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EPIMORPHIC FLAT MAPS*

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Introduction

In this note, we derive a necessary and sufficient condition for a flat map of (commutative) rings to be a flat epimorphism. Flat epimorphisms $\phi: A \to B$ (i.e., ϕ is an epimorphism in the category of rings, and the ring B is flat as an A-module) have been studied by several authors in different forms. Flat epimorphisms generalize many of the results that hold for localizations with respect to a multiplicatively closed set (see, for example [**6**]). In a geometric formulation, D. Lazard [**3**, Chapitre IV, Proposition 2.5] has shown that isomorphism classes of flat epimorphisms from a ring A are in 1-1 correspondence with those subsets of Spec A such that the sheaf structure induced from the canonical sheaf structure of Spec A yields an affine scheme. N. Popescu and T. Spircu [**4**, Théorème 2.7] have given a characterization for a ring homomorphism to be a flat epimorphism, but our characterization, under the assumption of flatness is easier to apply. For corollaries, we can obtain known results due to D. Lazard, T. Akiba, and M. F. Jones, and generalize a geometric theorem of D. Ferrand.

Lazard has also shown that flat epimorphic extensions of a ring are equivalent to flat extensions contained in the maximal flat epimorphic extension. The corresponding result does not hold for flat extensions contained in the maximal *rational* extension, as our condition (and an example) shows.

The torsion theoretic background for this paper may be found in the text [**7**, chapters VI, IX, and XI].

The main theorem

In the following, we shall use the equivalent formulation of epimorphism that the canonical map $\phi \otimes 1: B \to B \otimes_A B$ is an isomorphism. Observe that $\phi \otimes 1$ is always an injection since it is split by the map $\gamma: B \otimes_A B \to B$ defined by $\gamma(b_1 \otimes b_2) = b_1 b_2$. But to show $\phi \otimes 1$ is a surjection (hence ϕ is an epimorphism) it suffices to show that for each cyclic A-submodule C of coker ϕ we have $C \otimes_A B = 0$. By a *dense* ideal I of A it is meant that the annihilator of I in A is zero. The proof of the theorem is based on a method used by F. Richman [**5**].

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Theorem 1. *Let $\phi: A \to B$ be a ring homomorphism. Then ϕ is a flat epimorphism if and only if ϕ is flat and, for all $b \in B$, there is a finitely generated dense ideal I of $\phi(A)$ with $Ib \subseteq \phi(A)$.*

Proof. We first treat the case when B is an extension of A, i.e., $A \subseteq B$ and ϕ is the inclusion map. To prove ϕ is a flat epimorphism, by the above observation, we may prove that for each $b \in B$, if $I = (A:b)_A$ then $IB = B$. Now suppose that B is A-flat and there is (by hypothesis) a finitely generated ideal $J = (x_1, \ldots, x_n) \subseteq I$ with zero annihilator in A. Then the annihilator of J in B is also zero (for if $\alpha: A \to A^n$ is given by $\alpha(a) = (ax_1, \ldots, ax_n)$ then α is injective, whence $\alpha \otimes 1: B \to B^n$ is injective by flatness). Define $\beta: A^2 \to A^n$ by $\beta(a, a')$ in the ith position to be $ax_ib - a'x_i$. Note that $x_ib \in A$ so β is well-defined and that $K = \ker \beta = \{(a, ab) \mid a \in I\}$ since J has zero annihilator in B.

Tensoring the exact sequence $0 \to K \to A^2 \to A^n$ with B, we obtain the exact sequence $0 \to K \otimes B \to B^2 \to B^n$. But the image of $K \otimes B$ in B^2 is just $IB(1, b)$. On the other hand, $(1, b) \in \ker(\beta \otimes 1) = IB(1, b)$. Comparison of first components yields $B = IB$ as desired.

The converse for this case follows readily from [4] where it is shown that flat epimorphisms are rings of quotients with respect to a Gabriel topology of finite type. The general case is immediate, for if $\phi: A \to B$ is a homomorphism and B is A-flat then B is $\phi(A)$-flat, while if B is an epimorphic extension of $\phi(A)$, then ϕ is an epimorphism.

Remarks. If, in the above proof of the extension case, we do not have the finitely generated dense ideal at our disposal, but B is Mittag–Leffler and a ring of quotients with respect to a Gabriel topology, then we may draw the same conclusion by defining a map $\beta': A^2 \to A^I$, using $\beta' \otimes 1: B^2 \to A^I \otimes B \cong B^I$ to argue as above. That B is a ring of quotients ensures that I is a dense ideal of A. We thus obtain as a corollary to the proof of Theorem 1 the known result:

Corollary 2 [2, Prop. 2.1]. *If $B \supseteq A$ is a flat ring of quotients of A that is Mittag–Leffler, then B is a flat epimorphic extension of A.*

An immediate consequence of the theorem is a result essentially due to Richman [5].

Corollary 3. *If B is a ring between a domain A and its quotient field, and B is A-flat, then B is a flat epimorphic extension of A.*

As another application of the theorem, we clarify one of the ideas in the series of papers of T. Akiba [1]. An extension $B \supseteq A$ is called *rational* if for each $b \in B$ there exists a dense ideal I of A with $Ib \subseteq A$. Every rational extension, up to isomorphism, is contained in the maximal rational extension $Q_{rat}(A)$, the ring of quotients of A with respect to the Gabriel topology consisting of the dense ideals of A. Flat epimorphic extensions of A are clearly rational. On the other hand, Lazard has shown that there is a maximal flat epimorphic extension $M(A)$ of A, hence $A \subseteq M(A) \subseteq Q_{rat}(A)$. In addition, he proves that flat extensions of A contained in $M(A)$ are epimorphic. However, this last result is not true for flat extensions contained in $Q_{rat}(A)$. For an example of a flat rational extension that is not epimorphic, we turn to von Neumann rings, rings deficient in *finitely generated* dense ideals.

Example 4. Let F be a field and let R be the subring of $\prod_{i=1}^{\infty} F$ consisting of all eventually constant sequences. Since R is von Neumann regular, every finitely generated ideal is generated by an idempotent, and every module is flat. It follows from the theorem that $R=M(R)$. There is only one proper dense ideal $\mathbf{m}=\oplus F$, so that $Q_{rat}(R)=\mathrm{Hom}(\mathbf{m},\mathbf{m})=\Pi F$. Hence, $Q_{rat}(R)\supsetneq R$ is a flat and rational, but not epimorphic, extension of R.

Our final application is to algebraic-geometry. If G is a subset of Spec A, we denote by $Q_G(A)$ the ring of quotients with respect to the Gabriel topology $\{I\subseteq A \mid \forall p\in G, I\subsetneq p\}$. Recall that if $\phi: A\to Q_G(A)$ is the natural map then ker ϕ and coker ϕ are torsion, while $Q_G(A)$ is torsion free. The following generalizes a geometric result of D. Ferrand [3, Proposition 5.12, p. 121].

Corollary 5. *Let G be a quasi-compact subset of* Spec A. *If the natural map $\phi: A\to Q_G(A)$ is flat then ϕ is a flat epimorphism. In this case G imbeds in* Spec $Q_G(A)$.

Proof. Let $q\in Q_G(A)$ and $I=(\phi(A):q)_A$. Since coker ϕ is torsion, I is in the Gabriel topology. Thus for each $p\in G$, there exists an element $x\in I\backslash p$, i.e., the basic open sets $D(x)$, $x\in I$, cover G. Select a finite subcover $\{D(x_1),\dots,D(x_n)\}$ and let $J=(x_1,\dots,x_n)\subseteq I$. It follows that J is in the topology and $Jq\subseteq\phi(A)$. Since $\phi(A)\subseteq Q_G(A)$ is torsion free, $\phi(J)$ is a dense ideal of $\phi(A)$ and satisfies the hypotheses of the theorem.

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