. The ring $\mathbf{C}[[z]]$ of n-variable complex formal power series is topologized with its $\mathfrak{M}$-adic topology [**9**] where $\mathfrak{M}$ is the ideal generated by $z_1, \ldots, z_n$.

LEMMA. *Let D be a polydisc containing the origin in $\mathbf{C}^n$, and f_i for $i \geqq 0$ be a sequence of power series converging in D, with coefficients in a subfield K of $\mathbf{C}$, and satisfying*

(1) $\quad \sup f_i(z) < \infty, \quad z \in D, i \geqq 0.$

Suppose that there is a subsequence $f_{i(j)}$ for $j \geqq 1$ of the f_i which is compact and such that the set of limit points of the subsequence is not a finite set of algebraic functions. Let ρ_i for $i \geqq 1$ be a sequence of integers all larger than one. Let $z_0 = (z_{01}, \ldots, z_{0n}) \in \mathbf{C}^n$ be a point with $0 < |z_{0i}| < 1$ for $i = 1, \ldots, n$ and such that the $|z_{0i}|$ are multiplicatively independent. Then, for every $p \in \mathbf{N}^+$, there is a polynomial $\mathscr{A}(z, y) \in K[z_1, \ldots, z_n, y]$ of degree at most p in each of its $n + 1$ variables and such that for infinitely many $k \in \mathbf{N}^+$, one has the inequality

(2) $\quad 0 \neq |\mathscr{A}(z_0^{(k)}, f_k(z_0^k))| < \exp(-c_1 p^{1+n}\rho^{(k)})$

where $\rho^{(k)} = \rho_1\rho_2 \ldots \rho_k$, $z_0^{(k)} = (z_{01}^{\rho^{(k)}}, \ldots, z_{0n}^{\rho^{(k)}})$, and $c_1 > 0$ is a positive real number independent of both p and k.

Proof. Since $F = \{f_{i(j)}\}$ is compact, its elements lie in at most finitely many cosets of $\mathbf{C}[[z]]$ modulo $\mathfrak{M}^\kappa$ where $\mathfrak{M} = (z_1, \ldots, z_n)$ and $\kappa = [p^{1+n^{-1}}] + 1$. Therefore, there is an $m \in \mathbf{N}^+$ such that the subsequence $F \cap \{f_{i(m)} + \mathfrak{M}^\kappa\}$ has a set of limit points which is other than a finite set of algebraic functions.

There is a non-zero polynomial $\mathscr{A}(z, y) \in K[z, y]$ of degree at most p in each of the $n + 1$ variables and such that for every $h \in \mathbf{C}[[z]]$ with $h \equiv f_{i(m)} (\mathrm{mod}\ \mathfrak{M}^\kappa)$, one has $\mathscr{A}(z, h(z)) \in \mathfrak{M}^\kappa$. In fact, comparing coefficients in

$$\mathscr{A}(z, h(z)) = \sum_{i_1 \ldots i_n} a_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n},$$

one can express the $a_{i_1 \ldots i_n}$ as linear forms in the $(p + 1)^{n+1}$ coefficients of $\mathscr{A}(z, y)$. Further, if $0 \leqq i_1 + \ldots i_n \leqq p^{1+n^{-1}}$, then the coefficients of the form describing $a_{i_1 \ldots i_n}$ do not depend on the choice of h in the coset $f_{i(m)} + \mathfrak{M}^\kappa$. Now there are at most $(p^{1+n^{-1}} + 1)^n$ such n-tuples of indices, and

$$(p^{1+n^{-1}} + 1)^n < (p + 1)^{n+1}.$$

Therefore the existence of $\mathscr{A}(z, y)$ amounts to the existence of a non-trivial solution of a homogeneous system of linear equations with more variables than equations.

By the choice of m, there is a subsequence $\{f_{j(k)}\}$ of $F \cap \{f_{i(m)} + \mathfrak{M}^\kappa\}$ which converges $\mathfrak{M}$-adically to a formal power series g which is not a root of $\mathscr{A}(z, y) = 0$. The $g_i(z) = \mathscr{A}(z, f_i(z))$ are defined for $z \in D$ and are bounded above in absolute value uniformly in D and independently of i by Equation (1). By Cauchy's inequality, there exist positive real numbers $R_0, R_1, \ldots, R_n$ with

$$|g_{ih_1\ldots h_n}| \leqq R_0 R_1^{h_1} \ldots R_n^{h_n}$$

for all $h_1, \ldots, h_n \geqq 0$ where $g_{ih_1\ldots h_n}$ denotes the coefficient of $z_1^{h_1} \ldots z_n^{h_n}$ in $g_i(z)$.

Let S be the set of non-zero monomials of g partially ordered via $Az_1^{h_1} \ldots z_n^{h_n} \leqq Bz_1^{j_1} \ldots z_n^{j_n} \Leftrightarrow h_i \leqq j_i$ for $i = 1, \ldots, n$. The set T of minimal elements of S is finite [**1**, Lemma 3]. Since the $|z_{01}|, \ldots, |z_{0n}|$ are multiplicatively independent, one can order T via

$$Az_1^{h_1} \ldots z_n^{h_n} \leqq Bz_1^{j_1} \ldots z_n^{j_n} \Leftrightarrow \prod_{k=1}^{n} |z_{0k}|^{h_k - j_k} \leqq 1.$$

Let $Az_1^{m_1} \ldots z_n^{m_n}$ denote the largest element of T.

If one defines $z_0^{\,t} = (z_{01}^{\,t}, \ldots, z_{0n}^{\,t})$, then

$$(3) \quad \lim_{\substack{k\to\infty \\ t\to\infty}} \frac{g_{j(k)}(z_0^{\,t})}{A\,(z_{01}^{m_1} \ldots z_{0n}^{m_n})^t} = 1.$$

In fact, there is a positive integer k_0 such that for all $k \geqq k_0$, one has $f_{j(k)} \equiv g \pmod{\mathfrak{M}^\tau}$ where τ is some fixed integer larger than the total degrees of all the elements of T. Letting

$$H_{i_1\ldots i_n}(z) = \sum_{j_1=i_n}^{\infty} \ldots \sum_{j_n=i_n}^{\infty} R_0 R_1^{j_1} \ldots R_n^{j_n} z_1^{j_1-i_1} \ldots z_n^{j_n-i_n},$$

one has the obvious majorization

$$\left| \frac{g_{j(k)}(z_0^{\,t})}{A\,(z_{01}^{m_1} \ldots z_{0n}^{m_n})^t} - 1 \right| \leqq \sum \frac{\prod_{r=1}^{n} |z_{0r}|^{(i_r - m_r)t}}{|A|} H_{i_1\ldots i_n}(|z_0|^t) + \sum_{r=1}^{n} \frac{|z_{0r}|^t}{|A|} H_{m_1 m_2\ldots m_r+1\ldots m_n}(|z_0|^t)$$

for $k \geqq k_0$ where $|z_0| = (|z_{01}|, \ldots, |z_{0n}|)$ and the first summation is over all $Bz_1^{i_1} \ldots z_n^{i_n}$ in T except for $Az_1^{m_1} \ldots z_n^{m_n}$. Since the z_{0j} are in the open unit disc, the $H_{i_1\ldots i_n}(|z_0|^t)$ and the $H_{m_1\ldots m_r+1\ldots m_n}(|z_0|^t)$ are bounded for t large, and $|z_{0r}|^t \to 0$ as $t \to \infty$. By the choice of the m_i, we also have $(\prod_{r=1}^{n} |z_{0r}|^{i_r - m_r})^t \to 0$ as $t \to \infty$ which proves Equation (3).

By the construction of the auxiliary polynomial $\mathscr{A}(z, y)$, one knows that $m_1 + \ldots + m_n \geqq \kappa > p^{1+n^{-1}}$. Therefore

$$|A| \prod_{i=1}^{n} |z_{0i}|^{m_i \rho^{(j(k))}} \geqq |A| \exp\left(-c_2 p^{1+n^{-1}} \rho^{(j(k))}\right) \geqq \exp\left(-2\, c_2 p^{1+n^{-1}} \rho^{(j(k))}\right)$$

where $c_2 > 0$ is independent of p and k, and where the second inequality is claimed only for k larger than some function of p. Combining this last inequality with Equation (3) gives the assertion of the lemma.

THEOREM. *Let D, K, f_i, ρ_i and z_0 be as in the statement of the lemma and suppose in addition that K is a number field and the z_{0j} are algebraic numbers. For $i \geqq 1$, let $T_i : \mathbf{A}^n \times \mathbf{P}^m \to \mathbf{A}^n \times \mathbf{P}^m$ be a rational map of the product of affine n-space $\mathbf{A}^n$ and projective m-space $\mathbf{P}^m$. Assume that T_i is defined by*

$$(z_1, \ldots, z_n, w_0, \ldots, w_m) \mapsto (z_1^{\rho_i}, \ldots, z_n^{\rho_i}, t_{i0}(z, w), \ldots, t_{im}(z, w))$$

where the $t_{ij}(z, w) \in K[z, w]$ are of total degree at most b in the variables $z_1, \ldots, z_n$, have algebraic integer coefficients in K, and are forms in the variables $w_0, \ldots, w_m$ of degree d_i. Suppose, in addition, that the maximum B_i of the absolute values of the conjugates of the coefficients of the $t_{ij}(w)$ satisfies

(4) $\log B_i \ll \rho^{(i)}$

where $\rho^{(i)} = \rho_1 \rho_2 \ldots \rho_i$, and that there is a $\lambda > 1$ with

(5) $\rho_i / d_i \geqq \lambda > 1.$

Let $w_0 = (w_{01}, \ldots, w_{0m})$ be such that for all $k \geqq 0$,

$$T^{(k)}(z_0, w_0) = T_k \circ \ldots \circ T_1(z_0, w_0) = (z_{01}^{\rho^{(k)}}, \ldots, z_{0n}^{\rho^{(k)}}, w_{01}^{(k)}, \ldots, w_{0m}^{(k)})$$

is defined and

(6) $f_k(z_{01}^{\rho^{(k)}}, \ldots, z_{0n}^{\rho^{(k)}}) = w_{01}^{(k)} / w_{00}^{(k)}.$

Then w_0 is a transcendental point.

Proof. If not, then by extending K if necessary, it may be assumed that K contains the coordinates of z_0 and w_0. By multiplying through by a common denominator, it may be assumed that the w_{0j} are algebraic integers. For each positive integer p, apply the lemma to obtain an auxiliary polynomial $\mathscr{A}(z, y)$. Clearly by multiplying $\mathscr{A}(z, y)$ by a common denominator of its coefficients, it may be assumed that $\mathscr{A}(z, y)$ has algebraic integer coefficients.

The idea of the proof being to use the Liouville inequality to obtain a contradiction with Equation (2), we make a slight digression to review the properties of size. Recall [3] that the size $s(\alpha)$ of $\alpha \in K$ is defined by

$$s(\alpha) = \max (\log \operatorname{den} \alpha, \log |\overline{\alpha}|)$$

where den α is the denominator of α and $|\overline{\alpha}|$ is the maximum of the absolute values of the conjugates of α. If $\alpha_1, \ldots, \alpha_r \in K$, then clearly

$$(7) \qquad s\left(\prod_{i=1}^{r} \alpha_i\right) \leqq \sum_{i=1}^{r} s(\alpha_i).$$

If, in addition, there is a set of $P \geqq 1$ primes containing every prime divisor of $\prod_{i=1}^{r} \operatorname{den} \alpha_i$, then it is easy to verify that

$$(8) \qquad s\left(\sum_{i=1}^{r} \alpha_i\right) \leqq P \max_i s(\alpha_i) + \log r.$$

Finally, the fact that the norm of a non-zero algebraic integer is no smaller than 1 implies the Liouville inequality [3]

$$(9) \qquad \log |\alpha| \geqq -2\,[K:\mathbf{Q}]\, s(\alpha)$$

for $\alpha \in K\backslash(0)$.

If $h \in \overline{\mathbf{Q}}[X_1, \ldots, X_r]$ and $k \in \mathbf{R}[X_1, \ldots, X_r]$ are polynomials with algebraic and real coefficients respectively, then we write $h \ll k$ to indicate that the maximum of absolute values of the conjugates of each coefficient of $h(X)$ is no larger than the corresponding coefficient of $k(X)$. The c_i appearing below are assumed to be appropriately chosen positive real numbers not depending on the integer parameters p and k.

By composing the T_i's, one obtains

$$T^{(k)}(z, w) = (z_1^{\rho(k)}, \ldots, z_n^{\rho(k)}, t_0^{(k)}(z, w), \ldots, t_m^{(k)}(z, w)).$$

If $B_i' = (m+1)B_i$ and $L(z) = 1 + z_1 + \ldots + z_n$, then

$$t_{ij}(z, w) \ll \frac{1}{m+1} B_i' L(z)^b (w_0 + \ldots + w_m)^{d_i}$$

for $i \geqq 1$ and $j = 0, \ldots, m$. By induction on k, it follows that

$$t_j^{(k)}(z, w) \ll \frac{1}{m+1} \left(\prod_{i=1}^{k} B_i'^{d_k d_{k-1} \ldots d_{i+1}}\right) \left\{\prod_{i=0}^{k-1} L(z^{\rho(i)})^{d_k \ldots d_{i+1}}\right\}^b \times (w_0 + \ldots + w_m)^{d_1 \ldots d_k}$$

where $z^{\rho(j)} = (z_1^{\rho(j)}, \ldots, z_n^{\rho(j)})$.

Note that Equation (5) implies that

$$\frac{\sum_{j=1}^{k} \prod_{i=1}^{j} (\rho_i/d_i)}{\prod_{i=1}^{k} (\rho_i/d_i)} = \sum_{j=1}^{k} \prod_{i=j+1}^{k} (d_i/\rho_i) \leqq \sum_{j=1}^{k} \lambda^{j-k} = \frac{\lambda^{-k} - 1}{\lambda^{-1} - 1} \leqq \frac{1}{1 - \lambda^{-1}} = c_3$$

is bounded independently of k. Therefore, since clearly

$$L(z^{\rho(j)}) \ll L(z)^{\rho(j)},$$

one has

$$\prod_{j=0}^{k-1} L(z^{\rho^{(j)}})^{d_k \ldots d_{j+1}} \ll L(z)^{\sigma}$$

where

$$\sigma = (d_1 \ldots d_k) \sum_{j=1}^{k} \prod_{i=1}^{j} (\rho_i/d_i) \leqq c_3\, \rho^{(k)}.$$

Similarly, using Equation (4), one obtains

$$\prod_{i=1}^{k} B_i'^{d_k \ldots d_{i+1}} \leqq \left(\prod_{i=1}^{k} B_i'^{(d_1 \ldots d_i)^{-1}} \right)^{d_1 \ldots d_k}$$
$$\leqq \exp \left((d_1 \ldots d_k) \sum_{i=1}^{k} \{c_4\, \rho^{(i)}/(d_1 \ldots d_i)\} \right)$$
$$\leqq \exp\, (c_3\, c_4\, \rho^{(k)}).$$

Substituting these estimates into the result of the last paragraph gives

$$t_j^{(k)}(z, w) \ll \frac{1}{m+1} \exp\, (c_3\, c_4\, \rho^{(k)}) L(z)^{c_3 b \rho^{(k)}} (w_0 + \ldots + w_m)^{d_1 \ldots d_k}.$$

In particular, one has

(10) $\quad |\overline{t_j^{(k)}(z_0, w_0)}| \leqq \exp\, (c_5\, \rho^{(k)})$

for $k \geqq 0$ and $j = 0, 1, \ldots, m$.

The prime factors of the denominators of the $t_j^{(k)}(z_0, w_0)$ are amongst those of the denominators of the z_{0i} and so are contained in a set of, say, $P \geqq 1$ primes. Tracing through the argument of the last paragraph to estimate the exponents of these primes in the denominators of the $t_j^{(k)}(z_0, w_0)$, one easily obtains

$$\log \text{den}\; t_j^{(k)}(z_0, w_0) \ll \rho^{(k)},$$

and hence

(11) $\quad s(t_j^{(k)}(z_0, w_0)) \leqq c_6\, \rho^{(k)}.$

Let the polynomials $A_i(z)$ be defined by

$$\mathscr{A}(z, y) = \sum_{i=0}^{p} A_i(z) y^i.$$

Then the $A_i(z)$ are of degree at most p in each variable z_i and have algebraic integer coefficients lying in K. If S denotes the maximum of the sizes of the coefficients of the $A_i(z)$, then the majorization

$$A_i(z^{\rho^{(k)}}) \ll e^S \prod_{i=1}^{n} (1 + z_i^{\rho^{(k)}})^p \ll e^S \left\{ \prod_{i=1}^{n} (1 + z_i) \right\}^{p \rho^{(k)}}$$

implies that

$$\overline{|A_i(z_0{}^{\rho^{(k)}})|} \leqq \exp\,(S + c_7\, p\rho^{(k)})$$

and

$$\log \text{den } A_i(z_0{}^{\rho^{(k)}}) \ll p\rho^{(k)}.$$

Therefore,

$$(12) \quad s(A_i(z_0{}^{\rho^{(k)}})) \leqq S + c_8\, p\rho^{(k)}$$

where S depends on p but not k.

Define the quantity $E_p(k)$ by

$$(13) \quad E_p(k) = t_0{}^{(k)}(z_0, w_0)^p \mathscr{A}(z_0{}^{\rho^{(k)}}, f_k(z_0{}^{\rho^{(k)}}))$$
$$= \sum_{i=0}^{p} A_i(z_0{}^{\rho^{(k)}}) t_1{}^{(k)}(z_0, w_0)^i t_0{}^{(k)}(z_0, w_0)^{p-i}$$

where we have used Equation (6). By Equations (7, 8, 11, 12), one has the estimate

$$s(E_p(k)) \leqq P\{S + c_8\, p\rho^{(k)} + c_6\, p\rho^{(k)}\} + \log\,(p + 1).$$

Therefore, for all k larger than some function of p, one has

$$(14) \quad s(E_p(k)) \leqq c_9\, p\rho^{(k)}.$$

Finally the Liouville inequality (9) together with Equations (10, 14) imply that for all k larger than some function of p and satisfying Equation (2), one has

$$\log |\mathscr{A}(z_0{}^{\rho^{(k)}}, f_k(z_0{}^{\rho^{(k)}}))| = \log |E_p(k)| - p \log |t_0{}^{(k)}(z_0, w_0)|$$
$$\geqq -2[K : \mathbf{Q}]c_9\, p\rho^{(k)} - c_5\, p\rho^{(k)} = -c_{10}\, p\rho^{(k)},$$

and so

$$|\mathscr{A}(z_0{}^{\rho^{(k)}}, f_k(z_0{}^{\rho^{(k)}}))| \geqq \exp\,(-c_{10}\, p\rho^k).$$

But if p is chosen sufficiently large, this contradicts Equation (2), and so the theorem is proved.

2. Linear functional equations. The most interesting applications of the theorem occur when the functional equations are linear. To fix the notation, let $f_0, f_1, f_2, \ldots$ be an infinite sequence of power series converging in a neighborhood D of the origin in $\mathbf{C}^n$, having coefficients in a number field K, and satisfying Equation (1). Assume further that the f_i satisfy functional equations of the form

$$(15) \quad f_i(z) = a_i(z) f_{i+1}(\Omega_{i+1} z) + b_i(z)$$

where $\Omega_{i+1} = \rho_{i+1} I$ is a scalar matrix with $\rho_{i+1} \geqq 2$ an integer, $\Omega_{i+1} z$ is the map defined in the introduction, and the $a_i(z)$ and $b_i(z)$ are rational functions with coefficients in K and $a_i(0) \neq 0$. Note that Equation (15) is a substitute for the map T_{i+1} of the theorem; more precisely, the corresponding map $T_{i+1} : \mathbf{A}^n \times \mathbf{P}^1 \to \mathbf{A}^n \times \mathbf{P}^1$ is defined by

$$(z_1, \ldots, z_n, w_0, w_1) \mapsto (z_1^{\rho_{i+1}}, \ldots, z_n^{\rho_{i+1}}, c_i(z)w_0, \\ c_i(z)a_i(z)^{-1}(w_1 - b_i(z)w_0))$$

where $c_i(z)$ is a common denominator for $a_i(z)^{-1}$ and $a_i(z)^{-1}b_i(z)$.

The functional equations (15) can be composed to obtain relations

(16) $f_i(z) = A_{ir}(z)f_{i+r}(\Omega^{(i,r)}z) + B_{ir}(z)$

where

(17) $A_{ir}(z) = \prod_{j=0}^{r-1} a_{i+j}(\Omega^{(i,j)}z)$

(18) $B_{ir}(z) = \sum_{j=0}^{r-1} A_{ij}(z)b_{i+j}(\Omega^{(i,j)}z)$

(19) $\Omega^{(i,r)}z = \Omega_{i+r-1}\,\Omega_{i+r-2}\ldots\Omega_i = \left(\prod_{j=0}^{r-1} \rho_{i+j}\right)I = \rho^{(i,r)}I.$

Since the f_i and a_i are holomorphic at the origin, Equation (15) shows that the b_i satisfy the same condition, and hence so do the $A_{ir}(z)$ and the $B_{ir}(z)$. Now Equations (16, 19) shows that

(20) $f_i(z) \equiv A_{ir}(z)f_{i+r}(0) + B_{ir}(z) \pmod{\mathfrak{M}^{\rho^{(i,r)}}}$.

This last congruence allows one to construct transcendental numbers by judicious repetition of the $a_i(z)$, $b_i(z)$ and Ω_i in the functional equation (15). For example, let

$$f_i(z) = \prod_{j=1}^{\infty} (1 - z^{\rho^{(i+1,j)}}/n_{i+j})$$

where the $n_j \ll 2^j$ and the $\rho_j \geqq 2$ are rational integers. The $f_i(z)$ are defined in the open unit disc, are dominated in absolute value by

$$\prod_{j=0}^{\infty} (1 + z^{2^j}) = (1 - z)^{-1}$$

and satisfy the functional equations

$$f_i(z) = (1 - z^{\rho_{i+1}}/n_{i+1})f_{i+1}(z^{\rho_{i+1}}).$$

Now the function $h(z) = \prod_{j=0}^{\infty} (1 - z^{\rho^j})$ is transcendental [**6**] for every fixed integer $\rho \geqq 2$. Therefore, if we suppose that there is an integer $\rho \geqq 2$ such that the sequence of (n_j, ρ_j) for $j \geqq 1$ contains arbitrarily long segments of repetitions of the term $(1, \rho)$, then the sequence $f_i(z)$ has $h(z)$ as an $\mathfrak{M}$-adic limit point by Equation (20). By the theorem, it follows that if z_0 is an algebraic point of the punctured open unit disc satisfying $z_0^{\rho^{(1,j)}} \neq n_j$ for all j, then $f(z_0)$ is a transcendental number.

https://doi.org/10.4153/CJM-1977-065-2 Published online by Cambridge University Press

Suppose in addition to the hypotheses of the first paragraph of this section that the set

$$\{a_i(z), b_i(z), \Omega_{i+1}, f_{i+1}(0) | i \geqq 0\}$$

is finite. Then for each fixed $r \geqq 0$, the set

$$\{A_{ir}(z) f_{i+r}(0) + B_{ir}(z) | i \geqq 0\}$$

is also finite and hence $\{f_i(z)\}$ is compact in the $\mathfrak{M}$-adic topology. Suppose that the set of $\mathfrak{M}$-adic limit points of $\{f_i(z)\}$ consists of finitely many power series, say $g_1, \ldots, g_m$. Let $\{f_{i(j)}\}$ be a subsequence of $\{f_i\}$ converging $\mathfrak{M}$-adically to g_1. By induction on N, one can choose a sequence $\{k(s) | s \geqq 1\}$ of non-negative integers such thas for each $N \geqq 1$, there are infinitely many j with

$$(a_{i(j)-s}, b_{i(j)-s}, \Omega_{i(j)+1-s}) = (a_{k(s)}, b_{k(s)}, \Omega_{k(s)+1})$$

for $s = 1, 2, \ldots, N$. By Equation (16), one has for these j,

$$f_{i(j)-N}(z) = A_{i(j)-N,N}(z) f_{i(j)}(\Omega^{(i(j)-N,N)} z) + B_{i(j)-N,N}(z)$$

where $A_{i(j)-N,N}$, $B_{i(j)-N,N}$, and $\Omega^{(i(j)-N,N)}$ are independent of j. It follows that these $f_{i(j)-N}(z)$ converge $\mathfrak{M}$-adically to

$$h_N(z) = A_{i(j)-N,N}(z) g_1(\Omega^{(i(j)-N,N)} z) + B_{i(j)-N,N}(z),$$

and so there is a $t(N)$ with $h_N(z) = g_{t(N)}(z)$. If v is an index occurring more than once in the sequence $\{t(N)\}$, then

$$g_v(z) = A_{ir}(z) g_v(\Omega^{(i,r)} z) + B_{ir}(z)$$

where A_{ir}, B_{ir}, $\Omega^{(i,r)}$ are obtained from a sequence of (a_j, b_j, Ω_{j+1}) of length r whose corresponding functional equation sequence occurs infinitely often in (15). The above discussion together with the theorem implies the following result.

COROLLARY. *Let f_i, a_i, b_i, D, Ω_i, A_{ir}, B_{ir}, and $\Omega^{(i,r)}$ be as in the first two paragraphs of this section. Assume that the set*

$$\{a_i(z), b_i(z), \Omega_{i+j}, f_i(0) | i \geqq 0\}$$

is finite and that there are no algebraic solutions of any of the functional equations

$$(21) \quad f(z) = A(z) f(\Omega z) + B(z)$$

where (A, B, Ω) ranges through the triples occurring infinitely often in the doubly indexed sequence of $(A_{ir}, B_{ir}, \Omega^{(i,r)})$. Let $z_0 = (z_{01}, \ldots, z_{0n})$ be an algebraic point with each z_{0i} lying in the punctured open unit disc and with $|z_{01}|, \ldots, |z_{0n}|$ multiplicatively independent. Suppose that for every $k \geqq 0$, the point $z_0^{\rho^{(k)}} = (z_{01}^{\rho^{(k)}}, \ldots, z_{0n}^{\rho^{(k)}})$ is not a root of the numerator nor of the denominator of any $a_j(z)$ and is not a root of the denominator of any $b_j(z)$. Then the $f_i(z_0)$ are transcendental numbers.

For example, suppose that $n = 1$, the $a_i(z)$ are non-zero constants, and the $b_i(z)$ are non-constant polynomials of degree less than ρ_{i+1}. By replacing the $f_i(z)$ and $b_i(z)$ by $f_i(z) - f_i(0)$ and $b_i(z) - b_i(0)$ respectively, we may suppose that $f_i(0) = b_i(0) = 0$ for all $i \geqq 0$. By Equations (17, 18), we know that the $A_{ir}(z)$ are non-zero constant polynomials and the $B_{ir}(z)$ are non-zero polynomials of degree less then $\rho^{(i,r)}$. In fact, the degree condition on b_{i+j} implies that each term of $b_{i+j}(\Omega^{(i,j}z)$ in Equation (18) has degree in the range $[\rho^{(i,j)}, \rho^{(i,j+1)})$ and so $B_{ir}(z) \not\equiv 0$. By [**1**, Proposition 3], the functional equation (21) therefore can have an algebraic solution only if one (and hence all) its solutions are rational. Counting poles in each member of Equation (21) shows that rational solutions are polynomial, and degree considerations rule out polynomial solutions. Thus we conclude that the functional equations (21) have no algebraic solutions, and so the corollary may be applied. As a special case, one obtains the assertion made in the introduction.

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