19

Basic nuclear structure

The goal is to compare experimental electron scattering data with a theoretical picture of the hadronic target, and in so doing, develop an understanding of that system. We start the discussion within the traditional non-relativistic many-body description of the nucleus.

If the nucleus is modeled as a quantum mechanical system of point nucleons with intrinsic magnetic moments, then one knows how to construct the charge density, convection current density, and intrinsic magnetization density from basic quantum mechanics. In first quantization these quantities are given by

$$\begin{aligned}
\hat{\rho}_{\mathrm{N}}(\mathbf{x}) &= \sum_{j=1}^{A} e(j)\delta^{(3)}(\mathbf{x}-\mathbf{x}_j) \\
\hat{\mathbf{J}}_{\mathrm{c}}(\mathbf{x}) &= \sum_{j=1}^{A} e(j)\{\frac{\mathbf{p}(j)}{m},\delta^{(3)}(\mathbf{x}-\mathbf{x}_j)\}_{\mathrm{sym}} \\
\hat{\boldsymbol{\mu}}(\mathbf{x}) &= \sum_{j=1}^{A} \mu(j)\frac{\boldsymbol{\sigma}(j)}{2m}\delta^{(3)}(\mathbf{x}-\mathbf{x}_j)
\end{aligned} \tag{19.1}$$

Here $\mathbf{p} \equiv (1/i)\nabla$ and $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Thus for a single particle, for example

$$\langle\hat{\rho}_{\mathrm{N}}(\mathbf{x})\rangle = \int \psi^*(\mathbf{x}_p)\delta^{(3)}(\mathbf{x}-\mathbf{x}_p)\psi(\mathbf{x}_p)\,d^3x_p \quad = \quad |\psi(\mathbf{x})|^2 \tag{19.2}$$

and also

$$\begin{aligned}
\langle\hat{\mathbf{J}}_{\mathrm{c}}(\mathbf{x})\rangle &= \int \psi^*(\mathbf{x}_p)\frac{1}{2im}[\nabla_p\delta^{(3)}(\mathbf{x}-\mathbf{x}_p)+\delta^{(3)}(\mathbf{x}-\mathbf{x}_p)\nabla_p]\psi(\mathbf{x}_p)\,d^3x_p \\
&= \frac{1}{2im}\{\psi^*(\mathbf{x})\nabla\psi(\mathbf{x})-[\nabla\psi(\mathbf{x})]^*\psi(\mathbf{x})\}
\end{aligned} \tag{19.3}$$

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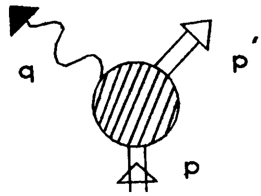

Fig. 19.1. Electromagnetic vertex for a free nucleon.

A partial integration has been used in obtaining the last equality. The charge and magnetic moment of the nucleon are given by

$$\begin{aligned} e(j) &\equiv \frac{1}{2}[1+\tau_3(j)] \\ \mu(j) &\equiv \lambda_p\frac{1}{2}[1+\tau_3(j)]+\lambda_n\frac{1}{2}[1-\tau_3(j)] \end{aligned} \tag{19.4}$$

The *anomalous* magnetic moment $\lambda'(j)$ of the nucleon is defined by

$$\begin{aligned} \lambda'(j) &= \lambda'_p\frac{1}{2}[1+\tau_3(j)]+\lambda_n\frac{1}{2}[1-\tau_3(j)] \\ \mu(j) &= e(j)+\lambda'(j) \end{aligned} \tag{19.5}$$

This discussion presents a consistent non-relativistic treatment in a picture where the nucleus is made up of point nucleons with appropriate charges and intrinsic magnetic moments; however, a central goal of nuclear physics is the measurement and calculation of nuclear electromagnetic transition densities out to momentum transfers $q^2 = O(m^2)$ and well beyond. It is essential to consider corrections to the non-relativistic current operator as one moves into this regime. In order to do this, a fully relativistic treatment of the interacting many-body system is required, and the next section is devoted to this topic. For the present, we simply consider the nuclear current density arising from the full relativistic electromagnetic vertex of a free nucleon [Mc62].

The relativistic electromagnetic vertex of a free nucleon is illustrated in Fig. 19.1. The most general structure of the matrix element of the current for a free nucleon is given by [Bj65, Wa95]

$$\langle \mathbf{p}'\sigma'\rho'|J_\mu(0)|\mathbf{p}\sigma\rho\rangle = \frac{i}{\Omega}\bar{u}(\mathbf{p}',\sigma')\eta^\dagger_{\rho'}[F_1\gamma_\mu+F_2\sigma_{\mu\nu}q_\nu]\eta_\rho u(\mathbf{p},\sigma) \tag{19.6}$$

Here the spin and isospin quantum numbers have been made explicit; $\bar{u}, u$ are Dirac spinors and η_p, η_n are two-component Pauli isospinors. The four-momentum transfer is defined by $q = p - p'$, and the form factors $F_{1,2}$ are functions of q^2. The isospin structure of the form factors must be

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of the form

$$F_i = \frac{1}{2}(F_i^S + \tau_3 F_i^V) \qquad ; \; i = 1,2 \tag{19.7}$$

Relevant numerical values are

$$\begin{aligned} F_1^S(0) &= F_1^V(0) = 1 \\ 2mF_2^S(0) &= \lambda_p' + \lambda_n = -0.120 \\ 2mF_2^V(0) &= \lambda_p' - \lambda_n = +3.706 \end{aligned} \tag{19.8}$$

To construct the *nuclear* current density one carries out the following steps:

1. Substitute the explicit form of the Dirac spinors for a free nucleon

$$u(\mathbf{p}, \sigma) = \left(\frac{E_p + m}{2E_p}\right)^{1/2} \begin{bmatrix} \chi_\sigma \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p + m}\chi_\sigma \end{bmatrix} \tag{19.9}$$

Here $\chi_\uparrow, \chi_\downarrow$ are two-component Pauli spinors for spin up and down along the z-axis. Now expand the matrix element in Eq. (19.6) consistently to order $1/m^2$. The result is[1]

$$\langle \mathbf{p}'\sigma'\rho' | J_\mu(0) | \mathbf{p}\sigma\rho \rangle = \frac{1}{\Omega} \eta_{\rho'}^\dagger \chi_{\sigma'}^\dagger \mathscr{M}_\mu \chi_\sigma \eta_\rho \tag{19.10}$$

$$\boldsymbol{\mathscr{M}} = F_1 \frac{1}{2m}(\mathbf{p} + \mathbf{p}') + (F_1 + 2mF_2)\left[\frac{-i\boldsymbol{\sigma} \times \mathbf{q}}{2m}\right] + O(\frac{1}{m^3})$$

$$\mathscr{M}_0 = F_1 - (F_1 + 4mF_2)\left[\frac{\mathbf{q}^2}{8m^2} - \frac{i\mathbf{q}\cdot(\boldsymbol{\sigma} \times \mathbf{p})}{4m^2}\right] + O(\frac{1}{m^3})$$

Here $\mathscr{M}_\mu = (\boldsymbol{\mathscr{M}}, i\mathscr{M}_0)$.

2. Take the prescription for constructing the nuclear current density operator at the origin, in second quantization, to be

$$\hat{J}_\mu(0) = \sum_{\mathbf{p}'\sigma'\rho'} \sum_{\mathbf{p}\sigma\rho} c_{\mathbf{p}'\sigma'\rho'}^\dagger \langle \mathbf{p}'\sigma'\rho' | J_\mu(0) | \mathbf{p}\sigma\rho \rangle c_{\mathbf{p}\sigma\rho} \tag{19.11}$$

where the single-particle matrix element is that of Eq. (19.6).

3. Use the general procedure for passing from first quantization to second quantization [Fe71]. If, in first quantization the one-body nuclear density operator has the form

$$\hat{J}_\mu(\mathbf{x}) = \sum_{i=1}^{A} \{ J_\mu^{(1)}(i) \delta^{(3)}(\mathbf{x} - \mathbf{x}_i) \} \tag{19.12}$$

[1] It is assumed that both q_0 and F_2 are $O(1/m)$.

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then in second quantization the operator density is

$$\hat{J}_\mu(\mathbf{x}) = \sum_{\mathbf{p}'\sigma'\rho'} \sum_{\mathbf{p}\sigma\rho} c^\dagger_{\mathbf{p}'\sigma'\rho'} \langle \mathbf{p}'\sigma'\rho' | J_\mu(\mathbf{x}) | \mathbf{p}\sigma\rho \rangle c_{\mathbf{p}\sigma\rho} \tag{19.13}$$

with

$$\langle \mathbf{p}'\sigma'\rho' | J_\mu(\mathbf{x}) | \mathbf{p}\sigma\rho \rangle = \int d^3y \, \phi^\dagger_{\mathbf{p}'\sigma'\rho'}(\mathbf{y}) \{ J^{(1)}_\mu(\mathbf{y}) \delta^{(3)}(\mathbf{x}-\mathbf{y}) \} \phi_{\mathbf{p}\sigma\rho}(\mathbf{y}) \tag{19.14}$$

4. The discussion in chapter 9 shows that physical rates and cross sections are expressed in terms of the Fourier transform of the transition matrix element of the current

$$\int e^{-i\mathbf{q}\cdot\mathbf{x}} \langle f | \hat{J}_\mu(\mathbf{x}) | i \rangle \, d^3x \tag{19.15}$$

Here $q = p - p'$, and in electron scattering $q = k_2 - k_1$. Define

$$\langle f | \hat{J}_\mu(\mathbf{x}) | i \rangle \equiv J_\mu(\mathbf{x})_{fi} \tag{19.16}$$

and observe that by partial integration in Eq. (19.15), with localized densities, one can make the replacement

$$\nabla \leftrightarrow i\mathbf{q} \tag{19.17}$$

The presence of terms in $i\mathbf{q}$ in the elementary nucleon amplitudes are then anticipated by defining

$$\begin{aligned} \mathbf{J}(\mathbf{x})_{fi} &\equiv \mathbf{J}_c(\mathbf{x})_{fi} + \nabla \times \boldsymbol{\mu}(\mathbf{x})_{fi} \\ \rho(\mathbf{x})_{fi} &\equiv \rho_N(\mathbf{x})_{fi} + \nabla \cdot \mathbf{s}(\mathbf{x})_{fi} + \nabla^2 \phi(\mathbf{x})_{fi} \end{aligned} \tag{19.18}$$

The use of Eq. (19.13) evaluated at $\mathbf{x} = 0$ now permits the identification of the nuclear density operators in first quantization, which give rise to the required result in second quantization of Eq. (19.10). The operators take the form

$$\begin{aligned} \hat{\mathbf{J}}(\mathbf{x}) &= \hat{\mathbf{J}}_c(\mathbf{x}) + \nabla \times \hat{\boldsymbol{\mu}}(\mathbf{x}) \\ \hat{\rho}(\mathbf{x}) &= \hat{\rho}_N(\mathbf{x}) + \nabla \cdot \hat{\mathbf{s}}(\mathbf{x}) + \nabla^2 \hat{\phi}(\mathbf{x}) \end{aligned} \tag{19.19}$$

Here the densities are defined by Eqs. (19.1), (19.4), (19.5), and

$$\hat{\phi}(\mathbf{x}) = \sum_{j=1}^{A} s(j) \frac{1}{8m^2} \delta^{(3)}(\mathbf{x} - \mathbf{x}_j) \tag{19.20}$$

$$\hat{\mathbf{s}}(\mathbf{x}) = \sum_{j=1}^{A} s(j) \frac{1}{4m^2} \boldsymbol{\sigma}(j) \times \{ \mathbf{p}(j), \delta^{(3)}(\mathbf{x} - \mathbf{x}_j) \}_{\text{sym}}$$

where in the static limit

$$s(j) \equiv e(j) + 2\lambda'(j) \tag{19.21}$$

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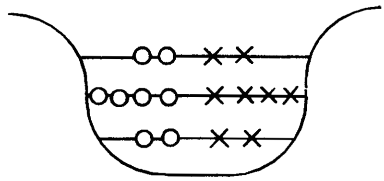

Fig. 19.2. Basis of Hartree–Fock states.

5. It is an empirical result that in the nuclear domain[2]

$$\frac{F_1(q^2)}{F_1(0)} \approx f_{\rm SN}(q^2) \approx \frac{F_2(q^2)}{F_2(0)}$$
$$f_{\rm SN}(q^2) = \frac{1}{(1+q^2/0.71\ {\rm GeV}^2)^2} \tag{19.22}$$

At finite q^2, the quantity $f_{\rm SN}(q^2)$ enters as an overall factor in the electromagnetic vertex, and it can be included by using an overall *effective* Mott cross section $\bar{\sigma}_{\rm M}$ in the $({\rm e},{\rm e}')$ cross section

$$\bar{\sigma}_{\rm M} \equiv \sigma_{\rm M}|f_{\rm SN}(q^2)|^2 \tag{19.23}$$

The use of this effective Mott cross section represents an approximate way of taking into account in the nuclear domain the spatial extent of the *internal* charge and magnetization densities of a single constituent nucleon.

The present analysis gives the leading relativistic corrections to the nuclear current, assuming it is a one-body operator. It *neglects*, among other things: meson exchange currents, other multibody currents, relativistic terms in the *wave functions*, and off-shell corrections to the nucleon vertex in the nuclear medium. We shall return to many of these points.

Consider next the many-body matrix elements of the current [Fe71, Wa95]. Introduce a complete basis of Hartree–Fock states as illustrated in Fig. 19.2. Assume a central field and label the quantum numbers by

$$\alpha = (nl\tfrac{1}{2}jm_j;\tfrac{1}{2}m_t)$$
$$\equiv (a; m_j, m_t) \tag{19.24}$$

[2] A more accurate representation of the experimental data for the proton and neutron out to very large q^2 is given by [Ba73, Wa84]

$$G_{\rm M}(q^2) \equiv F_1 + 2mF_2 = f_{\rm SN}(q^2)G_{\rm M}(0)$$
$$G_{\rm E}(q^2) \equiv F_1 - (q^2/2m)F_2 = f_{\rm SN}(q^2)G_{\rm E}(0)$$

although $G_{\rm E}^n(q^2)$ remains to be measured well. Elastic scattering from the nucleon is discussed in more detail in chapter 20.

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Then

$$-\alpha \equiv (a; -m_j, -m_t) \tag{19.25}$$

We shall also need the phase defined by

$$S_\alpha \equiv (-1)^{j_\alpha - m_{j\alpha}}(-1)^{1/2 - m_{t\alpha}} \tag{19.26}$$

Any many-body multipole operator of the above current can now be written in second quantization as [Fe71]

$$\hat{\mathscr{T}}_{JM_J;TM_T}(\kappa) = \sum_\alpha \sum_\beta c_\alpha^\dagger \langle \alpha | T_{JM_J;TM_T}(\kappa) | \beta \rangle c_\beta \tag{19.27}$$

$c_\alpha^\dagger, c_\beta$ are creation and destruction operators for the single-particle Hartree–Fock states. The single-particle matrix elements are calculated using the wave functions of this basis, the current densities above, and the appropriate multipole projections of chapter 9.

Within the present framework, an arbitrary matrix element between exact eigenstates $|\Psi_i\rangle$ and $|\Psi_f\rangle$ of the nuclear many-body system can be written

$$\begin{aligned} \langle \Psi_f | \hat{\mathscr{T}}_{JM_J;TM_T}(\kappa) | \Psi_i \rangle &= \sum_\alpha \sum_\beta \langle \alpha | T_{JM_J;TM_T}(\kappa) | \beta \rangle \; \psi_{\alpha\beta}^{fi} \\ \psi_{\alpha\beta}^{fi} &= \langle \Psi_f | c_\alpha^\dagger c_\beta | \Psi_i \rangle \end{aligned} \tag{19.28}$$

The quantities $\psi_{\alpha\beta}^{fi}$ are simply *numerical coefficients*. This result has the following features:

- It assumes the current is a one-body operator — exchange currents, for example, are neglected;
- Any shell-model calculation, no matter how complicated, must give an answer of this form. The exact many-body matrix element is a sum of single-particle matrix elements with numerical coefficients;
- This is an *exact* statement within the traditional non-relativistic nuclear many-body problem.

Let us extract the angular momentum properties of the operators involved in the above. Suppress isospin for the moment; it will be restored at the end. The angular momentum operator for the system is

$$\begin{aligned} \hat{\mathbf{J}} &= \sum_\alpha \sum_\beta c_\alpha^\dagger \langle \alpha | \mathbf{J} | \beta \rangle c_\beta \\ &= \sum_{nlj} \sum_{m'} \sum_m c_{nljm'}^\dagger \langle jm' | \mathbf{J} | jm \rangle c_{nljm} \end{aligned} \tag{19.29}$$

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Note that the single-particle matrix elements of **J** are diagonal in (nl) and independent of (nl). Now make use of the basic anti-commutation relations

$$\begin{aligned} \{c_\alpha, c_\beta^\dagger\} &= \delta_{\alpha\beta} \\ \{c_\alpha, c_\beta\} &= \{c_\alpha^\dagger, c_\beta^\dagger\} = 0 \end{aligned} \tag{19.30}$$

It is then a matter of algebra to establish the relation[3]

$$[\hat{\mathbf{J}}, c_{nljm}^\dagger] = \sum_{m'} \langle jm'|\mathbf{J}|jm\rangle c_{nljm'}^\dagger \tag{19.31}$$

Hence $c_\alpha^\dagger$ is an irreducible tensor operator (ITO) of rank j [Ed74].

Use of the Wigner–Eckart theorem allows one to establish the following relations [Ed74]

$$\begin{aligned} \langle jm|J_{1q}|jm'\rangle &= (-1)^{m'-m+1}\langle j,-m'|J_{1q}|j,-m\rangle \\ \sum_m \langle jm|J_{1q}|jm\rangle &= 0 \end{aligned} \tag{19.32}$$

This permits the angular momentum operator to be written in an equivalent form [recall Eq. (19.26)]

$$\begin{aligned} \hat{\mathbf{J}} &= \sum_\alpha \sum_\beta (S_{-\alpha}c_{-\alpha})\langle\alpha|\mathbf{J}|\beta\rangle(S_{-\beta}c_{-\beta})^\dagger \qquad (19.33) \\ &= \sum_{nlj}\sum_{m'}\sum_m [(-1)^{j+m'}c_{nlj,-m'}]\,\langle jm'|\mathbf{J}|jm\rangle\,[(-1)^{j+m}c_{nlj,-m}^\dagger] \end{aligned}$$

Hence one concludes the $S_{-\alpha}c_{-\alpha}$ is an ITO by the same proof as above.

Now restore isospin (treated in an exactly analogous fashion) and assume the initial and final many-body target states are eigenstates of angular momentum and isospin. Use of Eq. (19.27) and the Wigner–Eckart theorem on both the many-body and single-particle matrix elements, and a change of dummy indices, leads to[4]

$$\begin{aligned} &\langle J_f M_f T_f \bar{M}_f|\hat{\mathscr{T}}_{JM_J;TM_T}|J_i M_i T_i \bar{M}_i\rangle = \\ &\quad (-1)^{J_f-M_f}\begin{pmatrix} J_f & J & J_i \\ -M_f & M_J & M_i \end{pmatrix} \times [J \rightleftharpoons] \times \langle J_f T_f ⁞⁞ \hat{\mathscr{T}}_{J,T} ⁞⁞ J_i T_i\rangle \\ &= \sum_{a,b}\langle a ⁞⁞ \hat{\mathscr{T}}_{J,T} ⁞⁞ b\rangle\langle J_f M_f T_f \bar{M}_f| \left\{ \sum_{m_{j_\alpha} m_{j_\beta}} \langle j_\alpha m_{j_\alpha} j_\beta m_{j_\beta}|j_\alpha j_\beta J M_J\rangle \right. \end{aligned}$$

[3] See [Fe71, Wa95].

[4] The symbol $\langle ⁞⁞ O ⁞⁞ \rangle$ indicates a matrix element reduced with respect to both angular momentum and isospin.

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$$\times \sum_{m_{t_\alpha} m_{t_\beta}} \langle t_\alpha m_{t_\alpha} t_\beta m_{t_\beta} | \frac{1}{2}\frac{1}{2} T M_T \rangle (-1)^{j_\beta + m_{j_\beta}} (-1)^{1/2+m_{t_\beta}}$$

$$\times \frac{1}{\sqrt{(2J+1)(2T+1)}} c_\alpha^\dagger c_{-\beta} \Bigg\} |JM_i T\bar{M}_i\rangle \qquad (19.34)$$

One now identifies the *tensor product* $[c_\alpha \odot S_{-\beta}c_{-\beta}]_{JM_J;TM_T}$ of two ITO [Ed74]. Use of the Wigner–Eckart on that quantity gives, upon cancellation of common factors, the following expression for the matrix element of a multipole operator

$$\langle J_f T_f \vdots\vdots \hat{\mathscr{T}}_{J,T}(\kappa) \vdots\vdots J_i T_i \rangle = \sum_{a,b} \langle a \vdots\vdots \mathscr{T}_{J,T}(\kappa) \vdots\vdots b \rangle \psi^{fi}_{J,T}(a,b) \qquad (19.35)$$

$$\psi^{fi}_{J,T}(a,b) = \frac{1}{\sqrt{(2J+1)(2T+1)}} \langle J_f T_f \vdots\vdots \hat{\zeta}^\dagger(ab;JT) \vdots\vdots J_i T_i \rangle$$

$$\hat{\zeta}^\dagger(ab;JM_J,TM_T) \equiv \sum_{m_{j_\alpha} m_{j_\beta}} \langle j_\alpha m_{j_\alpha} j_\beta m_{j_\beta} | j_\alpha j_\beta J M_J \rangle$$

$$\times \sum_{m_{t_\alpha} m_{t_\beta}} \langle t_\alpha m_{t_\alpha} t_\beta m_{t_\beta} | \frac{1}{2}\frac{1}{2} T M_T \rangle c_\alpha^\dagger [S_{-\beta} c_{-\beta}]$$

This is our principal result for the many-body matrix element. It has the following features:

- It is doubly reduced with respect to angular momentum and isospin;
- It expresses the many-body matrix element as a sum of single-particle matrix elements;
- It assumes a one-body current;
- It is *exact* within the traditional non-relativistic nuclear many-body problem.

Consider the isospin dependence in more detail. The previous single-particle densities, and hence the single-particle multipole operators, all have the form

$$\mathscr{T}_{JM_J} = \frac{1}{2}\mathscr{T}^{(0)}_{JM_J} + \frac{1}{2}\tau_3 \mathscr{T}^{(1)}_{JM_J}$$

$$\equiv I_{00}\, \mathscr{T}^{(0)}_{JM_J} + I_{10}\, \mathscr{T}^{(1)}_{JM_J} \qquad (19.36)$$

It follows from this definition that $\langle \frac{1}{2} || I_T || \frac{1}{2} \rangle = [(2T+1)/2]^{1/2}$ [Ed74]. The many-body multipole operators thus have the corresponding isospin structure

$$\hat{\mathscr{T}}_{JM_J} = \hat{\mathscr{T}}_{JM_J;00} + \hat{\mathscr{T}}_{JM_J;10} \qquad (19.37)$$

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In electron scattering (e, e'), as well as real photon transitions, the third component of isospin of the target cannot change, and hence $\bar{M}_f = \bar{M}_i$. Now use the Wigner–Eckart theorem on the isospin dependence of the many-particle matrix elements to obtain the doubly reduced matrix elements. The basic result in Eq. (19.35) can then be employed. The isospin dependence of the single-particle matrix elements in Eq. (19.36) *factors*, and thus the doubly reduced single-particle matrix elements are particularly simple. It follows that the many-body matrix elements that enter into the cross sections and rates must take the form

$$\langle \Psi_f || \hat{\mathscr{T}}_J(\kappa) || \Psi_i \rangle = \langle J_f ; T_f \bar{M}_i || \hat{\mathscr{T}}_J(\kappa) || J_i ; T_i \bar{M}_i \rangle \tag{19.38}$$
$$= (-1)^{T_f - \bar{M}_i} \begin{pmatrix} T_f & 0 & T_i \\ -\bar{M}_i & 0 & \bar{M}_i \end{pmatrix} \sum_{a,b} \frac{1}{\sqrt{2}} \langle a || \mathscr{T}_J^{(0)} || b \rangle \psi^{fi}_{J,0}(ab)$$
$$+ (-1)^{T_f - \bar{M}_i} \begin{pmatrix} T_f & 1 & T_i \\ -\bar{M}_i & 0 & \bar{M}_i \end{pmatrix} \sum_{a,b} \sqrt{\frac{3}{2}} \langle a || \mathscr{T}_J^{(1)} || b \rangle \psi^{fi}_{J,1}(ab)$$

This is the basic multipole matrix element for the transition $|J_i ; T_i \bar{M}_i\rangle \to |J_f ; T_f \bar{M}_i\rangle$. It relates the many-body reduced matrix element to a sum of single-particle reduced matrix elements; the isospin dependence of the transition multipoles has now been explicitly exhibited. The many-body physics is in the numerical coefficients $\psi^{fi}_{J,T}(ab)$. Again, this is a *general result* within the current framework.

Once one has a set of coefficients $\psi^{fi}_{J,T}(ab)$ from the many-body analysis (several examples are discussed in chapter 20), the problem is reduced to computation of the single-particle reduced matrix elements of the multipole operators [Wi63]. The tables in [Do79, Do80] are a substantial aid here since all the angular momentum algebra of computing the reduced matrix element of a tensor product in a coupled basis [Ed74] has already been carried out. There are two sets of tables. The first [Do79] is in a harmonic oscillator single-particle basis where the wave functions can be written in analytic form [Fe71, Wa95]. In this case, the required radial integrals can all be done analytically in terms of hypergeometric functions [de66]. The result is of the form $\exp(-y) \times$ polynomial in y where

$$y \equiv \left(\frac{\kappa b_{\text{osc}}}{2}\right)^2$$
$$\hbar\omega_{\text{osc}} = \frac{\hbar^2}{m b_{\text{osc}}^2} \tag{19.39}$$

In the second set of tables [Do80], the calculation is carried out for arbitrary radial wave functions up to the point where a final radial integral must be done numerically.

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If one inserts the single-particle densities from Eqs. (19.1, 19.4, 19.5, 19.20), then through $O(1/m)$ the single-particle multipole operators take the form

$$\begin{aligned}
M_{JM_J}(\kappa) &= M_J^{M_J}(\kappa\mathbf{x})\,\frac{1}{2}(1+\tau_3) \\
iT^{\rm mag}_{JM_J}(\kappa) &= \frac{\kappa}{m}\left\{\mathscr{M}^{M_J}_{JJ}(\kappa\mathbf{x})\cdot\frac{1}{\kappa}\nabla\,\frac{1}{2}(1+\tau_3)\right. \\
&\quad \left.-\frac{1}{2}\left[\frac{1}{i\kappa}\nabla\times(\mathscr{M}^{M_J}_{JJ})\right]\cdot\boldsymbol{\sigma}\,\left[\frac{1}{2}(\mu_p+\mu_n)+\frac{1}{2}\tau_3(\mu_p-\mu_n)\right]\right\} \\
T^{\rm el}_{JM_J}(\kappa) &= \frac{\kappa}{m}\left\{\left[\frac{1}{i\kappa}\nabla\times(\mathscr{M}^{M_J}_{JJ})\right]\cdot\frac{1}{\kappa}\nabla\,\frac{1}{2}(1+\tau_3)\right. \\
&\quad \left.+\frac{1}{2}(\mathscr{M}^{M_J}_{JJ})\cdot\boldsymbol{\sigma}\,\left[\frac{1}{2}(\mu_p+\mu_n)+\frac{1}{2}\tau_3(\mu_p-\mu_n)\right]\right\} \qquad (19.40)
\end{aligned}$$

Here

$$\begin{aligned}
M_J^{M_J}(\kappa\mathbf{x}) &\equiv j_J(\kappa x)Y_{JM_J}(\Omega_x) \\
\mathscr{M}^{M_J}_{JJ} &\equiv j_J(\kappa x)\mathscr{Y}^{M_J}_{JJ1}(\Omega_x) \qquad (19.41)
\end{aligned}$$

Note the isospin dependence is now explicit and one can read off $\mathscr{T}^{(0)}_{JM_J}$ and $\mathscr{T}^{(1)}_{JM_J}$ in Eq. (19.36). Furthermore, it is no longer necessary to symmetrize the convection current since $\nabla\cdot\mathscr{M}^{M_J}_{JJ}=\nabla\cdot[\nabla\times\mathscr{M}^{M_J}_{JJ}]=0$.

A notation which identifies the various pieces of the multipole operators in Eqs. (19.40) is introduced in [Do79]

$$\begin{aligned}
M_{JM_J}(\kappa) &= \frac{1}{2}(1+\tau_3)\,M_J^{M_J}(\kappa\mathbf{x}) \qquad (19.42) \\
iT^{\rm mag}_{JM_J}(\kappa) &\equiv \frac{\kappa}{m}\left\{\frac{1}{2}(1+\tau_3)\,\Delta_J^{M_J}\right. \\
&\quad \left.+\left[\frac{1}{2}(\mu_p+\mu_n)+\frac{1}{2}\tau_3(\mu_p-\mu_n)\right]\left(-\frac{1}{2}\right)\,\Sigma_J^{\prime M_J}\right\} \\
T^{\rm el}_{JM_J}(\kappa) &\equiv \frac{\kappa}{m}\left\{\frac{1}{2}(1+\tau_3)\,\Delta_J^{\prime M_J}\right. \\
&\quad \left.+\left[\frac{1}{2}(\mu_p+\mu_n)+\frac{1}{2}\tau_3(\mu_p-\mu_n)\right]\left(\frac{1}{2}\right)\,\Sigma_J^{M_J}\right\}
\end{aligned}$$

The quantities $(\Delta,\Delta',\Sigma,\Sigma')$ follow from comparison with Eqs. (19.40). One can now directly employ the tables in [Do79, Do80].

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