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Asymptotics determined by pairs of additive polynomials*

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Introduction

This article determines the asymptotic behavior of a class of two variable 'simultaneous' lattice point problems in $\mathbb{R}^n$ for any $n \geqslant 2$. An earlier article [Li-3] has studied a similar problem for a large class of such problems in $\mathbb{R}^2$.

In general, it seems fair to say that the precise asymptotic behavior of such problems is much less clearly understood than that of their classical counterpart in 1 variable. Primarily, this is due to the greater difficulty in understanding the polar structure of a 2 variable Dirichlet series, whose Mellin transform in 2 variables counts the number of lattice points of interest.

In [Li-1] a general class of simultaneous lattice point problems was defined with the hope that their asymptotic analysis could be carried out in a manner that generalized nicely the well known and standard 1 variable method (see [La, Li-4, Ma, Sa]). Roughly speaking, the idea is the following. Suppose P_1, P_2 are two polynomials on $\mathbb{R}^n$ that satisfy the growth condition of 'hypoellipticity' on $[1, \infty)^n$ (see Section 1). One may then assume that each is positive outside a compact subset of $[1, \infty)^n$. One is interested in describing the precise asymptotic behavior of the function

$$N(t_1, t_2) = \#\mathbb{N}^n \cap \{P_1 \leqslant t_1\} \cap \{P_2 \leqslant t_2\} \quad \text{as } t_1, t_2 \to \infty.$$

To do so, one introduces the Dirichlet series (setting $\mathbf{s} = (s_1, s_2)$)

$$D(\mathbf{s}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}^n \\ P_1 \cdot P_2(\mathbf{m}) \neq 0}} \frac{1}{P_1(\mathbf{m})^{s_1} P_2(\mathbf{m})^{s_2}},$$

for which $N(t_1, t_2)$ is a Mellin transform. It is not difficult to see that $D(\mathbf{s})$ is analytic in a region of the form $\{\sigma_1, \sigma_2 > c, \}, c \gg 1$, where $\sigma_i = \mathrm{Re}(s_i)$ for each

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i. Moreover, $D(\mathbf{s})/s_1 s_2$ admits an analytic continuation to $\mathbb{C}^2$ as a meromorphic function, for which the real part of its domain of analyticity is bounded by a polygon, denoted subsequently by Γ (see [Li-1]). Evidently, Γ is the natural generalization of the largest pole of a Dirichlet series determined by one hypoelliptic polynomial. Let $\mathcal{V}$ resp. $\mathcal{L}$ denote the set of vertices resp. complex lines that contain a face of Γ (bounded or unbounded). It is then natural to ask if Γ contributes effectively to the dominant asymptotics of $N(t_1, t_2)$. The most important ingredient of an affirmative answer to this question is a nonvanishing property for an iterated residue, stated as follows:

CONJECTURE. For each $v \in \mathcal{V}$ and each $L \in \mathcal{L}$, for which $v \in L$,

$$\mathrm{Res}_v\, \mathrm{Res}_L (t_1^{s_1} t_2^{s_2} D(\mathbf{s})/s_1 s_2 \,\mathrm{d}s_1\, \mathrm{d}s_2) \neq 0. \tag{0.1}$$

[Li-3] verified this conjecture for the class of nondegenerate and hypoelliptic polynomials on $[1, \infty)^2$.

If one verifies this conjecture for a vertex v, it follows from the analysis in [Li-1,3], that there exist unbounded semi-algebraic regions $\mathcal{R}(v)$ in which $N(t_1, t_2)$ grows like a monomial $A_v t_1^{v_1} t_2^{v_2}$, where $A_v > 0$ and $v = (v_1, v_2)$. (Of course, one may need to include $\log t_i$ factors if the multiplicity of D along L is greater than one.)

To verify the conjecture for a particular vertex v, one is led to replace $D(\mathbf{s})$ by the integral

$$I(\mathbf{s}) = \int_{[1,\infty)^n} \frac{1}{P_1(x)^{s_1} P_2(x)^{s_2}}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

[Li-2] shows that (0.1) holds for v iff the iterated residue, obtained by replacing $D(\mathbf{s})$ by $I(\mathbf{s})$ is nonzero. This observation now permits one to employ methods of singularity theory to analyze $I(\mathbf{s})/s_1 s_2$ in a neighborhood of each vertex of Γ.

On the other hand, there is, as yet, no general result that insures that for any pair of hypoelliptic P_1, P_2, the nonvanishing property (0.1) holds at any vertex of the polygon Γ. To encourage belief that such a result indeed exists, it is helpful to have some good supporting evidence. Whereas [Li-3] gave an affirmative solution to the conjecture for a class of polynomials in two variables, it is also instructive to show that an affirmative solution exists for pairs of polynomials in more than two variables. The class studied in this paper is the following.

Let $(b_1, \ldots, b_n) \neq (c_1, \ldots, c_n)$ be tuples of positive integers. One defines:

$$P_1(x) = \sum_{i=1}^{n} x_i^{b_i}, \qquad P_2(x) = \sum_{i=1}^{n} x_i^{c_i}.$$

The conjecture's solution for this class of pairs of 'additive' polynomials has a simple and elegant form. To formulate it precisely, suppose that co-

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ordinates are chosen so that $b_1/c_1 \leqslant b_2/c_2 \leqslant \cdots \leqslant b_n/c_n$. Further, define $(b_{n+1}, c_{n+1}) = (1, 0)$. Suppose, for simplicity here, that these n ratios are distinct. One now defines the following points and regions of $[1, \infty)^2$;

$$v_0 = \left(\sum_{j=1}^{n} 1/b_j, 0\right), \qquad v_i = \left(\sum_{j=i+1}^{n} 1/b_j, \sum_{j=1}^{i} 1/c_j\right), \quad i \in [1, n-1],$$

$$v_n = \left(0, \sum_{j=1}^{n} 1/c_j\right),$$

$$\mathcal{R}(v_0) = \{t_1^{c_1/b_1} < t_2\}, \qquad \mathcal{R}(v_i) = \{t_1^{c_{i+1}/b_{i+1}} < t_2 < t_1^{c_i/b_i}\}, \quad i \geqslant 1.$$

Let (v_{1i}, v_{2i}) denote the coordinates of v_i.

Next, one introduces two types of subsets of these regions. For each point $v = v_0, \ldots, v_n$, one uses the notation $\mathcal{R}_\infty(v)$ to denote any unbounded, connected, semialgebraic subset of $\mathcal{R}(v)$ such that

$$d\left((t_1, t_2), \partial\mathcal{R}(v)\right) \to \infty \quad \text{if } (t_1, t_2) \to (\infty, \infty), \quad (t_1, t_2) \in \mathcal{R}_\infty(v).$$

In addition, if γ is any unbounded analytic arc lying in $\mathcal{R}(v)$, one says that γ is asymptotic to $\partial\mathcal{R}(v)$ at infinity iff

$$d\left(|\gamma|, \partial\mathcal{R}(v) \cap \{\|(t_1, t_2)\| \leqslant t\}\right) \to 0 \quad \text{as } t \to \infty.$$

Given this geometric data, the paper establishes a precise description for the dominant behavior of $N(t_1, t_2)$, as follows.

THEOREM. (1) *For each i there exists an effectively determined $B_i > 0$, so that, given any subregion $\mathcal{R}_\infty(v_i)$, a positive θ exists such that*

$$N(t_1, t_2) = B_i t_1^{v_{1i}} t_2^{v_{2i}} + O(t_1^{v_{1i}-\theta} t_2^{v_{2i}-\theta}),$$
$$\textit{if } (t_1, t_2) \to (\infty, \infty) \quad \textit{and} \quad (t_1, t_2) \in \mathcal{R}_\infty(v_i).$$

(2) *Let γ denote an arc asymptotic to $\partial\mathcal{R}(v_i)$ at infinity. Let t be a parameter for γ so that for $t \gg 1$*

$$t_1(t) = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \cdots \quad \textit{where } \alpha_1 > \alpha_2 > \cdots,$$
$$t_2(t) = a_1' t^{\beta_1} + a_2' t^{\beta_2} + \cdots \quad \textit{where } \beta_1 > \beta_2 > \cdots.$$

There exists a positive number $\epsilon = \epsilon(\gamma)$ so that $t \gg 1$ implies

$$N(t_1, t_2)|_\gamma = B_i t^{v_i \cdot (\alpha_1, \beta_1)} + O\left(t^{v_i \cdot (\alpha_1, \beta_1) - \epsilon}\right).$$

If there are $r < n$ distinct ratios among the b_j/c_j, then there will only be $r + 1$ distinct points v_i and regions $\mathcal{R}(v_i)$ that determine the dominant asymptotic terms

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of $N(t_1, t_2)$, the form for which is similar to that above (see (5.1), (5.4), and (6.2) for the general statement).

As the proof of the Theorem makes clear, the points $v_0, v_1, \ldots, v_n$ are *precisely* the vertices of the polygon Γ, associated to the Dirichlet series, defined above for the pair of polynomials. In addition, for $i \leqslant n-1$, the slopes of the 2 sides that intersect at v_i are $-b_i/c_i$ and $-b_{i+1}/c_{i+1}$. For v_n the two sides consist of a vertical ray and a segment with slope $-b_n/c_n$. Moreover, $D(\mathbf{s})/s_1s_2$ has a simple pole along each line containing a face of Γ. Similar properties hold if there are $r < n$ distinct values (see Theorem 6.2). As a result, the Conjecture is verified for this pair of polynomials.

The principal idea of the proof is sketched in Section 1 (see [Li-3] also). Essentially, one needs to construct a set of simplicial cones, satisfying (1.4)(i–iii) *and* (1.7.1), (1.7.2). Sections 2–5 carry out this construction. Section 6 gives the proof of the Theorem. Since the details are a little intricate, Section 7 provides an example to help guide the reader through the arguments. The reader is encouraged to consult first the definitions given in Part 1 of Section 2, and then the example, before tackling the details of the general argument. Section 8 gives explicit estimates inside any $\mathcal{R}_\infty(v_i)$ of the difference between $N(t_1, t_2)$ and the volume of the region $\{P_1 \leqslant t_1\} \cap \{P_2 \leqslant t_2\} \cap [1, \infty)^n$.

1. Preliminaries

This section briefly summarizes the results obtained in [Li-1, 2, 3] that will be needed in the paper.

Additive polynomials are both hypoelliptic on $[1, \infty)^n$ [Hö, ch. 11] and nondegenerate with respect to their polyhedra at infinity [Sa]. These are growth conditions that enable the Theorem to be proved using geometric-analytic methods.

The first point is that the series $D(\mathbf{s})$ can essentially be replaced by the integral

$$I(\mathbf{s}) = \int_{[1,\infty)^n} \frac{1}{P_1^{s_1} P_2^{s_2}}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n\,.$$

It is standard to see that $I(\mathbf{s})$ possesses analytic properties similar to those of $D(\mathbf{s})$ stated in the Introduction. Thus, the boundary of the real part of the domain of analyticity of $I(\mathbf{s})/s_1s_2$ is also a polygon, denoted below as $\Gamma(\mathbf{P})$. One now uses:

THEOREM 1.1. (1) *The polygon* Γ, *defined in the Introduction, equals* $\Gamma(\mathbf{P})$.

(2) *There exists an open neighborhood of* Γ *that is unbounded in the imaginary directions of* $\mathbb{C}^2$, *such that* $D - I$ *is analytic in the neighborhood.*

Remark. This is shown in [Li-2]. (1) follows from hypoellipticity of each $P_i|_{[1,\infty)^n}$. (2) implies that for any vertex v of Γ and line L containing a face of Γ, one has

$$\operatorname{Res}_v \operatorname{Res}_L \bigl(t_1^{s_1} t_2^{s_2} I(\mathbf{s})/s_1s_2 \; \mathrm{d}s_1 \mathrm{d}s_2\bigr) = \operatorname{Res}_v \operatorname{Res}_L \bigl(t_1^{s_1} t_2^{s_2} D(\mathbf{s})/s_1s_2 \; \mathrm{d}s_1 \mathrm{d}s_2\bigr)\,.$$

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Thus, using the arguments of [Li-1, Sect. 6] and [Li-3, Sect. 1, appx], the asymptotic for $N(t_1, t_2)$ inside each $\mathcal{R}_\infty(v_i)$ follows from Theorem 1.1 *and* the following:

$$\mathrm{Res}_{v_i}\, \mathrm{Res}_L \left(t_1^{s_1} t_2^{s_2} I(\mathbf{s})/s_1 s_2 \; \mathrm{d}s_1 \mathrm{d}s_2\right) \neq 0 \text{ for } (t_1, t_2) \in \mathcal{R}(v_i). \tag{1.2}$$

It is *not* a priori clear that (1.2) should occur. To show nonvanishing, one needs a precise description of the Laurent series of $I(\mathbf{s})/s_1 s_2$ at v_i. The following suffices for this paper. Its easy proof is left to the reader.

PROPOSITION 1.3. *Let v be a vertex of Γ. Suppose the following conditions hold at v :*

(A) *Exactly two components L_1, L_2 of the polar locus of $I(\mathbf{s})/s_1 s_2$ contain v.*
(B) *Let L_i be defined by the form ℓ_i. In a neighborhood of v one has*

$$\frac{I(\mathbf{s})}{s_1 s_2} = \frac{I^*(\mathbf{s})}{\ell_1 \ell_2} + (\text{ analytic function }),$$

where $I^*(\mathbf{s})$ is analytic at v and $I^*(v) > 0$.

Then for each $j = 1, 2$

$$\mathrm{Res}_v\, \mathrm{Res}_{L_j} \left(t_1^{s_1} t_2^{s_2} I(\mathbf{s})/s_1 s_2 \; \mathrm{d}s_1 \mathrm{d}s_2\right) \neq 0 \text{ for all } (t_1, t_2) \in \mathcal{R}(v)\,. \qquad \square$$

Let the polyhedron of P_i at infinity be denoted Γ_i. Given a direction vector $\xi \in (\mathbb{R}^n_+)^*$ and $j = 1, 2$, set

$$M_j(\xi) = \max\{\xi \cdot x : x \in \Gamma_j\} \quad \text{and} \quad \mathcal{K}_j(\xi) = \{x \in \Gamma_j : \xi \cdot x = M_j(\xi)\}\,.$$

One now partitions $(\mathbb{R}^n_+)^*$ as follows.

PROPOSITION 1.4. *There exist closed simplicial cones $C_i = \langle a_1(i), \ldots, a_n(i)\rangle_{\mathbb{R}_+}, i = 1, \ldots, R$, such that*

(i) $(\mathbb{R}^n_+)^* = \cup_{i=1}^R C_i$;
(ii) $\dim C_i \cap C_j < n$ if $i \neq j$;
(iii) $a_1(i), \ldots, a_n(i) \in \mathbb{Z}^n_+$, are linearly independent, and $\bigcap_{k=1}^n \mathcal{K}_j(a_k(i)) \neq \emptyset$, $j = 1, 2$.

To each vector $a_r(i)$, in the 1-skeleton of C_i, one sets $|a_r(i)| =$ sum of coordinates of $a_r(i)$, and defines the line resp. ‘upper’ halfplane as follows:

$$\begin{aligned} L\, a_r(i) &= \{(s_1, s_2) \in \mathbb{C}^2 :\; M_1(a_r(i))s_1 + M_2(a_r(i))s_2 = |a_r(i)|\}, \\ L^+ a_r(i) &= \{(\sigma_1, \sigma_2) \in \mathbb{R}^2 :\; M_1(a_r(i))\sigma_1 + M_2(a_r(i))\sigma_2 \geqslant |a_r(i)|\}\,. \end{aligned} \tag{1.5}$$

Let $\mathcal{T}$ denote a set of cones satisfying (1.4)(i–iii). Define the polygon

$$\hat{\Gamma}(\mathcal{T}) = \partial \left(\bigcap_{i=1}^R \bigcap_{r=1}^n L^+ a_r(i) \cap \mathbb{R}^2_+ \right).$$

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(A similar polygon was defined in [Li-3, p. 715], but an error appears in the statement of the region $\mathcal{E}$ whose boundary is the polygon. One should replace the union over j, β by the intersection.)

Proved in [ibid] is the following invariance property, due solely to the non-degeneracy of each P_i.

PROPOSITION 1.6.

(1) *For any collection* $\mathcal{T}$ *satisfying* (1.4)(i–iii), $\hat{\Gamma}(\mathcal{T}) = \Gamma(\mathbf{P})$.
(2) $\hat{\Gamma}(\mathcal{T}) = \Gamma$.

One now states two properties (which may or may not be true) about the collection of lines $\{La_j(i)\}_{i,j}$. To do so, one defines for each vertex v of Γ, family $\mathcal{T}$ satisfying (1.4), and $i = 1, ..., R$, the sets $\mathcal{L}_v(i) = \{r : v \in La_r(i)\}$. The properties are as follows:

(1.7.1) For each vertex $v \in \Gamma$, not lying on a coordinate axis, there exist i and $r_1 \neq r_2$ such that:

(a) $L\,a_{r_1}(i), L\,a_{r_2}(i)$ contain the segments of Γ intersecting at v;
(b) $v \notin L\,a_r(i)$ if $r \neq r_1, r_2$;
(c) For any i' for which $\mathcal{L}_v(i') \neq \emptyset$, any line in this set must contain a face of Γ.

(1.7.2) For each vertex v lying on a coordinate axis, $v \in La_r(i)$ implies $La_r(i)$ contains a face of Γ.

It is then easy to see ([Li-3, Sect. 1]):

PROPOSITION 1.8. *If conditions* (1.7.1), (1.7.2) *are true at each vertex, then the hypotheses* (A), (B) *of Proposition* 1.3 *are satisfied at each vertex.*

2. Ordering of lines with given slope

There are two parts to this section. The first contains notations and definitions of basic objects that will be needed in the rest of the article. The second contains some elementary ordering properties of the lines from which one forms Γ.

Part 1: Notations

(2.1) Given the ordering $b_1/c_1 \leqslant \cdots \leqslant b_n/c_n$, set $r = \#\{b_1/c_1, \ldots, b_n/c_n\}$, and write the distinct elements of this set as $\{\rho_1 < \rho_2 < \cdots < \rho_r\}$.

(2.2) Define $k_{r+1} = n + 1$, and for $j = 1, \ldots, r$ set $k_j = \min\{i : b_i/c_i = \rho_j\}$, and $\mathcal{I}_j = [k_j, k_{j+1} - 1]$.

(2.3) An n-*chain* of subsets of $\{1, 2, \ldots, n\}$ is a chain $C : A(1) \subset A(2) \subset \cdots \subset A(n)$ such that $|A(i)| = i, i = 1, 2, \ldots, n$. The elements of $A(u)$ will be written

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as $A(u) = \{i_1(u) < i_2(u) < \cdots < i_u(u)\}$. In particular, $i_1(u)$ always denotes $\min A(u)$, for each $u = 1, \ldots, n$.

(2.4) The collection of all n-chains is denoted $\mathcal{F}$. Given $C \in \mathcal{F}$, the unique number in $A(1)$ is called the *root* of C. The subset of chains rooted at i is denoted $\mathcal{F}_i$.

(2.5) Given any chain C, and $\ell \in \{1, \ldots, n\}$, set $\iota(\ell) =$ unique j for which $i_1(\ell) \in \mathcal{I}_j$.

(2.6) For $C \in \mathcal{F}_i$ define the collection of subsets of $\{1, \ldots, n\}$:

$$\begin{aligned}\mathcal{S}(C) = \{\{u_1 < u_2 < \cdots < u_k\} : \iota(u_1) = \iota(1) \quad \text{and} \\ \iota(u_1) > \iota(u_2) > \cdots > \iota(u_k)\}.\end{aligned}$$

(2.7.1) Given any $\mu = \{u_1 < \cdots < u_k\} \in \mathcal{S}(C)$, one defines a sequence $\mu^* = \{u_1^* < \cdots < u_k^*\}$ by the rule:

$$\text{for each } j, \quad u_j^* = \max\{\ell : \ell \geqslant u_j \quad \text{and} \quad \iota(\ell) = \iota(u_j)\}.$$

The sequence μ^* is called the *maximal sequence* for μ.

(2.7.2) Given the chain C, set

$$\begin{array}{ll}\kappa_1 = \max\{\iota(\ell) : \ell \in [1, n]\} & u_1^*(C) = \max\{\ell : \iota(\ell) = \kappa_1\} \\ \kappa_2 = \max\{\iota(\ell) : \ell \geqslant u_1^*(C) + 1\} & u_2^*(C) = \max\{\ell : \iota(\ell) = \kappa_2\} \\ \quad \vdots & \quad \vdots \\ \kappa_d = 1 & u_d^*(C) = n.\end{array}$$

The κ_j are strictly decreasing while the $u_j^*(C)$ are strictly increasing. The index d is therefore the *smallest* integer ℓ so that $\kappa_\ell = 1$. Its value is then well defined (and depends upon C). One writes it as d. This integer is called the *depth* of the chain C. The sequence $\mu^*(C) \stackrel{\text{def}}{=} \{u_j^*(C)\}$ is called the *maximal sequence* of C. One then partitions $\{1, \ldots, n\}$ by setting $u_0^*(C) = 0$, and for each $w = 1, \ldots, d$, one defines $\mathcal{U}_w(C) = [u_{w-1}^*(C) + 1, u_w^*(C)]$.

(2.8) For $k \geqslant 2$ and $1 \leqslant j_1 < \cdots < j_k \leqslant n$, define the direction vector

$$e_{j_1,\ldots,j_k} = \sum_{q=1}^{k} \left(\prod_{i \neq q} b_{j_i} \right) \mathbf{e}_{j_q},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the unit basis vectors for $\mathbb{R}^n$. (If $k = 1$, then $e_{j_1} = \mathbf{e}_{j_1}$.)

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(2.9) To each chain C, whose elements one writes as in (2.3), one associates

$$\text{the vectors} \qquad E_u \stackrel{\text{def}}{=} e_{i_1(u),\dots,i_u(u)}, \quad u = 1,\dots,n,$$
$$\text{and closed cone} \qquad \mathcal{C}(C) = \langle E_1,\dots,E_n\rangle_{\mathbb{R}_+}.$$

(2.10) For each $k = 1,\dots,n$, set $I_k = b_k\mathbf{e}_k$ and $J_k = c_k\mathbf{e}_k$. The face of Γ_1 resp. Γ_2 spanned by $I_{j_1},\dots,I_{j_k}$ resp. $J_{j_1},\dots,J_{j_k}$ is denoted $[I_{j_1},\dots,I_{j_k}]$ resp. $[J_{j_1},\dots,J_{j_k}]$. Define

$$[I_{j_1},\dots,I_{j_k}]^\perp = \{\xi \in (\mathbb{R}^n_+)^* : \mathcal{K}_1(\xi) = [I_{j_1},\dots,I_{j_k}]\},$$
$$[J_{j_1},\dots,J_{j_k}]^\perp = \{\xi \in (\mathbb{R}^n_+)^* : \mathcal{K}_2(\xi) = [J_{j_1},\dots,J_{j_k}]\}.$$

(so, $e_{j_1,\dots,j_k} \in [I_{j_1},\dots,I_{j_k}]^\perp$ for all $j_1 < \cdots < j_k$).

(2.11) For each i, $C \in \mathcal{F}_i$, and $\mu = \{u_1 < \cdots < u_k\} \in \mathcal{S}(C)$, one defines the subcone of $(\mathbb{R}^n_+)^*$

$$\Sigma(\mu) = \left[\bigcup_{i=1}^{k} \{J_a\}_{a\in A(u_i)\cap\mathcal{I}_{\iota(u_i)}}\right]^\perp.$$

(2.12) For a nonvertical line L with nonzero slope, $h(L)$ resp. $v(L)$ denotes the s_1 resp. s_2 axis intercept. For two such lines L, L' which are also parallel, one defines the ordering

$$L' \preceq L \quad \text{iff} \quad h(L') \leqslant h(L) \quad \text{iff} \quad v(L') \leqslant v(L).$$

One writes $L' \prec L$ if there is strict inequality in the axis intercepts.

Part 2: Elementary aspects of the ordering of lines

The argument proceeds by constructing a set of simplicial cones that satisfies (1.4)(i–iii), and then showing that (1.7.1), (1.7.2) are also satisfied. Such cones will be constructed in Section 6, using an operation of subdivision that is based upon defining recurrently, new direction vectors, starting from the E_u, (see (2.14), (2.16)). In order to describe Γ, one must then understand how the lines, determined by the vectors as in (1.5), are ordered. This will be the subject of Section 2, part 2, and Section 3. The proofs of the lemmas in this section are all elementary (see Addendum for details).

One first observes:

LEMMA 2.13.

(1) *The collection of cones* $\{\mathcal{C}(C) : C \in \mathcal{F}\}$ *satisfies properties* (i–ii) *of* (1.4).
(2) *Let* $i_1(1)$ *denote the root of chain* C. *Then* $\bigcap_{j=1}^n \mathcal{K}_1(E_j) = \{I_{i_1(1)}\}$.

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(3) *The following properties are satisfied for any* $C : A(1) \subset \cdots A(n) \in \mathcal{F}$, *and* $u = 1, \ldots, n$:

(a) *For any* j, *and* $b, c \in \mathcal{I}_j$, $\quad E_u \cdot J_b = E_u \cdot J_c \quad$ *if* $b, c \in A(u) \cap \mathcal{I}_j$;
(b) $M_1(E_u) = \prod_{q \in A(u)} b_q = E_u \cdot I_\ell \quad$ *for all* $\ell \in A(u)$;
(c) *If* $i_1(u) \in \mathcal{I}_j$, *then*

$$M_2(E_u) = c_{i_1(u)} \cdot \prod_{\ell \neq 1} b_{i_\ell(u)} = E_u \cdot J_q \quad \textit{for all } q \in A(u) \cap \mathcal{I}_j;$$

(d) $\iota(u) > \iota(v)$ *implies* $i_1(u) > i_1(v)$, *and* $E_v \cdot (J_{i_1(v)} - J_{i_1(u)}) > 0$. *However,* $\iota(u) = \iota(v)$ *implies* $E_v \cdot J_{i_1(v)} = E_v \cdot J_{i_1(u)}$. *Further,* $E_u \cdot J_{i_e(v)} = 0$ *if* $i_e(v) \in \mathcal{I}_{\iota(v)}$.
(e) *For* $u \leqslant n - 1$,

$$\frac{|E_u|}{M_1(E_u)} < \frac{|E_{u+1}|}{M_1(E_{u+1})};$$

(f) $\mathrm{sl}\, LE_u = -b_{i_1(u)}/c_{i_1(u)}$;
(g) *If* $u < v$ *and* $\iota(u) = \iota(v)$, *then* $LE_u \prec LE_v$.

Note. From (2.13)(3c), it follows that no refinement of $\mathcal{C}(C)$ is needed whenever C is rooted at $i \in \mathcal{I}_1$. However, if the root belongs to $\mathcal{I}_j$, $j > 1$, then a refinement is needed to insure (1.4)(iii) holds with respect to Γ_2. In the following, one will therefore assume $C \in \mathcal{F}_i$ for some $i \geqslant k_2$. □

Given chain $C, u < v$, and the vectors E_u, E_v (see (2.9)) one constructs a third vector $f(E_u, E_v)$, as follows. Define

$$f(E_u, E_v) = \begin{cases} [E_v \cdot (J_{i_1(v)} - J_{i_1(u)})]E_u + [E_u \cdot J_{i_1(u)}]E_v & \text{if } \iota(u) > \iota(v) \\ 0 & \text{if } \iota(u) = \iota(v). \end{cases} \tag{2.14}$$

Note. When no confusion can result, one writes this vector as $f(u, v)$. □

One then verifies:

LEMMA 2.15. *If* $u < v$ *and* $\iota(u) > \iota(v)$, *then:*

(i) $f(u, v) \in \Sigma(u, v)$;
(ii) $\mathrm{sl}\, Lf(u, v) = \mathrm{sl}\, LE_u$;
(iii) $LE_u \prec Lf(u, v)$.

The operation of forming $f(u, v)$ can be iterated and thereby extended to any element $\mu = \{u_1 < \cdots < u_k\} \in \mathcal{S}(C)$. One defines the sequence of direction vectors $f_k(u_1, \ldots, u_k)$ recurrently as follows:

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$$\begin{aligned} &\text{for } u_1 < u_2 \quad f_1(u_1) = E_{u_1}, \quad f_2(u_1,u_2) = f(E_{u_1}, E_{u_2}),\\ &\text{for } k \geqslant 3, \quad f_k(\mu) \overset{\text{def}}{=} f_k(u_1,\dots,u_k) = f(E_{u_1}, f_{k-1}(u_2,\dots,u_k)). \end{aligned} \tag{2.16}$$

The line, determined by $f_k(\mu)$, is denoted $Lf_k(\mu)$ (see (1.5)).

Note. By the expression in (2.16), one means the following. By (2.15), when $k = 3$ the vector $f_2(u_2,u_3) \in \Sigma(u_2,u_3)$. So, for any $a \in A(u_2) \cap \mathcal{I}_{\iota(u_2)}$, $b \in A(u_3) \cap \mathcal{I}_{\iota(u_3)}$,

$$\begin{aligned} f_2(u_2,u_3)\cdot J_a &= f_2(u_2,u_3)\cdot J_b > f_2(u_2,u_3)\cdot J_c,\\ &\qquad \text{for any } c \notin (A(u_2)\cap\mathcal{I}_{\iota(u_2)}) \cup (A(u_3)\cap\mathcal{I}_{\iota(u_3)}). \end{aligned}$$

In particular, $i_1(u_1) \notin \Big(A(u_2)\cap\mathcal{I}_{\iota(u_2)}\Big) \cup \Big(A(u_3)\cap\mathcal{I}_{\iota(u_3)}\Big)$. The vector $f_3(u_1,u_2,u_3)$ is then defined to equal

$$[f_2(u_2,u_3)\cdot(J_{i_1(u_2)} - J_{i_1(u_1)})]E_{u_1} + [E_{u_1}\cdot J_{i_1(u_1)}]\cdot f_2(u_2,u_3)$$

whenever $\{u_1 < u_2 < u_3\} \in \mathcal{S}(C)$. One verifies easily, that $f_3(u_1,u_2,u_3) \in \Sigma(\{u_1,u_2,u_3\})$. So, it can be used to define any f_4 in the same way, and so forth. □

One then observes:

LEMMA 2.17. *If* $C \in \mathcal{F}_i$ *and* $\mu = \{u_1 < \cdots < u_k\} \in \mathcal{S}(C)$, *then:*

(i) $f_k(\mu) \in \Sigma(\mu)$;
(ii) $\mathrm{sl}\, Lf_k(\mu) = \mathrm{sl}\, Lf_1(u_1)$;
(iii) $Lf_1(u_1) \prec Lf_k(\mu)$.

The lines $Lf_k(\mu), Lf_{k-1}(\mu - \{u_k\})$ are now known to be parallel. But one does not yet know their order with respect to $\prec$. This is given in the next lemma.

LEMMA 2.18. *Let* $k \geqslant 2$ *and* $\mu \in \mathcal{S}(C)$. *Then* $Lf_{k-1}(\mu - \{u_k\}) \prec Lf_k(\mu)$.

To complete the study of the ordering of lines produced by the recurrence (2.16) within a given chain C, one also needs to understand the relation among the lines determined by vectors of the form $f_k(u'_1,\dots,u'_k), f_k(u_1,\dots,u_k)$ if $u'_j \leqslant u_j$, and $\iota(u'_j) = \iota(u_j)$ for all j. To this end, the following property suffices.

LEMMA 2.19. *For any* k *and* $e \leqslant k$ *for which* $u'_e < u_e$ *and* $\iota(u'_e) = \iota(u_e)$, *one has*

$$Lf_k(u_1,\dots,u_{e-1},u'_e,u_{e+1},\dots,u_k) \prec Lf_k(u_1,\dots,u_{e-1},u_e,u_{e+1},\dots,u_k).$$

A corollary of (2.19) serves to simplify the work below (see (2.7)).

LEMMA 2.20. *Let* μ^* *be the maximal sequence for* μ. *Then:*

(1) $u_k^* = n$;
(2) $Lf(\mu) \prec Lf(\mu^*)$.

3. The maximal line in $\mathcal{F}_i$

To determine explicitly the polygon Γ, the analysis in Section 2 does not seem sufficient. It appears necessary to find more precise expressions for a vector whose corresponding line is maximal with slope $-b_i/c_i$. This section identifies, for each i, a chain $\mathcal{C}_i$ for which $Lf(\mu^*(\mathcal{C}_i)) = \max\{Lf(\mu) : \mu \in \mathcal{S}(C), C \in \mathcal{F}_i\}$ (see (3.6)).

Thus, let $C : A(1) \subset \cdots \subset A(n)$ belong to $\mathcal{F}_i$ and $\mu = \{u_1 < \cdots < u_k\} \in \mathcal{S}(C)$. One needs to determine explicitly positive integers $\lambda_1, \ldots, \lambda_k$, depending upon the choice of μ, so that $\sum_{j=1}^k \lambda_j E_{u_j} \in \Sigma(\mu)$. Thus, they must satisfy the equations:

$$\begin{aligned}\lambda_k E_{u_k} \cdot J_{i_1(u_k)} &= \lambda_{k-1} E_{u_{k-1}} \cdot J_{i_1(u_{k-1})} + \lambda_k E_{u_k} \cdot J_{i_1(u_{k-1})} \\ &= \cdots = \sum_{j=1}^{k} \lambda_j E_{u_j} \cdot J_{i_1(u_1)}. \end{aligned} \tag{3.1}$$

Solutions of (3.1) can be found by hand. To write them concisely, introduce the

NOTATION.

(1) For each $j \in [1, k-1]$, set $\delta_{j,j+1} = c_{i_1(u_{j+1})} b_{i_1(u_j)} - c_{i_1(u_j)} b_{i_1(u_{j+1})}$.
(2) Set $B_k = 1$, and for each $j \leqslant k-1$, set (see (2.3)):

$$B_j = \prod_{q \in A'(u_k) - A(u_j)} b_q.$$

Note. Although these quantities depend upon the chain C and element μ of $\mathcal{S}(C)$, the dependence will not be emphasized in the notation for the sake of simplicity. This will hopefully not lead to any confusion on the reader's part. □

One notes that $\delta_{j,j+1} > 0$, for any $\mu \in \mathcal{S}(C)$.

LEMMA 3.2. *Let* $\mu = \{u_1 < \cdots < u_k\}$. *Then:*

(1) *Positive integral solutions to* (3.1) *are given by*

$$\lambda_k = \prod_{j=1}^{k-1} c_{i_1(u_j)}, \qquad \lambda_{k-1} = \prod_{j=1}^{k-2} c_{i_1(u_j)} \cdot B_{k-1} \cdot \delta_{k-1,k},$$

and for $r \geqslant 2$

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$$\lambda_{k-r} = \left(\prod_{\substack{j=1 \\ j\neq k-r,k-r+1}}^{k} c_{i_1(u_j)} \right) \cdot B_{k-r} \cdot \delta_{k-r,k-r+1}.$$

(2) *Let* $\xi_1, \ldots, \xi_k$, *be the positive integers, determined by* (2.16), *such that*

$$f_k(\mu) = \sum_{j=1}^{k} \xi_j E_{u_j}.$$

Then there exists $c > 0$ *such that* $(\lambda_1, \ldots, \lambda_k) = c \cdot (\xi_1, \ldots, \xi_k)$.

Proof. Part (2) follows from a simple analysis of (i) the recurrence (2.16), which shows that $f_k(\mu)$ lies in the interior of the cone $\langle E_{u_1}, \ldots, E_{u_k}\rangle_{\mathbb{R}_+}$, and (ii) linear algebra, using the fact that the $k \times k$ matrix whose (ℓ, j)th entry equals $E_{u_\ell} \cdot J_{i_1(u_j)}$ is lower diagonal and has rank k. It suffices to prove part (1).

Starting with the first equation in (1), one notes that λ_{k-1}, λ_k must satisfy the equation

$$\lambda_k \cdot B_{k-1} \cdot \delta_{k-1,k} = \lambda_{k-1} c_{i_1(u_{k-1})}.$$

Solutions are $\lambda_k = c_{i_1(u_{k-1})}, \lambda_{k-1} = B_{k-1} \cdot \delta_{k-1,k}$, and therefore also,

$$\lambda_k = \prod_{j=1}^{k-1} c_{i_1(u_j)} \quad \text{and} \quad \lambda_{k-1} = \prod_{j=1}^{k-2} c_{i_1(u_j)} \cdot B_{k-1} \cdot \delta_{k-1,k}.$$

Arguing by induction, one assumes that for given $r - 1 \geqslant 1$, the above expressions for $\lambda_{k-(r-1)}, \lambda_{k-(r-2)}, \ldots, \lambda_k$ give solutions to the first $r - 1$ equations of (3.1). Thus, one assumes that the integers

$$\lambda_{k-(r-1)} = \left(\prod_{\substack{j=1 \\ j\neq k-r+1,k-r+2}}^{k} c_{i_1(u_j)} \right) \cdot B_{k-(r-1)} \cdot \delta_{k-(r-1),k-(r-1)+1},$$

satisfy the $r - 1$ equations

$$\lambda_k E_{u_k} \cdot J_{i_1(u_k)} = \cdots = \sum_{j=k-(r-1)}^{k} \lambda_j E_{u_j} \cdot J_{i_1(u_{k-(r-1)})}.$$

One now proceeds to show that this property extends upon replacing $r - 1$ by r.

To do so, one solves for λ_{k-r} so that

$$\lambda_k E_{u_k} \cdot J_{i_1(u_k)} = \sum_{j=k-r}^{k} \lambda_j E_{u_j} \cdot J_{i_1(u_{k-r})}.$$

Equality holds if λ_{k-r} satisfies the following equation:

$$\lambda_{k-r} E_{u_{k-r}} \cdot J_{i_1(u_{k-r})}$$
$$= \lambda_k E_{u_k} \cdot (J_{i_1(u_k)} - J_{i_1(u_{k-r})}) - \sum_{j=k-(r-1)}^{k-1} \lambda_j E_{u_j} \cdot J_{i_1(u_{k-r})}.$$

Thus,

$$\lambda_{k-r}\, c_{i_1(u_{k-r})} \cdot \prod_{q \in A'(u_{k-r})} b_q$$
$$= \lambda_k \left(c_{i_1(u_k)} \cdot \prod_{q \in A'(u_k)} b_q - c_{i_1(u_{k-r})} \cdot \prod_{\substack{q \in A(u_k) \\ q \neq i_1(u_{k-r})}} b_q \right)$$
$$- c_{i_1(u_{k-r})} \cdot \sum_{j=1}^{r-1} \lambda_{k-j} \cdot \prod_{\substack{q \in A(u_{k-j}) \\ q \neq i_1(u_{k-r})}} b_q.$$

Replacing each λ_{k-j} in the sum (over j) by the expression one assumes to hold by hypothesis, one observes, after some simplification left to the reader, that

$$\prod_{j=1}^{k-r} c_{i_1(u_j)} \cdot \prod_{\substack{q \in A'(u_k) \\ q \neq i_1(u_{k-r})}} b_q$$

is a factor common to each term appearing in the right side of this equation. Factoring it out, the *other* factor is then seen to equal:

$$(c_{i_1(u_k)} b_{i_1(u_{k-r})} - c_{i_1(u_{k-r})} b_{i_1(u_k)}) \prod_{j=k-r+1}^{k-1} c_{i_1(u_j)}$$
$$- c_{i_1(u_{k-r})} \sum_{j=1}^{r-1} \delta_{k-j,k-j+1} \prod_{\substack{v=k-r+1 \\ v \neq k-j,k-j+1}}^{k} c_{i_1(u_v)}. \tag{3.3}$$

One now rewrites the term (3.3) in the form $\sum_{j=0}^{r-1} (L_j - R_j)$, where

$$L_0 = c_{i_1(u_k)} b_{i_1(u_{k-r})} \cdot \prod_{j=k-r+1}^{k-1} c_{i_1(u_j)},$$
$$R_0 = c_{i_1(u_{k-r})} b_{i_1(u_k)} \cdot \prod_{j=k-r+1}^{k-1} c_{i_1(u_j)},$$

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and for each $j \geqslant 1$

$$L_j = c_{i_1(u_{k-r})} \cdot c_{i_1(u_{k-j})} \cdot b_{i_1(u_{k-(j-1)})} \cdot \prod_{\substack{v=k-r+1 \\ v \neq k-j, k-(j-1)}}^{k} c_{i_1(u_v)},$$

$$R_j = c_{i_1(u_{k-r})} \cdot c_{i_1(u_{k-(j-1)})} \cdot b_{i_1(u_{k-j})} \cdot \prod_{\substack{v=k-r+1 \\ v \neq k-j, k-(j-1)}}^{k} c_{i_1(u_v)}.$$

One then observes that for $j = 0, 1, \ldots, r-2, L_{j+1} = R_j$. It follows that (3.3) telescopes to $L_0 - R_{r-1}$, which is easily checked to equal

$$\delta_{k-r,k-r+1} \cdot \prod_{j=k-r+2}^{k} c_{i_1(u_j)}.$$

This shows that

$$\lambda_{k-r}\, c_{i_1(u_{k-r})} \prod_{q \in A'(u_{k-r})} b_q = \prod_{\substack{q \in A'(u_k) \\ q \neq i_1(u_{k-r})}} b_q \cdot \prod_{\substack{j=1 \\ j \neq k-r+1}}^{k} c_{i_1(u_j)} \cdot \delta_{k-r,k-r+1}.$$

Thus, one concludes

$$\lambda_{k-r} = B_{k-r} \cdot \delta_{k-r,k-r+1} \cdot \prod_{\substack{j=1 \\ j \neq k-r, k-r+1}}^{k} c_{i_1(u_j)}. \qquad \square$$

NOTATION. Given $\mu = \{u_1 < \cdots < u_k\}$ and the sequence $\lambda_1, \ldots, \lambda_k$, produced from (3.2), one defines

$$F_k(\mu) = F_k(u_1, \ldots, u_k) \overset{\text{def}}{=} \sum_{j=1}^{k} \lambda_j E_{u_j}. \tag{3.4}$$

Since $F_k(\mu)$ and $f_k(\mu)$ point in the same direction, it follows that

$$F_k(u_1, \ldots, u_k) \in \Sigma(\{u_1, \ldots, u_k\}) \quad \text{and}$$
$$LF_k(u_1, \ldots, u_k) = Lf_k(u_1, \ldots, u_k). \qquad \square$$

Left to the reader is the elementary proof, using Lemmas 2.13, 3.2, of the following formulae.

LEMMA 3.5. *For each* $\mu \in \mathcal{S}(C),\ C \in \mathcal{F}_i,$

$$M_1(F_k(u_1, \ldots, u_k)) = F_k(u_1, \ldots, u_k) \cdot I_{i_1(u_k)}$$

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$$\begin{aligned} &= b_{i_1(u_1)} \cdot \prod_{q \in A'(u_k)} b_q \cdot \prod_{j=2}^{k} c_{i_1(u_j)}, \\ M_2(F_k(u_1,\dots,u_k)) &= F_k(u_1,\dots,u_k) \cdot J_{i_1(u_k)} \\ &= \prod_{q \in A'(u_k)} b_q \cdot \prod_{j=1}^{k} c_{i_1(u_j)}. \qquad \square \end{aligned}$$

The following chains are the most important ones for this paper:

DEFINITION 3.6. For each $i = 1, 2, \dots, n$, define the chain $\mathcal{C}_i$ as follows:

$$\begin{aligned} \mathcal{A}(j) &= \{i, i+1, \dots, i+j-1\}, \qquad j = 1, \dots, n-i, \\ \mathcal{A}(n-i+j) &= \{i-j+1, i-j+2, \dots, n\}, \quad j = 1, \dots, i. \end{aligned} \tag{3.6.1}$$

Set $d(i)$ to denote the depth of $\mathcal{C}_i$ (see (2.7.2)). The maximal sequence of $\mathcal{C}_i$ is denoted $\mu^*(i)$ and its elements written $u_j^*(i), j = 1, \dots, d(i)$. One sets

$$F_i \stackrel{\text{def}}{=} F_{d(i)}(\mu^*(i)). \tag{3.6.2}$$

LEMMA 3.7. *For each* i *and* $C \in \mathcal{F}_i$, $\quad LF_i = \max\{Lf(\mu) : \mu \in \mathcal{S}(C)\}$.

Proof. Let $C : A(1) \subset \cdots \subset A(n)$ be any chain *other than* $\mathcal{C}_i$ in $\mathcal{F}_i$. Let $\mu = \{u_1 < \cdots < u_d\} \in \mathcal{S}(C)$ and μ^* its maximal sequence (see (2.7.1)). The Lemma follows from (2.20) by showing $LF_d(\mu^*) \preceq LF_i$.

By the definition of $\mathcal{C}_i$, there exist $j_1 < j_2 < \cdots < j_d$, such that (see (2.3))

$$\min \mathcal{A}(j_\ell) = \min A(u_\ell^*) \quad \text{for all } \ell = 1, \dots, d. \tag{3.8}$$

Set $\nu_i = \{j_1 < \cdots < j_d\}$. Evidently, $\nu_i \in \mathcal{S}(\mathcal{C}_i)$.

The proof then has two parts. One shows

(A) $LF_d(\mu^*) \preceq LF_d(\nu_i)$;
(B) $LF_d(\nu_i) \preceq LF_i$.

Proof of (A). By (2.20) one knows $u_d^* = n$. Thus, it follows that $j_d = n$. As a result, it is not difficult to see from the above definitions and (3.5) that $M_1\left(F_d(\mu^*)\right) = M_1\left(F_d(\nu_i)\right)$. Now set

$$\Delta(\mu^*) = h\left(LF_d(\mu^*)\right) - h\left(LF_d(\nu_i)\right).$$

One wants to show that $\Delta(\mu^*) \leqslant 0$.

By (3.2) and (3.4), there are numbers $\lambda_k, \lambda_k', k = 1, \dots, d$ such that

$$F_d(\mu^*) = \sum_{k=1}^{d} \lambda_k E_{u_k^*} \quad \text{and} \quad F_d(\nu_i) = \sum_{k=1}^{d} \lambda_k' E_{j_k}.$$

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Thus,

$$\Delta(\mu^*) = \frac{1}{M_1(F_d(\nu_i))} \sum_{k=1}^{d} [\lambda_k |E_{u_k^*}| - \lambda'_k |E_{j_k}|].$$

By (3.8), the formulae of (3.2), and a simple calculation, one sees easily that the kth term in the sum on the right equals

$$(+)_k \cdot (B_k(\mu^*) \cdot |E_{u_k^*}| - B_k^* \cdot |E_{j_k}|), \tag{3.9}$$

where $B_d(\mu^*) = B_d^* = 1$, $(+)_k$ denotes a positive constant depending upon k (whose expression is not needed in the following) and for $k < d$

$$B_k(\mu^*) = \prod_{q \in A'(n) - A(u_k^*)} b_q \quad \text{and} \quad B_k^* = \prod_{q \in \mathcal{A}'(n) - \mathcal{A}(j_k)} b_q.$$

By definition,

$$|E_{u_k^*}| = \sum_{v=1}^{u_k^*} \prod_{\substack{q \in A(u_k^*) \\ q \neq i_v(u_k^*)}} b_q \quad \text{and} \quad |E_{j_k}| = \sum_{v=1}^{j_k} \prod_{\substack{q \in \mathcal{A}(j_k) \\ q \neq i_v(j_k)}} b_q.$$

Thus,

$$B_k(\mu^*) \cdot |E_{u_k^*}| = \sum_{v=1}^{u_k^*} \prod_{\substack{q \in \mathcal{A}'(n) \\ q \neq i_v(u_k^*)}} b_q \quad \text{and} \quad B_k^* \cdot |E_{j_k}| = \sum_{v=1}^{j_k} \prod_{\substack{q \in \mathcal{A}'(n) \\ q \neq i_v(j_k)}} b_q.$$

One now observes that the definition of $\mathcal{C}_i$ and (3.8) imply $A(u_\ell^*) \subset \mathcal{A}(j_\ell)$ for all $\ell = 1, \ldots, d$. One then concludes that the term in (3.9) is nonpositive for each k and is strictly negative if $A(u_k^*) \subsetneq \mathcal{A}(j_k)$. Thus, $\Delta(\mu^*) \leqslant 0$. This completes the proof of part A. Part B follows from (2.20). □

Remark 3.10. This argument also shows (see (3.6.2)) that

$$LF_d(\mu^*) = LF_i(\mu^*(i)) \quad \text{iff} \quad \mu^* = \mu^*(i). \qquad \square$$

One next observes:

LEMMA 3.11. *If $i_1 \neq i_2$ satisfy the property that $\iota(i_1) = \iota(i_2)$, then $F_{i_1} = F_{i_2}$.*

Proof. It suffices to observe that $d(i_1) = d(i_2)$ and $\mathcal{A}(u_\ell^*(i_1)) = \mathcal{A}(u_\ell^*(i_2))$ for all $\ell = 1, 2, \ldots$. That is, the elements of the chain $\mathcal{C}_{i_1}$ resp. $\mathcal{C}_{i_2}$, whose indices belong to the maximal sequence $\mu^*(i_1)$ resp. $\mu^*(i_2)$, *are* the same. Using the procedure described by Lemma 3.2, it follows that the vectors F_{i_1}, F_{i_2} must therefore be equal. □

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The chains $\mathcal{C}_{k_\ell}, \ell = 1, \ldots, r$ will be needed in Section 6. Set (see (2.2), (3.6))

$$F(\ell) \stackrel{\text{def}}{=} F_{k_\ell}, \quad \ell = 1, \ldots, r. \tag{3.12}$$

In particular, since $k_1 = 1,\ F(1) = e_{1,2,\ldots,n}$.

The next Lemma summarizes useful properties of $\mathcal{C}_{k_\ell}$ and $F(\ell)$ that follow from Sections 2 and 3.

LEMMA 3.13. *For each* $\ell = 1, \ldots, r$:

(1) *The maximal sequence* $\mu^*(k_\ell)$ *contains* ℓ *elements, given by the integers* $u_j^*(k_\ell) = n - k_{\ell-j+1} + 1,\ j = 1, \ldots, \ell$.
(2) $i_1(u_j^*(k_\ell)) = k_{\ell-j+1}$ *and* $\iota(u_j^*(k_\ell)) = \ell - j + 1$ *for each* $j = 1, \ldots, \ell$.
(3) *The depth* $d(k_\ell)$ *of* $\mathcal{C}_{k_\ell}$ *equals* ℓ.
(4) *The slope of* $LF(\ell)$ *equals* $-\rho_\ell$ *and* $F(\ell) \in \left[\{J_a : a \in \cup_{q=1}^{\ell} \mathcal{I}_q\}\right]^{\perp}$.

4. Two useful properties of $F(\ell)$

Applying Lemma 3.5 to each $F(\ell)$, one obtains concise expressions for $M_1F(\ell)$, $M_2F(\ell)$. However, one also needs convenient expressions for $|F(\ell)|$ to construct the polygon, determined by the lines $LF(\ell)$, and, in particular, to derive formulae for its vertices.

Set $\quad \beta_j = \prod_{\substack{q=2 \\ q \neq j}}^{n} b_q, \quad j = 2, \ldots, n.$

LEMMA 4.1. *For each* $\ell = 1, \ldots, r$,

$$|F(\ell)| = \prod_{j=2}^{n} b_j \cdot \sum_{v=1}^{\ell} \prod_{\substack{q=1 \\ q \neq v}}^{\ell} c_{k_q} + \sum_{v=1}^{\ell-1} \left(\prod_{\substack{w=1 \\ w \neq v}}^{\ell} c_{k_w} \cdot b_{k_v} \cdot \sum_{j \in \mathcal{I}_v - \{k_v\}} \beta_j \right) + \prod_{j=1}^{\ell-1} c_{k_j} \cdot b_{k_\ell} \cdot \sum_{v \geqslant k_\ell + 1} \beta_v.$$

Remark 4.2. (1) If $\mathcal{I}_v = \{k_v\}$, then the factor consisting of the sum of β values is understood to be zero.

(2) Since ℓ is fixed in the argument below, a simpler notation can be used. The elements of the maximal indexing sequence $\mu^*(k_\ell)$ will be written, *in the following proof only*, as u_j (vs. $u_j^*(k_\ell)$). It is useful to observe that (see (2.3))

$$\begin{aligned} &\mathcal{A}'(u_1) = \{k_\ell + 1, k_\ell + 2, \ldots, n\} \quad \text{and} \\ &\mathcal{A}'(u_{j+1}) - \mathcal{A}(u_j) = \mathcal{I}_{\ell-j} - \{k_{\ell-j}\}, \quad 1 \leqslant j \leqslant \ell - 1. \end{aligned} \tag{4.3}$$

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Proof of (4.1). By (3.13), $d(k_\ell) = \ell$. By (3.2), (3.4), the positive integers $\lambda_j = \lambda_j(\ell)$, for which $F(\ell) = \sum_{j=1}^{\ell} \lambda_j(\ell) E_{u_j}$, are given as follows

$$\lambda_\ell(\ell) = \prod_{j=1}^{\ell-1} c_{i_1(u_j)}, \qquad \lambda_{\ell-1}(\ell) = \prod_{j=1}^{\ell-2} c_{i_1(u_j)} \cdot B_{\ell-1} \cdot \delta_{\ell-1,\ell},$$

and for $r \geqslant 2$

$$\lambda_{\ell-r}(\ell) = \prod_{\substack{j=1 \\ j \neq \ell-r, \ell-r+1}}^{\ell} c_{i_1(u_j)} \cdot B_{\ell-r} \cdot \delta_{\ell-r,\ell-r+1}.$$

It follows that the contribution to $|F(\ell)|$ from $\lambda_\ell(\ell)|E_{u_\ell}|$ equals

$$\prod_{j=1}^{\ell-1} c_{i_1(u_j)} \cdot \sum_{i=1}^{n} \prod_{q \neq i} b_q = \prod_{j=1}^{\ell-1} c_{i_1(u_j)} \cdot \left[\prod_{j=2}^{n} b_j + b_{i_1(u_\ell)} \sum_{i \in \mathcal{A}'(u_\ell)} \beta_i \right].$$

Using the expressions for B_j, B_{j+1}, given in Section 3 (with the chain (3.6.1) and $i = k_\ell$), one notes that $j < \ell$ implies $1 \notin \mathcal{A}(u_j)$ and

$$B_j|E_{u_j}| = \prod_{q \in \mathcal{A}'(u_\ell) - \mathcal{A}(u_j)} b_q \cdot \sum_{t=1}^{u_j} \prod_{\substack{w \in \mathcal{A}(u_j) \\ w \neq i_t(u_j)}} b_w = \sum_{i \in \mathcal{A}(u_j)} \beta_i.$$

One then observes by a simple calculation that the term

$$\prod_{\substack{k=1 \\ k \neq \ell}}^{\ell} c_{i_1(u_k)} \cdot b_{i_1(u_\ell)} \cdot \sum_{w \in \mathcal{A}(u_{\ell-1})} \beta_w$$

appears twice in the expression for $\lambda_{\ell-1}(\ell)|E_{u_{\ell-1}}| + \lambda_\ell(\ell)|E_{u_\ell}|$, but with opposite signs. Thus, a certain amount of cancellation occurs. The expression obtained for the contribution to $|F(\ell)|$ from the sum of the $(\ell-1)^{\text{st}}$ and ℓth terms equals (after some rearranging for consistency with the general pattern to be described below) the sum of three terms $\alpha_\ell(1) + \alpha_\ell(2) + \alpha_\ell(3)$ where

$$\alpha_\ell(1) = \prod_{q=2}^{n} b_q \left(\prod_{\substack{j=1 \\ j \neq \ell}}^{\ell} c_{i_1(u_j)} + \prod_{\substack{j=1 \\ j \neq \ell-1}}^{\ell} c_{i_1(u_j)} \right),$$

$$\alpha_\ell(2) = \prod_{\substack{j=1 \\ j \neq \ell}}^{\ell} c_{i_1(u_j)} \cdot b_{i_1(u_\ell)} \cdot \sum_{t \in \mathcal{A}'(u_\ell) - \mathcal{A}(u_{\ell-1})} \beta_t,$$

$$\alpha_\ell(3) = \prod_{\substack{j=1 \\ j \neq \ell-1}}^{\ell} c_{i_1(u_j)} \cdot b_{i_1(u_{\ell-1})} \cdot \sum_{t \in \mathcal{A}'(u_{\ell-1})} \beta_t.$$

One now repeats this reasoning with consecutive pairs $\lambda_{\ell-2e-1}(\ell)|E_{u_{\ell-2e-1}}|$, $\lambda_{\ell-2e}(\ell)|E_{u_{\ell-2e}}|$, $e \in [1, \ell/2] \cap \mathbb{N}$, with the convention that a term with nonpositive index equals 0. Assuming for the moment that $\ell - 2e - 1 \geqslant 1$, one sets $t = \ell - 2e$, and proceeds as above. One observes that the term

$$\prod_{\substack{j=1 \\ j \neq t}}^{\ell} c_{i_1(u_j)} \cdot b_{i_1(u_t)} \cdot \sum_{q \in \mathcal{A}(u_{t-1})} \beta_q$$

appears twice in the expression for $\lambda_{t-1}(\ell)|E_{u_{t-1}}| + \lambda_t(\ell)|E_{u_t}|$, but with opposite signs. So again, some cancellation of terms occurs. One concludes, after some straightforward calculation, left to the reader, that $t \geqslant 2$ implies:

$$\lambda_{t-1}(\ell)|E_{u_{t-1}}| + \lambda_t(\ell)|E_{u_t}| = \alpha_t(1) + \alpha_t(2) + \alpha_t(3),$$

where

$$\begin{aligned}
\alpha_t(1) &= \prod_{q=2}^{n} b_q \left(\prod_{\substack{j=1 \\ j \neq t-1}}^{\ell} c_{i_1(u_j)} + \prod_{\substack{j=1 \\ j \neq t}}^{\ell} c_{i_1(u_j)} \right), \\
\alpha_t(2) &= \prod_{\substack{j=1 \\ j \neq t}}^{\ell} c_{i_1(u_j)} \cdot b_{i_1(u_t)} \cdot \sum_{w \in \mathcal{A}'(u_t) - \mathcal{A}(u_{t-1})} \beta_w, \\
\alpha_t(3) &= \prod_{\substack{j=1 \\ j \neq t-1}}^{\ell} c_{i_1(u_j)} \cdot b_{i_1(u_{t-1})} \cdot \sum_{w \in \mathcal{A}'(u_{t-1})} \beta_w \\
&\quad - \prod_{\substack{j=1 \\ j \neq t+1}}^{\ell} c_{i_1(u_j)} \cdot b_{i_1(u_{t+1})} \cdot \sum_{w \in \mathcal{A}(u_t)} \beta_w.
\end{aligned}$$

The term to the right of the minus sign in $\alpha_t(3)$ is due to the presence of factors $\delta_{*,*}$ in both of the coefficients $\lambda_{\ell-2e-1}(\ell), \lambda_{\ell-2e}(\ell)$ since $e \geqslant 1$.

If $t \leqslant 1$, then the only contribution to consider occurs for $t = 1$. Here, $\lambda_1(\ell)|E_{u_1}| = \alpha_1(1) + \alpha_1(2) + \alpha_1(3)$ where

$$\alpha_1(1) = \prod_{q=2}^{n} b_q \cdot \prod_{j \neq 1}^{\ell} c_{i_1(u_j)}, \qquad \alpha_1(2) = \prod_{j \neq 1}^{\ell} c_{i_1(u_j)} \cdot b_{i_1(u_1)} \cdot \sum_{w \in \mathcal{A}'(u_1)} \beta_w,$$

$$\alpha_1(3) = -\prod_{j \neq 2}^{\ell} c_{i_1(u_j)} \cdot b_{i_1(u_2)} \cdot \sum_{w \in \mathcal{A}(u_1)} \beta_w.$$

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Set $T=\{\ell-2e : e\in[0,\ell/2]\cap\mathbb{N},\ \text{and}\ \ell-2e\geqslant 1\}$. So, $1\in T$ iff ℓ is odd, but 2 is the smallest element of T if ℓ is even. It is simple to see that

$$\sum_{t\in T}\alpha_t(1)=\prod_{j=2}^{n} b_j\cdot\sum_{v=1}^{\ell}\prod_{\substack{q=1\\ q\neq v}}^{\ell} c_{k_q}.$$

This takes care of the first term in the asserted formula for $|F(\ell)|$.

To write succinctly the expressions for the other sums, one sets $u_0=u_{-1}=b_0=0$, and $\mathcal{A}(u_0)=\mathcal{A}(u_{-1})=\emptyset$. One determines the sums of the $\alpha_t(2),\alpha_t(3)$, using the above expressions and a straightforward calculation. One finds:

$$\sum_{t\in T}\alpha_t(2) = \sum_{t\in T}\prod_{j\neq t}^{\ell} c_{i_1(u_j)}\cdot b_{i_1(u_t)}\cdot\sum_{w\in\mathcal{A}'(u_t)-\mathcal{A}(u_{t-1})}\beta_w,$$

$$\sum_{t\in T}\alpha_t(3) = \sum_{t\in T}\prod_{j\neq t-1}^{\ell} c_{i_1(u_j)}\cdot b_{i_1(u_{t-1})}\cdot\sum_{w\in\mathcal{A}'(u_{t-1})-\mathcal{A}(u_{t-2})}\beta_w.$$

Using (3.13)(2), (4.3), and the formula (needed when $t=2$)

$$\prod_{j\neq 1}^{\ell} c_{i_1(u_j)}=c_{k_{\ell-1}}c_{k_{\ell-2}}\cdots c_{k_1},$$

one deduces the formula in the statement of the Lemma. □

A relation of importance, needed in the proof of the Theorem, exists between the vectors $F(\ell)$ and $F(\ell+1)$ and indeed, more generally, between the vectors $F_k(u_1,\ldots,u_k)$ and $F_{k-1}(u_2,\ldots,u_k)$ for $\{u_1<\cdots<u_k\}\in\mathcal{S}(C), C\in\mathcal{F}_i$, and $i\geqslant k_2$. One first observes that (3.13)(1) implies:

$$u_v^*(k_\ell)=u_{v+1}^*(k_{\ell+1})\quad \text{for}\, v=1,\ldots,\ell,\quad \ell=1,\ldots,r-1. \tag{4.4}$$

An elementary argument, using (3.2) and (4.4), now shows

LEMMA 4.5.

(1) *For each* $\ell\in[1,r-1]$,

$$F(\ell+1)=c_{i_1(u_1^*(k_{\ell+1}))}F(\ell)+\lambda_1(\ell+1)E_{u_1^*(k_{\ell+1})}.$$

(2) *For any* $i\geqslant k_2$, *any* $C\in\mathcal{F}_i$, *and* $\{u_1<\cdots<u_k\}\in\mathcal{S}(C)$, *one has (see* (3.4)*)*

$$F_k(u_1,\ldots,u_k)=c_{i_1(u_1)}F_{k-1}(u_2,\ldots,u_k)+\lambda_1E_{u_1}.$$

One now defines the polygon

$$\Gamma^* = \partial\left(\bigcap_{\ell=1}^{r} L^+F(\ell) \cap \mathbb{R}^2_+\right). \tag{4.6}$$

5. Determination of the polygon Γ^*

The work done in earlier sections is rewarded here. It becomes a simple matter to determine Γ^*. There are two steps. Each is done rather easily. First, one shows that the lines $LF(\ell)$ and $LF(\ell+1)$ intersect in the first quadrant. Then, one shows that the polygon determined by three lines $LF(\ell), LF(\ell+1), LF(\ell+2)$ has three *bounded* faces. This leads to a complete description of Γ^*, given in (5.4).

Set $\mathcal{X}_\ell = LF(\ell) \cap LF(\ell+1)$, $\ell = 1, 2, \ldots, r-1$.

LEMMA 5.1. *For each* $\ell = 1, \ldots, r-1$, $\mathcal{X}_\ell = (\sum_{j=k_{\ell+1}}^{n} 1/b_j, \sum_{j=1}^{k_{\ell+1}-1} 1/c_j)$.

Proof. One uses the expressions given by Lemma 4.5. By (2.13)(3d), $E_{u_1^*(k_{\ell+1})} \cdot J_{k_\ell} = 0$. By (3.13)(4), $M_2(F(\ell)) = F(\ell) \cdot J_{k_{\ell-1}}, \ell \geqslant 2$. Thus, it follows that

$$\begin{aligned} M_1(F(\ell+1)) &= F(\ell+1) \cdot I_{k_{\ell+1}} \\ &= c_{i_1(u_1^*(k_{\ell+1}))} M_1(F(\ell)) + \lambda_1(\ell+1) E_{u_1^*(k_{\ell+1})} \cdot I_{k_{\ell+1}}, \\ M_2(F(\ell+1)) &= F(\ell+1) \cdot J_{k_\ell} = c_{i_1(u_1^*(k_{\ell+1}))} M_2(F(\ell)). \end{aligned}$$

Further, (4.5) evidently implies:

$$|F(\ell+1)| = c_{i_1(u_1^*(k_{\ell+1}))}|F(\ell)| + \lambda_1(\ell+1)|E_{u_1^*(k_{\ell+1})}|.$$

Using the fact that $E_{u_1^*(k_{\ell+1})} = e_{k_{\ell+1}, k_{\ell+1}+1, \ldots, n}$, and the above equations, one sees that

$$s_1(\mathcal{X}_\ell) = \frac{\begin{vmatrix} |F(\ell)| & M_2(F(\ell)) \\ |F(\ell+1)| & M_2(F(\ell+1)) \end{vmatrix}}{\begin{vmatrix} M_1(F(\ell)) & M_2(F(\ell)) \\ M_1(F(\ell+1)) & M_2(F(\ell+1)) \end{vmatrix}} = \frac{|E_{u_1^*(k_{\ell+1})}|}{E_{u_1^*(k_{\ell+1})} \cdot I_{k_{\ell+1}}} = \sum_{j=k_{\ell+1}}^{n} \frac{1}{b_j}.$$

Similarly, one finds $s_2(\mathcal{X}_\ell) = v(LF(\ell)) - \rho_\ell s_1(\mathcal{X}_\ell)$. One evaluates $v(LF(\ell))$, using (3.5), (4.1), (4.3). Thus, elementary calculation shows

$$v(LF(\ell)) = \sum_{j=1}^{\ell} \frac{1}{c_{k_j}} + \sum_{v=1}^{\ell-1} \frac{b_{k_v}}{c_{k_v}} \cdot \left(\sum_{j \in \mathcal{I}_v - \{k_v\}} \frac{1}{b_j}\right) + \frac{b_{k_\ell}}{c_{k_\ell}} \left(\sum_{q \geqslant k_{\ell+1}} \frac{1}{b_q}\right).$$

Subtracting $\rho_\ell s_1(\mathcal{X}_\ell)$, yields

$$s_2(\mathcal{X}_\ell) = \sum_{j=1}^{\ell} \frac{1}{c_{k_j}} + \sum_{v=1}^{\ell} \frac{b_{k_v}}{c_{k_v}} \cdot \left(\sum_{j \in \mathcal{I}_v - \{k_v\}} \frac{1}{b_j}\right).$$

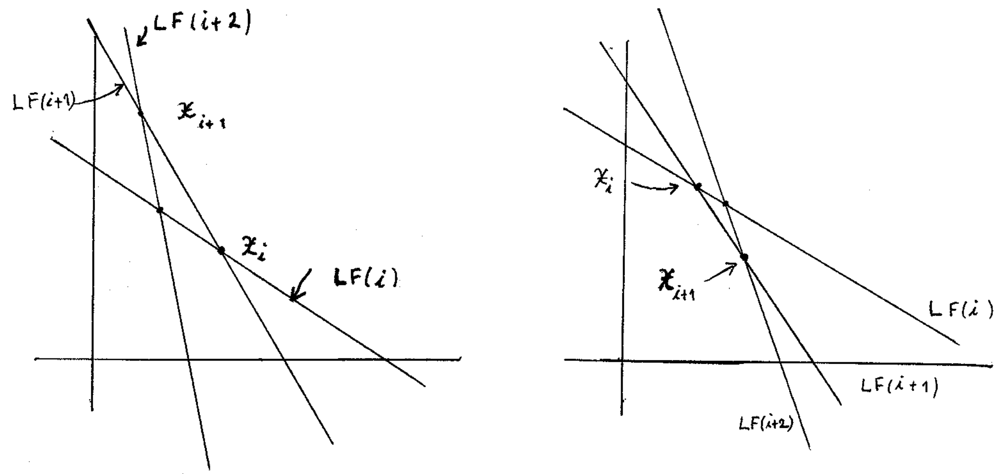

Figure 1.

To deduce the expression, asserted in the Lemma, one writes $1/b_j = c_j/b_j \cdot 1/c_j$ and uses the equations, nonvacuous for each v with $|\mathcal{I}_v| > 1$, $b_{k_v}/c_{k_v} = b_{k_v+1}/c_{k_v+1} = \cdots = b_{k_{v+1}-1}/c_{k_{v+1}-1}$. In this way, one sees that $b_{k_v}/c_{k_v} \cdot 1/b_j = 1/c_j$ for each $j \in \mathcal{I}_v - \{k_v\}$. Adding together all the reciprocals of the c_* yields the formula for $s_2(\mathcal{X}_i)$. □

Remark 5.2. Using the formulae of (5.1) one sees immediately that $h(LF(1)) = \sum_{j=1}^n 1/b_j$, and $v(LF(r)) = \sum_{j=1}^n 1/c_j$. □

From Lemma 5.1, one now concludes

COROLLARY 5.3. *For each* $\ell = 1, \ldots, r-2, \quad s_1(\mathcal{X}_\ell) - s_1(\mathcal{X}_{\ell+1}) > 0.$

By (5.1) and (5.3), one sees that the polygon determined by the lines $LF(\ell)$, $LF(\ell+1)$, $LF(\ell+2)$, must have three segments of slopes $-\rho_\ell > -\rho_{\ell+1} > -\rho_{\ell+2}$. If this were not the case, then it would necessarily follow that $s_1(\mathcal{X}_\ell) - s_1(\mathcal{X}_{\ell+1}) < 0$, as indicated by the diagram above.

A simple induction now shows

COROLLARY 5.4. *The polygon* Γ^*, *determined by the* r *lines* $LF(1), \ldots, LF(r)$, *has* r *bounded segments of slopes* $-\rho_1 > -\rho_2 > \cdots > -\rho_r$. *It has* $r-1$ *vertices in* $(0,\infty)^2$ *at the points* $v_1 = \mathcal{X}_1, \ldots, v_{r-1} = \mathcal{X}_{r-1}$, *and intercepts the axes at the points* $v_0 = \left(h(LF(1)), 0\right)$, $v_r = \left(0, v(LF(r))\right)$.

Remark 5.5. Evidently, the formulae given in (5.1), (5.2) for the vertices of Γ^* reduce to those given in the Introduction when $r = n$. □

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6. Proof of the Theorem

Following the outline indicated in Section 1, there are two parts to deriving the asymptotics for $N(t_1, t_2)$. One first constructs a collection of cones $\mathcal{T}$ satisfying (1.4)(i–iii). The construction will then imply that $\widehat{\Gamma}(\mathcal{T}) = \Gamma^*$, described by (5.4). One then shows that (1.7.1), (1.7.2) are also satisfied.

One now fixes an arbitrary chain $C \in \mathcal{F}_i, i \geqslant k_2$, and then shows how to refine the associated cone $\mathcal{C}(C)$ (see (2.9)) so that (1.4)(i–iii) hold for each cone produced by the refinement. To describe the procedure the following is useful.

NOTATIONS. (1) For each $i = 1, \ldots, r$ set $\mathcal{J}(i) = \{J_{k_i}, J_{k_i+1}, \ldots, J_{k_{i+1}-1}\}$.

(2) Let d denote the depth of the chain C. Let $\kappa_1, \ldots, \kappa_d$ denote the parameters of C defined in (2.7.2). If $\mathcal{K} = \langle g_1, \ldots, g_n \rangle$ is any closed simplicial subcone of $\mathcal{C}(C)$, set $\mathcal{V}_0\mathcal{K} = \{1, \ldots, n\}$ and for each $w = 1, \ldots, d-1$ set (see (1.4), (2.10))

$$\mathcal{V}_w\mathcal{K} = \{j : \mathcal{K}_2(g_j) \subset [\mathcal{J}(\kappa_d), \ldots, \mathcal{J}(\kappa_{w+1})].$$

LEMMA 6.1. *Given $i \geqslant k_2, C \in \mathcal{F}_i$, and $k \in [2, d]$, there exists a simplicial decomposition $\mathcal{C}(C) = \sum_{j=1}^{M_k} \mathcal{C}_j(k)$, so that for each j the following properties are satisfied for the cone $\mathcal{C}_j(k) = \langle g_1(j), \ldots, g_n(j) \rangle_{\mathbb{R}_+}$:*

(6.1.1) *For each $e \in \mathcal{V}_{d-k}\mathcal{C}_j(k)$, there exists $d' \leqslant d$ and $\{u_1 < u_2 < \cdots < u_{d'}\} \in \mathcal{S}(C)$, such that $g_e(j) = f_{d'}(u_1, u_2, \ldots, u_{d'})$.*

(6.1.2) *For each $e \notin \mathcal{V}_{d-k}\mathcal{C}_j(k)$, there exists a unique $u \in \bigcup_{\ell=1}^{d-k} \mathcal{U}_\ell(C)$ (see (2.7.2)) such that $g_e(j) = E_u$.*

(6.1.3) $\bigcap_{e \in \mathcal{V}_{d-k}\mathcal{C}_j(k)} \mathcal{K}_2(g_e(j)) \neq \emptyset$.

Proof. One first refines $\mathcal{C}(C)$ into a sum of cones $C_1(2), C_2(2)$, as follows. By (2.15), $f(u^*_{d-1}(C), u^*_d(C))$ lies in the interior of the face $\langle E_{u^*_{d-1}(C)}, E_{u^*_d(C)} \rangle$ of $\mathcal{C}(C)$. One then forms the two subcones

$$C_1(2) = \langle E_1, \ldots, E_{u^*_{d-1}(C)}, \ldots, E_{u^*_d(C)-1}, f(u^*_{d-1}(C), u^*_d(C)) \rangle,$$

$$C_2(2) = \langle E_1, \ldots, E_{u^*_{d-1}(C)-1}, f(u^*_{d-1}(C), u^*_d(C)), E_{u^*_{d-1}(C)+1}, \ldots, E_{u^*_d(C)} \rangle.$$

Remark. The refinement $\mathcal{C}(C) = C_1(2) + C_2(2)$ is referred to in the following as *the splitting of $\mathcal{C}(C)$ along the ray* $\langle f(u^*_{d-1}(C), u^*_d(C)) \rangle_{\mathbb{R}_+}$. This can be defined for any two dimensional cone with respect to a vector contained in its interior. □

For simplicity, rewrite the 1-skeletal vectors of these two cones, exhibited above, so that $C_j(2) = \langle g_1(j), \ldots, g_n(j) \rangle, j = 1, 2$. One now repeats the above procedure with any pair of vectors $g_{e'}(j), g_e(j)$ satisfying the properties (see (2.10)) $\mathcal{K}_2(g_{e'}(j)) \subset [\mathcal{J}(\kappa_{d-1})]$ and $\mathcal{K}_2(g_e(j)) \subset [\mathcal{J}(\kappa_d)]$, if these exist. That is, one

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then refines $C_j(2)$ by splitting it along the ray $\langle f(g_{e'}(j), g_e(j))\rangle_{\mathbb{R}_+}$. This is always permissible by (2.15). If no such vectors exist, then one has arrived at the property $\bigcap_{e \in \mathcal{V}_{d-2}C_j(2)} \mathcal{K}_2(g_e(j)) \neq \emptyset$. That is, one has completely separated the 1-skeleton direction vectors ξ for which $\mathcal{K}_2(\xi) \subset [\mathcal{J}(\kappa_{d-1})]$ from those for which $\mathcal{K}_2(\xi) \subset [\mathcal{J}(\kappa_d)]$.

If however two such vectors do exist then the above refinement has reduced by 1 the number of 1-skeleton vectors v in the cone $C_j(2)$ that lies in $\{v : \mathcal{K}_2(v) \subset [\mathcal{J}(\kappa_{d-1})]\} \cup \{v : \mathcal{K}_2(v) \subset [\mathcal{J}(\kappa_d)]\}$.

One now repeats, if needed, this procedure. After finitely many steps one will arrive at a decomposition $\mathcal{C}(C) = \sum_{j=1}^{M_2} \mathcal{C}_j(2)$ into simplicial subcones $\mathcal{C}_j(2) = \langle g_1(j), \dots, g_n(j)\rangle$, such that the following properties will be satisfied for each j, and $e \in \mathcal{V}_{d-2}\mathcal{C}_j(2)$:

(1) There exist $u_1 \in \mathcal{U}_{d-1}(C), u_2 \in \mathcal{U}_d(C)$ such that either $g_e(j) = f_2(u_1, u_2)$, or $g_e(j) = f_1(u_1)$, or $g_e(j) = f_1(u_2)$.
(2) $\bigcap_{e \in \mathcal{V}_{d-2}\mathcal{C}_j(2)} \mathcal{K}_2(g_e(j)) \neq \emptyset$.

Now, assuming the existence of a refinement $\mathcal{C}(C) = \sum_{j=1}^{M_{k-1}} \mathcal{C}_j(k-1)$, satisfying (6.1.1)–(6.1.3) for a given k such that $k-1 \geqslant 2$, one proceeds to refine each $\mathcal{C}_j(k-1)$.

By hypothesis, given $e' \notin \mathcal{V}_{d-k+1}\mathcal{C}_j(k-1)$ and $e \in \mathcal{V}_{d-k+1}\mathcal{C}_j(k-1)$ there exist:

(i) an index $u \in \bigcup_{w=1}^{d-k+1} \mathcal{U}_w(C)$, for which $g_{e'}(j) = E_u$;
(ii) an element $\{u_1 < \cdots < u_{d'}\} \in \mathcal{S}(C)$ such that $g_e(j) = f_{d'}(u_1, \dots, u_{d'})$.

One observes that $u < u_1$ and its value depends upon the vector $g_e(j)$. By definition, it follows that $\{u < u_1 < \cdots < u_{d'}\} \in \mathcal{S}(C)$. So, one can form the vector $f_{d'+1}(u, u_1, \dots, u_{d'})$. One then separates $g_{e'}(j)$ from the vector $g_e(j) = f_{d'}(u_1, \dots, u_{d'})$, by splitting $\mathcal{C}_j(k-1)$ along the ray $\langle f_{d'+1}(u, u_1, u_2, \dots, u_{d'})\rangle_{\mathbb{R}_+}$. This is possible since $f_{d'+1}(u, u_1, u_2, \dots, u_{d'}) \in$ Interior $\langle g_{e'}(j), g_e(j)\rangle_{\mathbb{R}_+}$, by (2.16). Arguing as in the case $k = 2$, after finitely many steps of this procedure, one arrives at a decomposition $\mathcal{C}(C) = \sum_{j=1}^{M_k} \mathcal{C}_j(k)$ such that each $\mathcal{C}_j(k)$ satisfies the properties (6.1.1)–(6.1.3) with $k-1$ replaced everywhere by k.

Once $k = d$, one has achieved a decomposition of $\mathcal{C}(C)$ such that each simplicial cone in the refinement satisfies (1.4)(i–iii). Moreover, each vector appearing in the 1-skeleton of any cone must be of the form $f_e(u_1, \dots, u_e)$ for some $\{u_1 < \cdots < u_e\} \in \mathcal{S}(C)$. This completes the proof of Lemma 6.1. □

The discussion in Sections 1–5 is now essentially summarized by

THEOREM 6.2. *Let $\mathcal{T}$ denote the set of all cones produced by* (6.1) *for all chains $C \in \mathcal{F}_i, i \geqslant k_2$. The following properties are satisfied.*

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(1) $\mathcal{T}$ *satisfies* (1.4)(i–iii).

(2) *Let* $\mathcal{S}k$ *denote the set of* 1*-skeletal vectors of the cones belonging to* $\mathcal{T}$. *For each* $\xi \in \mathcal{S}k$, *there exists an integer* $i \geqslant k_2$, *a chain* $C \in \mathcal{F}_i$, *and element* $\mu \in \mathcal{S}(C)$ *containing, say,* k *elements such that* $\xi = f_k(\mu)$.

(3) *The polygon* $\widehat{\Gamma}(\mathcal{T})$ *equals the polygon* Γ^* (*see* (4.6)). *Its vertices are therefore the points* v_ℓ, *defined in* (5.4).

(4) $\widehat{\Gamma}(\mathcal{T}) = \Gamma(\mathbf{P}) = \Gamma$ (*see* (1.1)), *and* $I(\mathbf{s})/s_1s_2$ *has a simple pole along each line containing a face of* Γ. *In particular, for each* $\ell = 1, \ldots, r-1$, *exactly two lines,* $LF(\ell-1), LF(\ell)$, *contain the vertex* v_ℓ.

(5) $LF(1)$ *resp.* $LF(r)$ *is the only polar component of* $I(\mathbf{s})$ *containing the vertex* v_0 *resp.* v_r.

Proof. (6.1) proves (1), (2). Using the chains $\mathcal{C}_{k_\ell}, \ell = 1, \ldots, r$, defined in Section 3 (see (3.6)), one forms the cones $\mathcal{C}(\mathcal{C}_{k_\ell})$. The cones belonging to $\mathcal{T}$ include those that refine each $\mathcal{C}(\mathcal{C}_{k_\ell})$. It follows that $F(\ell)$ will be a 1-skeletal vector of some cone that refines $\mathcal{C}(\mathcal{C}_{k_\ell})$, for each ℓ. Thus, the polygon $\widehat{\Gamma}(\mathcal{T})$ must equal Γ^*. This proves (3). (4) follows from (1.6) and the fact that if ξ is a vector other than some $F(\ell)$ such that ξ and $F(\ell)$ belong to the 1-skeleton of the *same* element of $\mathcal{T}$, then either $L\xi$ is not parallel to $LF(\ell)$ or it is but, in that case, $L\xi \prec LF(\ell)$ follows from the ordering properties established in Sections 2 and 3. This implies that the order of the pole along each $LF(\ell)$ equals 1. (5) follows from (3), (4). □

PROOF OF MAIN THEOREM. To finish the Theorem's proof, it suffices to show that $\mathcal{T}$ satisfies (1.7.1), (1.7.2). To do this, one identifies, for each $\ell \geqslant 2$, an element of $\mathcal{T}$ which contains both $F(\ell-1)$ and $F(\ell)$.

Using (2.9), one writes $\mathcal{C}(\mathcal{C}_{k_\ell}) = \langle E_1, \ldots, E_n \rangle$, where the subscripts of the E_j are determined by setting $i = k_\ell$ in (3.6.1). One notes that $E_n = F(1)$. Further, by (3.13), $d(\ell) = \ell$. Thus, (4.4) implies

$$u^*_{\ell-1}(\mathcal{C}_{k_\ell}) \overset{\text{def}}{=} u^*_{\ell-1}(k_\ell) = u^*_{\ell-2}(k_{\ell-1}) = \cdots = u^*_1(k_2).$$

Thus, by setting $\ell = 1$ in the statement of (4.5)(1), one sees that $F(2)$ is in the interior of $\langle F(1), E_{u^*_{\ell-1}(k_\ell)} \rangle_{\mathbb{R}_+}$. One now refines $\mathcal{C}(\mathcal{C}_{k_\ell})$ by splitting along the ray $F(2)$. Thus, $\mathcal{C}(\mathcal{C}_{k_\ell}) = C_1 + C_2$, where one chooses the indexing so that

$$C_2 = \langle E_1, \ldots, E_{u^*_{\ell-1}(k_\ell)-1}, F(2), E_{u^*_{\ell-1}(k_\ell)+1}, \ldots, F(1) \rangle_{\mathbb{R}_+}.$$

(4.4) also implies $u^*_1(k_3) = u^*_{\ell-2}(k_\ell)$. Thus, (4.5) with $\ell = 2$ implies that the vector $F(3)$ lies in the interior of the subcone $\langle E_{u^*_{\ell-2}(k_\ell)}, F(2) \rangle_{\mathbb{R}_+}$. Thus, one can split C_2 along the ray $F(3)$ to give $C_2 = C_3 + C_4$, with indexing chosen so that $C_4 = \langle E_1, \ldots, E_{u^*_{\ell-2}(k_\ell)-1}, F(3), E_{u^*_{\ell-2}(k_\ell)+1}, \ldots, F(2), \ldots, F(1) \rangle_{\mathbb{R}_+}$.

Proceeding inductively, it is now clear that one can assume the existence of a cone $C_{2(\ell-2)}$ that contains $F(\ell-1), F(\ell-2), \ldots, F(1)$ and $E_{u^*_1(k_\ell)}$ in its 1-

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skeleton. Using (4.4), (4.5) as above, one can then split this cone along $F(\ell-1)$ to form a new cone $C_{2(\ell-1)}$ that contains $F(\ell), F(\ell-1), \ldots, F(1)$ in its 1-skeleton. Evidently, $C_{2(\ell-1)}$ also belongs to $\mathcal{T}$. Thus, $\mathcal{T}$ also satisfies (1.7.1), (1.7.2). □

7. An example

Here are some details of an example that should help in following the general discussion. Part A will specify a distribution of intervals $\mathcal{I}_q$ that will be assumed in the other parts.

(A) Set $n = 7$ and $r = 3$. This means that three distinct values $\rho_1 < \rho_2 < \rho_3$ exist among the ratios b_i/c_i. Suppose that $k_1 = 1,\ k_2 = 3,\ k_3 = 6$. This determines the intervals (see (2.2)) $\mathcal{I}_1 = [1,2],\ \mathcal{I}_2 = [3,5],\ \mathcal{I}_3 = [6,7]$.

(B) Let C be the following chain rooted at 5:

$$\begin{array}{lll} A(1) = \{5\} & A(2) = \{5,7\} & A(3) = \{3,5,7\} \\ A(4) = \{3,5,6,7\} & A(5) = \{2,3,5,6,7\} & \\ A(6) = \{2,3,4,5,6,7\} & A(7) = \{1,2,3,4,5,6,7\}. & \end{array}$$

This means that $i_1(1) = i_1(2) = 5 \in \mathcal{I}_2,\ i_1(3) = i_1(4) = 3 \in \mathcal{I}_2,\ i_1(5) = i_1(6) = 2 \in \mathcal{I}_1$, and $i_1(7) = 1 \in \mathcal{I}_1$. Thus, $\iota(j) = 2$ for $j = 1,2,3,4$, and $\iota(j) = 1$ for $j = 5,6,7$. Further, the parameters $\kappa_i, u_i^*(C)$ (see (2.7.2)) are as follows.

$$\kappa_1 = 2 \quad u_1^*(C) = 4; \qquad \kappa_2 = 1 \quad u_2^*(C) = 7.$$

So, the depth equals 2, and the partition of $[1,7]$ associated to the maximal sequence of C is given by $\mathcal{U}_1(C) = [1,4], \mathcal{U}_2(C) = [5,7]$.

The set $\mathcal{S}(C)$ consists of all sets $\{u_1 < u_2\}$ such that $u_1 \in [1,4], u_2 \in [5,7]$. Given $\mu = \{(u_1 = 2) < (u_2 = 6)\} \in \mathcal{S}(C)$, one sees that $\mu^* = \{(u_1^* = 4) < (u_2^* = 7)\}$. A straightforward calculation shows that $Lf(2,6) \prec Lf(2,7) \prec Lf(4,7)$ (see (2.20)). Moreover, one sees that $\Sigma(\mu) = [J_5, J_2]^\perp$ and $\Sigma(\mu^*) = [J_5, J_3, J_2, J_1]^\perp$.

(C) The chain $\mathcal{C}_6$ (see (3.6.1)) equals:

$$\begin{array}{lll} \mathcal{A}(1) = \{6\} & \mathcal{A}(2) = \{6,7\} & \mathcal{A}(3) = \{5,6,7\} \\ \mathcal{A}(4) = \{4,5,6,7\} & \mathcal{A}(5) = \{3,4,5,6,7\} & \\ \mathcal{A}(6) = \{2,3,4,5,6,7\} & \mathcal{A}(7) = \{1,2,3,4,5,6,7\}. & \end{array}$$

The parameters (2.7.2) are as follows:

$$\kappa_1 = 3 \quad u_1^*(6) = 2; \qquad \kappa_2 = 2 \quad u_2^*(6) = 5; \qquad \kappa_3 = 1 \quad u_3^*(6) = 7.$$

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Thus, the depth of $\mathcal{C}_6$ is $d(6) = 3$. The sequence $\{u_j^*(6)\}$, defined in (3.6), is given by $\{u_1^*(6) = 2,\ u_2^*(6) = 5,\ u_3^*(6) = 7\}$. Thus, (3.6.2) means that $F_6 = F_3(2, 5, 7)$. One also shows (left to reader) that $F_6 = F_7$.

(D) Here, one has incorporated the notation of Section 3 to help the reader in following the proof in general. To give an explicit expression for $F(3) = F_6$ (see (3.12)), one notes that F_6 is a linear combination of E_2, E_5, E_7, constructed from $\mathcal{C}_6$. Thus (see (2.8), (2.9)),

$$F_6 = \lambda_1 e_{6,7} + \lambda_2 e_{3,4,5,6,7} + \lambda_3 e_{1,2,3,4,5,6,7}.$$

The expressions for the coefficients, determined by (3.2), are as follows. Using the element $\{2 < 5 < 7\} \in \mathcal{S}(\mathcal{C}_6)$, one sees that

$$\begin{aligned}
\lambda_1 &= c_{i_1(7)} \cdot B_1 \cdot \delta_{1,2} = c_1 \cdot \prod_{\substack{j \in \{2,3,\ldots,7\} \\ j \notin \{6,7\}}} b_j \cdot (c_{i_1(5)} b_{i_1(2)} - c_{i_1(2)} b_{i_1(5)}) \\
&= c_1 \cdot \prod_{j=2}^{5} b_j \cdot (c_3 b_6 - c_6 b_3), \\
\lambda_2 &= c_{i_1(2)} \cdot B_2 \cdot \delta_{2,3} = c_6 \cdot \prod_{\substack{j \in \{2,3,\ldots,7\} \\ j \notin \{3,4,\ldots,7\}}} b_j \cdot (c_{i_1(7)} b_{i_1(5)} - c_{i_1(5)} b_{i_1(7)}) \\
&= c_6 \cdot b_2 \cdot (c_1 b_3 - c_3 b_1), \\
\lambda_3 &= c_{i_1(2)} \cdot c_{i_1(5)} = c_3 \cdot c_6.
\end{aligned}$$

The idea of the main calculation in the proof of Lemma 3.2 can be seen in the derivation of the expression for λ_1, given these expressions for λ_2, λ_3. One looks for λ_1 so that

$$\lambda_1 (e_{6,7} \cdot J_6) = \lambda_3 \left(e_{1,2,3,4,5,6,7} \cdot (J_1 - J_6)\right) - \lambda_2 (e_{3,4,5,6,7} \cdot J_6).$$

Using the definitions of the vectors e_*, and the above expressions for λ_2, λ_3, this equation holds iff

$$\begin{aligned}
\lambda_1 \cdot c_6 \cdot b_7 &= c_3 c_6 \left[c_1 (b_2 b_3 b_4 b_5 b_6 b_7) - c_6 (b_1 b_2 b_3 b_4 b_5 b_7)\right] \\
&\qquad - \left[c_6 b_2 (c_1 b_3 - c_3 b_1)\right] \cdot \left[c_6 \cdot (b_3 b_4 b_5 b_7)\right] \\
&= c_6 (b_2 b_3 b_4 b_5 b_7) \left[(c_1 b_6 c_3 - c_6 b_1 c_3) + (c_6 c_3 b_1 - c_6 c_1 b_3)\right] \\
&= c_1 c_6 (b_2 b_3 b_4 b_5 b_7)(c_3 b_6 - c_6 b_3),
\end{aligned}$$

so, dividing out by $c_6 b_7$ gives $\lambda_1 = c_1 \cdot (b_2 b_3 b_4 b_5) \cdot \delta_{1,2}$, as claimed.

Using the general notation from Section 3, one sees that $u_1 = 2, u_2 = 5, u_3 = 7$, and $i_1(u_1) = 6, i_1(u_2) = 3, i_1(u_3) = 1$. One then can rewrite the above expressions as follows (using the particular ordering of terms from Section 3):

$$\lambda_1 \cdot c_{i_1(u_1)} \cdot \prod_{q \in A'(u_1)} b_q = c_{i_1(u_1)} \cdot \prod_{\substack{q \in A'(u_3) \\ q \neq i_1(u_1)}} b_q \cdot \left[(L_0 - R_0) + (L_1 - R_1)\right],$$

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where

$$L_0 = c_{i_1(u_3)} b_{i_1(u_1)} c_{i_1(u_2)}, \quad R_0 = c_{i_1(u_1)} b_{i_1(u_3)} c_{i_1(u_2)},$$
$$L_1 = c_{i_1(u_1)} c_{i_1(u_2)} b_{i_1(u_3)}, \quad R_1 = c_{i_1(u_1)} c_{i_1(u_3)} b_{i_1(u_2)}.$$

One observes that the cancellation of the term $c_3c_6b_1$ has occurred because $R_0 = L_1$.

(E) An example of the refinement procedure in Section 6 is now given for the cone $\mathcal{C}(\mathcal{C}_3)$, corresponding to the chain $\mathcal{C}_3$ and value $k_2 = 3$. Its elements are given by (3.6.1) setting $i = 3, n = 7$. The maximal sequence of the chain is easily seen to equal $\{u_1^*(3) = 5, u_2^*(3) = 7\}$. Thus, $F(3) = f(5,7)$. One starts with $\mathcal{C}(\mathcal{C}_3) = \langle E_1, \ldots, E_7\rangle$, where, by (2.10), $E_1 = e_3$ and

$$E_2 = e_{3,4}, \quad E_3 = e_{3,4,5}, \quad E_4 = e_{3,4,5,6}, \quad E_5 = e_{3,4,5,6,7},$$
$$E_6 = e_{2,3,4,5,6,7}, \quad E_7 = e_{1,2,3,4,5,6,7}.$$

The refinements need to separate E_6, E_7 from the other 5 vectors since $E_7 \in [J_1, J_2]^{\perp}, E_6 \in [J_2]^{\perp}$, but the other 5 vectors lie in $[J_3, J_4, J_5]^{\perp}$. Thus, one first splits along the ray in the direction of $f(5,7) \in [J_1, J_2]^{\perp}$ (the precise definition of which is given in Section 6). This produces the subcones

$$C_1(2) = \langle E_1, \ldots, E_6, f(5,7)\rangle \quad \text{and}$$
$$C_2(2) = \langle E_1, \cdots, E_4, f(5,7), E_6, E_7\rangle,$$

so that $\mathcal{C}(\mathcal{C}_3) = C_1(2) + C_2(2)$. One next splits $C_1(2)$ along the ray $f(5,6) \in [J_2, J_3, J_4, J_5]^{\perp}$ to give $C_1(2) = C_3(2) + C_4(2)$, where

$$C_3(2) = \langle E_1, \ldots, E_5, f(5,6), f(5,7)\rangle \quad \text{and}$$
$$C_4(2) = \langle E_1, \ldots, E_4, f(5,6), E_6, f(5,7)\rangle.$$

No further refinement of $C_3(2)$ is needed since J_2 is common to the $\mathcal{K}_2(\xi), \xi$ a 1-skeletal vector of $C_3(2)$. However, E_6 still appears in $C_4(2)$, so one must further refine, by splitting $C_4(2)$ along the ray in the direction of $f(4,6) \in [J_2, J_3, J_4, J_6]^{\perp}$. This gives $C_4(2) = C_5(2) + C_6(2)$, where

$$C_5(2) = \langle E_1, E_2, E_3, E_4, f(5,6), f(4,6), f(5,7)\rangle,$$
$$C_6(2) = \langle E_1, E_2, E_3, f(4,6), f(5,6), E_6, f(5,7)\rangle.$$

The pattern continues. Each cone $C_{2k-1}(2)$ needs no further refinement, but the cone $C_{2k}(2)$ does for $k = 3, 4, 5$, since E_6 continues to appear in its 1-skeleton. The last cone to be refined is $C_{10}(2) = C_{11}(2) + C_{12}(2)$, where

$$C_{11}(2) = \langle E_1, f(2,6), f(3,6), f(4,6), f(5,6), f(1,6), f(5,7)\rangle,$$
$$C_{12}(2) = \langle f(1,6), f(2,6), f(3,6), f(4,6), f(5,6), E_6, f(5,7)\rangle.$$

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The refinement $C_1(2) = \sum_{k=1}^{6} C_{2k-1}(2) + C_{12}(2)$ satisfies all the requirements of Lemma 6.1. One proceeds analogously for the cone $C_2(2)$. One then observes that the maximal line $LF(2) = Lf(5,7)$ appears exactly once in each of these seven cones.

8. An estimate for an error term

Define $\Omega(t_1,t_2) = \{P_1 \leqslant t_1\} \cap \{P_2 \leqslant t_2\} \cap [1,\infty)^n$. Let $V(t_1,t_2)$ denote its volume. If $(t_1,t_2) \in \mathcal{R}(v_i)$ is a regular value of $(P_1,P_2)|_{[0,\infty)^n}$, it follows, in particular, that $V(t_1,t_2)$ is a Mellin transform of $I(\mathbf{s})/s_1s_2$. The nonvanishing condition (1.2), established in the course of proving the Theorem, implies that $V(t_1,t_2)$ shares the same dominant asymptotic with $N(t_1,t_2)$.

Having thereby found the dominant asymptotic for $V(t_1,t_2)$ for any n, one can use a general description in [Dav] for the difference $|N(t_1,t_2) - V(t_1,t_2)|$ to give an *explicit* bound for this error term inside each $\mathcal{R}_\infty(v_i)$. This is possible if (t_1,t_2) is a regular value for $(P_1,P_2)|_{[0,\infty)^n}$, which will be assumed below. The estimate for this difference is expressed in terms of the maximum of volumes of projections of $\Omega(t_1,t_2)$ onto all lower dimensional coordinate planes. Since these are also determined by additive polynomials, the Theorem applies to the projections and enables the bound for the error to be given precisely.

Let $V_{\{j\}}(t_1,t_2)$ denote the volume of the projection of $\Omega(t_1,t_2)$ onto the hyperplane $x_j = 0$, for each $j = 1,\dots,n$. For each $i = 0,1,\dots,n$, set

$$V_i^*(t_1,t_2) = \max\{V_{\{j\}}(t_1,t_2) : j = 1,\dots,n\}|_{\mathcal{R}(v_i)}.$$

It is not difficult to see that $V_i^*(t_1,t_2)$ gives the largest contribution to the error term estimate of [Dav] inside any region $\mathcal{R}_\infty(v_i)$ (or even along any curve asymptotic to $\partial\mathcal{R}(v_i)$) for each i. Thus, an asymptotic for each V_i^* gives the *true order* of the estimate for the error term found by Davenport. To illustrate the point, this will now be done when $n = 3$. Similar analysis is possible for general n, but the actual writing down of the expressions becomes a little intricate. Restricting attention to the sets $\mathcal{R}_\infty(v_i)$ as the regions for (t_1,t_2), one has precisely:

THEOREM 8.2.

$$\begin{aligned}
&|N(t_1,t_2) - V(t_1,t_2)|\\
&\quad = O(\max\{t_1^{1/b_1+1/b_2}, t_1^{1/b_1+1/b_3}, t_1^{1/b_2+1/b_3}\}) \quad (t_1,t_2) \in \mathcal{R}_\infty(v_0),\\
&\quad = O(\max\{t_1^{1/b_2}t_2^{1/c_1}, t_1^{1/b_2+1/b_3}, t_1^{1/b_3}t_2^{1/c_1}\}) \quad (t_1,t_2) \in \mathcal{R}_\infty(v_1),\\
&\quad = O(\max\{t_1^{1/b_2}t_2^{1/c_1}, t_1^{1/b_3}t_2^{1/c_1}, t_1^{1/b_3}t_2^{1/c_2}\}) \quad (t_1,t_2) \in \mathcal{R}_\infty(v_2),\\
&\quad = O(\max\{t_2^{1/c_1+1/c_2}, t_2^{1/c_1+1/c_3}, t_2^{1/c_2+1/c_3}\}) \quad (t_1,t_2) \in \mathcal{R}_\infty(v_3).
\end{aligned}$$

CONCULDING REMARKS. (1) In the analytic number theory literature, one can find a few articles that have studied the representation of integers by sums of mixed powers (see [F], [H], [Va] for particular examples). These papers are in general devoted to the behavior of the counts

$$N(t) = \#\{\mathbf{m} \in \mathbb{N}^n : P(\mathbf{m}) = t\} \quad \text{as } t \to \infty,$$

when P is an additive polynomial of the type studied here. One is either interested in a precise asymptotic or a lower bound that grows with t. Such problems are evidently a natural generalization of Waring's problem, in which one allows the exponents to differ.

There is however, another type of extension of Waring's problem which has not been addressed so far in the literature, and which appears quite difficult. This asks for the number of *simultaneous representations* of vectors of integers by vectors of additive polynomials. In particular, given two additive polynomials in n variables, one can inquire about the asymptotic behavior of $\#\{P_1 = n_1\} \cap \{P_2 = n_2\} \cap \mathbb{N}^n, (n_1, n_2) \to (\infty, \infty)$. The Theorem proved in this paper gives non-trivial upper bounds to these counts inside the regions $\mathcal{R}_\infty(v_i),\ i = 1, \ldots, n-1$. More precisely, the Theorem immediately implies that for any $\mathcal{R}_\infty(v_i)$ there exists $\delta > 0$ such that

$$\begin{aligned} &\#\{P_1 = n_1\} \cap \{P_2 = n_2\} \cap \mathbb{N}^n \\ &\quad = O(n_1^{v_{1i}-\delta} n_2^{v_{2i}-\delta}) \quad (n_1, n_2) \to (\infty, \infty), \quad (n_1, n_2) \in \mathcal{R}_\infty(v_i). \end{aligned}$$

On the other hand, the asymptotic derived in $\mathcal{R}(v_0)$ resp. $\mathcal{R}(v_n)$ is a simple consequence of the weighted homogeneity of P_1 resp. P_2.

In principle, a value for δ can be given explicitly and shown to be smaller than 1. To increase δ to 1 would be interesting to establish and natural to expect. To achieve this, one will need to understand more precisely the behavior of the singular integral and series determined by P_1, P_2. It does not seem unreasonable to believe that the geometric analysis in this paper will be needed for such improvements.

On the other hand, going from good upper bounds to good lower bounds or precise asymptotics is not possible to accomplish by the geometric methods used here. A good deal of much finer arithmetic analysis would be required, especially in the case of two additive polynomials with no relation between their exponents.

(2) It appears reasonable to expect that one might be able to exploit the convexity of the polyhedra at infinity to verify the Conjecture, described in the Introduction, for any pair of polynomials, each nondegenerate with respect to its polyhedron at infinity. However, so far this seems to be difficult to do.

(3) The subject addressed in this paper is a particular case of the more general problem of the behavior of an integral over the smooth fibers of an algebraic or analytic morphism. Recent work [Li-5] on this problem for pairs of functions in two variables should be helpful in studying the conjecture, stated in the Introduction, for any pair of hypoelliptic polynomials on $\mathbb{R}^2$.

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Addendum: Details of proofs of lemmas in Section 2

PROOF OF LEMMA 2.15.

Proof of (i). One first checks that $f(u,v)\cdot J_a = f(u,v)\cdot J_b$ for any $a\in A(u)\cap \mathcal{I}_{\iota(u)}, b\in A(v)\cap\mathcal{I}_{\iota(v)}$. To do so, one notes that if $a\neq i_1(u)$ and $a\in A(u)\cap\mathcal{I}_{\iota(u)}$, then there exists $d<e$ such that $i_d(v)=i_1(u)$ and $i_e(v)=a$. Thus

$$E_v\cdot(J_a-J_{i_1(u)}) = \prod_{q\neq d,e}^{v} b_{i_q(v)}\cdot(c_{i_d(v)}b_{i_e(v)}-c_{i_e(v)}b_{i_d(v)})=0.$$

Hence, $f(u,v)\cdot J_a$ is constant for $a\in A(u)\cap\mathcal{I}_{\iota(u)}$. Further, if $b\in A(v)\cap\mathcal{I}_{\iota(v)}$, then $b\notin A(u)$. If this were not so, then there would exist d such that $b=i_d(u)$. Clearly, $d\neq 1$ since $\iota(u)>\iota(v)$. Moreover

$$E_u\cdot(J_b-J_{i_1(u)}) = c_{i_1(u)}\cdot c_{i_d(u)}\cdot\prod_{q\neq 1,d}^{u} b_{i_q(u)}\cdot(\rho_{\iota(u)}-\rho_{\iota(v)})>0.$$

But this inequality violates (2.13)(3d). Thus, one concludes that $E_u\cdot J_b=0$ for any $b\in A(v)\cap\mathcal{I}_{\iota(v)}$. It then follows that $f(u,v)\cdot J_b = f(u,v)\cdot J_{i_1(v)}$ for all such b. One concludes

$$f(u,v)\cdot J_a = f(u,v)\cdot J_b \quad \text{for any } a\in A(u)\cap\mathcal{I}_{\iota(u)}, b\in A(v)\cap\mathcal{I}_{\iota(v)}.$$

What needs to be verified next is that for any $c \notin \Big(A(u) \cap \mathcal{I}_{\iota(u)}\Big) \cup \Big(A(v) \cap \mathcal{I}_{\iota(v)}\Big)$, one has

$$f(u,v) \cdot J_c \leqslant f(u,v) \cdot J_e \quad \text{for any } e \in \Big(A(u) \cap \mathcal{I}_{\iota(u)}\Big) \cup \Big(A(v) \cap \mathcal{I}_{\iota(v)}\Big).$$

This is clear if $c \in A(u) - \mathcal{I}_{\iota(u)}$ because in that case $E_u \cdot J_c < E_u \cdot J_{i_1(u)}$ and $E_v \cdot J_c < E_v \cdot J_{i_1(v)}$. Suppose that $c \in A(v) - (A(u) \cup \mathcal{I}_{\iota(v)})$. Then $c \notin A(u)$ implies $E_u \cdot J_c = 0$ which implies the inequality to be proved. This completes the proof of (i).

Proof of (ii). One uses (2.13)(3b,c) to conclude

$$\begin{aligned}
M_1(f(u,v)) &= f(u,v) \cdot I_{i_1(u)} \\
&= (E_v \cdot J_{i_1(v)})(E_u \cdot I_{i_1(u)}) - (E_v \cdot J_{i_1(u)})(E_u \cdot I_{i_1(u)}) \\
&\quad + (E_u \cdot J_{i_1(u)})(E_v \cdot I_{i_1(u)}), \\
M_2(f(u,v)) &= f(u,v) \cdot J_{i_1(u)} = f(u,v) \cdot J_{i_1(v)} \\
&= (E_u \cdot J_{i_1(u)})(E_v \cdot J_{i_1(v)}).
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{M_1(f(u,v))}{M_2(f(u,v))} &= \frac{E_u \cdot I_{i_1(u)}}{E_u \cdot J_{i_1(u)}} \\
&\quad + \frac{\Big[(E_u \cdot J_{i_1(u)})(E_v \cdot I_{i_1(u)}) - (E_v \cdot J_{i_1(u)})(E_u \cdot I_{i_1(u)})\Big]}{(E_u \cdot J_{i_1(u)})(E_v \cdot J_{i_1(v)})}.
\end{aligned}$$

Since $i_1(u) = i_d(v)$ for some $d > 1$, a simple calculation shows that the bracketed term equals 0. This shows (ii).

Proof of (iii). One again uses the fact that $u < v$ implies $i_1(u) \in A(v)$. Thus

$$M_1(E_v) = E_v \cdot I_{i_1(u)}.$$

Since $M_1(E_u) = E_u \cdot I_{i_1(u)}$ by construction, it follows that $M_1(f(u,v)) = \alpha M_1(E_u) + \beta M_1(E_v)$, where the positive numbers α, β are given in (2.14). A simple calculation then shows

$$h(Lf(E_u,E_v)) - h(LE_u) = (+)[h(LE_v) - h(LE_u)],$$

where $(+)$ indicates a positive number. By (2.13)(3e), the difference is positive. □

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PROOF OF LEMMA 2.17.

Proof of (i). Set $\mu' = \{u_2 < \cdots < u_k\}$. By induction, one may assume that $f_{k-1}(\mu') \in \Sigma(\mu')$, and $f_{k-1}(\mu')$ is a linear combination of $E_{u_2}, E_{u_3}, \ldots, E_{u_k}$ with positive integer coefficients. Define α_k, β_k by the formulae

$$\begin{aligned} \alpha_k &= f_{k-1}(\mu') \cdot (J_{i_1(u_2)} - J_{i_1(u_1)}) = \cdots = f_{k-1}(\mu') \cdot (J_{i_1(u_k)} - J_{i_1(u_1)}), \\ \beta_k &= f_1(u_1) \cdot J_{i_1(u_1)}. \end{aligned} \tag{1}$$

Evidently, these two integers are positive. The expression on the second line in (2.16) is precisely the assertion that

$$f_k(\mu) = \alpha_k f_1(u_1) + \beta_k f_{k-1}(\mu').$$

Exactly as in the proof of (2.15)(i), one shows $f_k(\mu) \in \Sigma(\mu)$.

Proof of (ii). The proof of (2.15)(ii) extends straightforwardly to show that for each $k \geqslant 2$,

$$\operatorname{sl} L f_k(\mu) = \operatorname{sl} L f_1(u_1) = -b_{i_1(u_1)}/c_{i_1(u_1)}.$$

Proof of (iii). Since $i_1(u_1) \in A(u_\ell)$, for any $\ell \geqslant 1$, one has

$$M_1(f_k(\mu)) = \alpha_k M_1(f_1(u_1)) + \beta_k M_1(f_{k-1}(\mu')).$$

A straightforward calculation now verifies that

$$h(L f_k(\mu)) - h(L f_1(u_1)) > 0$$

follows from the induction hypothesis $h(L f_{k-1}(\mu')) - h(L f_1(u_1)) > 0$. This establishes (iii) and completes the proof. □

PROOF OF LEMMA 2.18. This uses the following preliminary result.

LEMMA. *Let* $\{u_1 < \cdots < u_d\} \in \mathcal{S}(C)$. *Set*

$$g_1 = \sum_{i=2}^{d-1} a_i E_{u_i}, \qquad g_2 = \sum_{i=2}^{d} b_i E_{u_i},$$

where a_i, b_j *are positive for any* i, j. *Then*

$$(g_2 \cdot \mathbf{e}_{i_1(u_1)})(g_1 \cdot \mathbf{e}_{i_1(u_2)}) = (g_1 \cdot \mathbf{e}_{i_1(u_1)})(g_2 \cdot \mathbf{e}_{i_1(u_2)}).$$

Proof. Since $i_1(u_1) = i_q(u_2)$ for some $q \geqslant 2$ the difference of the right and left side of the asserted equation equals

$$\sum_{\substack{i \neq j \\ i,j \geqslant 2}} a_i b_j [(E_{u_i} \cdot \mathbf{e}_{i_q(u_2)})(E_{u_j} \cdot \mathbf{e}_{i_1(u_2)}) - (E_{u_j} \cdot \mathbf{e}_{i_q(u_2)})(E_{u_i} \cdot \mathbf{e}_{i_1(u_2)})].$$

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(When $i=j$ it is clear that the factor of a_ib_i equals 0.) Let $\epsilon_{i,j}$ denote the coefficient of a_ib_j. One shows that $\epsilon_{i,j}=0$ as follows. Since $A(u_2)\subset A(u_i)\cap A(u_j)$, it follows that there exist s,t,v,w such that

$$\mathbf{e}_{i_q(u_2)}=\mathbf{e}_{i_s(u_i)}=\mathbf{e}_{i_t(u_j)},\qquad \mathbf{e}_{i_1(u_2)}=\mathbf{e}_{i_v(u_i)}=\mathbf{e}_{i_w(u_j)}.$$

Thus

$$\begin{aligned}\epsilon_{i,j} &= \left(\prod_{\ell\neq s}^{u_i} b_{i_\ell(u_i)}\right)\left(\prod_{k\neq w}^{u_j} b_{i_k(u_j)}\right)-\left(\prod_{\ell\neq v}^{u_i} b_{i_\ell(u_i)}\right)\left(\prod_{k\neq t}^{u_j} b_{i_k(u_j)}\right)\\ &= \left(\prod_{\ell\neq s,v}^{u_i} b_{i_\ell(u_i)}\right)\cdot\left(\prod_{k\neq t,w}^{u_j} b_{i_k(u_j)}\right)\left[b_{i_v(u_i)}\cdot b_{i_t(u_j)}-b_{i_s(u_i)}\cdot b_{i_w(u_j)}\right]\\ &= 0 \text{ by the above identifications.}\end{aligned}$$

This proves the lemma. □

One uses this lemma to prove (2.18) as follows. Set

$$g_{k-2}=f_{k-2}(u_2,\ldots,u_{k-1}),\qquad g_{k-1}=f_{k-1}(u_2,\ldots,u_k).$$

By (2.17), the lines Lg_{k-2} and Lg_{k-1} are parallel. Furthermore, since (2.15) takes care of the case $k=2$, one may assume by induction on the number of elements that $Lg_{k-2}\prec Lg_{k-1}$. Thus, it suffices to show

$$Lf(u_1,g_{k-2})\prec Lf(u_1,g_{k-1}).$$

By the formulae in (1) (see above), one sees that by defining

$$\begin{aligned}\theta_1 &= g_{k-2}\cdot(J_{i_1(u_2)}-J_{i_1(u_1)}),\\ \theta_2 &= E_{u_1}\cdot J_{i_1(u_1)},\\ \lambda_1 &= g_{k-1}\cdot(J_{i_1(u_2)}-J_{i_1(u_1)}),\end{aligned}\tag{2}$$

one obtains

$$f(u_1,g_{k-2})=\theta_1E_{u_1}+\theta_2g_{k-2}\quad\text{and}\quad f(u_1,g_{k-1})=\lambda_1E_{u_1}+\theta_2g_{k-1}.$$

Moreover, (2.17)(i) implies

$$\begin{aligned}M_2f(u_1,g_{k-2}) &= f(u_1,g_{k-2})\cdot J_{i_1(u_1)}=\cdots=f(u_1,g_{k-2})\cdot J_{i_1(u_{k-1})},\\ M_2f(u_1,g_{k-1}) &= f(u_1,g_{k-1})\cdot J_{i_1(u_1)}=\cdots=f(u_1,g_{k-1})\cdot J_{i_1(u_k)}.\end{aligned}$$

Using the expressions in (2), one sees that

$$M_2\,f(u_1,g_v)=M_2\,E_{u_1}\cdot M_2\,g_v,\quad v=k-2,k-1.$$

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Now define

$$\delta = v\left(Lf(u_1, g_{k-2})\right) - v\left(Lf(u_1, g_{k-1})\right) = \frac{|f(u_1, g_{k-2})|}{M_2\, f(u_1, g_{k-2})} - \frac{|f(u_1, g_{k-1})|}{M_2\, f(u_1, g_{k-1})}.$$

One must show $\delta < 0$, given that $Lg_{k-2} \prec Lg_{k-1}$.

Using (2), an elementary manipulation shows

$$\begin{aligned} \delta &= (+)[(g_{k-1} \cdot J_{i_1(u_1)})(g_{k-2} \cdot J_{i_1(u_2)}) - (g_{k-1} \cdot J_{i_1(u_2)})(g_{k-2} \cdot J_{i_1(u_1)})] \\ &\quad + \left[\frac{|g_{k-2}|}{M_2\, g_{k-2}} - \frac{|g_{k-1}|}{M_2\, g_{k-1}}\right], \end{aligned}$$

where $(+) = |E_{u_1}|/M_2\, E_{u_1} \cdot M_2\, g_{k-2} \cdot M_2\, g_{k-1}$. The Lemma above implies that the factor of $(+)$ equals zero. Thus, δ equals the difference in vertical axis intercepts of the lines Lg_{k-2}, Lg_{k-1}, which is negative by the induction hypothesis. This proves (2.18). □

PROOF OF LEMMA 2.19. Set

$$g_1 = f_{k-d+1}(u'_d, u_{d+1}, \ldots, u_k) \quad \text{and} \quad g_2 = f_{k-d+1}(u_d, u_{d+1}, \ldots, u_k).$$

It suffices to show

(i) $Lg_1 \prec Lg_2$;

(ii) $Lf(u, g_1) \prec Lf(u, g_2)$ for any index u such that $\{u < u'_d < \cdots < u_k\} \in \mathcal{S}(C)$.

It is easy to verify that (i) and (ii) follow from the same arguments used to establish (2.17), (2.18). □

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