Probability in the Engineering and Informational Sciences (2023), **37**:1 206–213
doi:10.1017/S0269964821000541

CAMBRIDGE
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RESEARCH ARTICLE

Extended generator and associated martingales for M/G/1 retrial queue with classical retrial policy and general retrial times

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Keywords: Extended generator, Martingale, Piecewise deterministic Markov process, Retrial queue

Abstract

A retrial queue with classical retrial policy, where each blocked customer in the orbit retries for service, and general retrial times is modeled by a piecewise deterministic Markov process (PDMP). From the extended generator of the PDMP of the retrial queue, we derive the associated martingales. These results are used to derive the conditional expected number of customers in the orbit in the transient regime.

1. Introduction

Retrial queueing models are specified by the following feature: an external arriving customer which finds all the servers busy joins a virtual waiting area, called orbit, instead of leaving the system and becomes a blocked customer. The blocked customers in the orbit are allowed to retry their luck to capture the service area after a random amount of time. Time intervals between these attempts are called retrial times or repeated times. Queueing systems with repeated times are regarded as a special part of queueing theory, which has a wide field of possible applications such as computer and communication networks. Telephone exchanges in a call center are a well-known application of retrial queues in the literature. This problem was tackled by Fayolle [14] in the case of a constant type of retrial policy where the recall rate is independent of the number of blocked customers. Fayolle [14] has analyzed the stationary distribution of the system states and the sojourn time distribution for an M/M/1 queue with a constant retrial policy. This analysis was extended to the case of an M/G/1 queue in Farahmand [13], as well as the case of a G/M/1 system in Lillo [19]. In the work of Kim *et al.* [18], the analysis of the G/M/1 system became simpler compared to Lillo [19] by using the matrix analytic technique. Alternatively, blocked customers may also attempt to get served at the same time. This refers to the classical retrial policy which states that each customer in the orbit tries to capture the server independently of the other customers. Thus, the retrial rate depends on the number of blocked customers. Retrial queues with classical retrial policy were extensively studied in the literature [11,12] .

All of the above-quoted works have supposed the exponential distribution for repeated time. Meanwhile, in our study, we consider general retrial time distribution. Research in this direction is so limited especially when dealing with the case of a blocked customer who acts independently of the other customers in the orbit. The first attempt to generalize the retrial time distribution was done by Kapyrin [16] for a classical retrial policy, but Falin [10] has shown later that his approach was incorrect.

In Kernane [17], the stability condition was determined for an M/G/1 retrial queue with general retrial times and classical retrial policy by assuming also that the process of service times is stationary and ergodic. In the references [2,20,21] , only some approximations and simulations are presented for models with phase-type retrial time. Therefore, dealing with general retrial times in the case of classical retrial policy is a challenging research problem in retrial queueing theory.

In this paper, we analyze the M/G/1 retrial queue as a piecewise deterministic Markov process (PDMP) instead of studying it using the traditional methods, such as the embedded Markov chain and the supplementary variable methods. This new approach makes it possible to study the dynamics of such a complicated model in greater depth as well as to derive the performance quantities, by means of martingales, in a transient regime which is difficult to investigate even for simple Markovian queues. In fact, PDMP modelization has been introduced to analyze several models in the literature including queueing and epidemic models [1,3,15] . However, the similarity between the dynamics of the SIR model with general infectious period distribution and those of the M/G/1 queue with general retrial times, which is clearly outstanding in Gómez-Corral and López-García [15], has most motivated us to carry out this work.

In Section 2, we recall the PDMP framework. In Section 3, we model the dynamics of the retrial queue with classical retrial policy and general retrial times by a PDMP. From the extended generator of the PDMP modeling the retrial queue, we derive the associated martingales in Section 4. In Section 5, we utilize these results to derive the conditional expected number of blocked customers in a transient regime. We end the paper with a conclusion in Section 6 giving some areas that can be studied for future works on the subject.

2. The PDMP framework

Piecewise deterministic Markov processes were introduced by Davis [7] as a general family of non-diffusion stochastic models. These Markov processes consist of a mixture of deterministic motion and random jumps. The class of PDMPs provides a framework for studying optimization problems especially in queueing systems [1,8] . As shown in Davis [7] and then extensively indicated in Davis [8], a PDMP can be explicitly determined by means of three parameters $(\mathfrak{X}, \lambda, Q)$. Let $X(t)$ be a PDMP in a state space $\mathbb{E}$ defined as follows. Let K be a countable set and $d : K \longrightarrow \mathbb{N}$ be a given function. For each $\upsilon \in K$, M_υ is an open set of $\mathbb{R}^{d(\upsilon)}$. Then, $\mathbb{E} = \bigcup_{\upsilon\in K} M_\upsilon = \{(\upsilon,\xi) : \upsilon \in K, \xi \in M_\upsilon\}$. Denote by $\mathcal{E}$ the σ-algebra generated by the Borel subsets of $\mathbb{E}$. The state of the process will be denoted by $X(t) = (\upsilon(t), \xi(t))$ and the characteristics $\mathfrak{X}$, λ and Q are defined as follows:

- $\mathfrak{X} = \{\mathfrak{X}_\upsilon, \upsilon \in K\}$ is a locally Lipschitz continuous vector fields in $\mathbb{E}$ with flow functions $\phi_\upsilon(t,\xi)$ for each $\xi \in M_\upsilon$.
- $\lambda : \mathbb{E} \longrightarrow \mathbb{R}_+$ is the jump rate. It is assumed that this function is measurable and for all $x = (\upsilon,\xi) \in \mathbb{E}$, there exists $\varepsilon > 0$ such that $\int_0^\varepsilon \lambda(\phi_\upsilon(t,\xi))\, dt$ exists.
- $Q : (\mathbb{E} \cup \partial^*\mathbb{E}) \times \mathcal{E} \longrightarrow [0,1]$, with $\partial^*\mathbb{E} = \bigcup_{\upsilon\in K} \partial^* M_\upsilon$ where $\partial^* M_\upsilon = \{\xi' \in \partial M_\upsilon : \phi_\upsilon(t,\xi) = \xi',\ \text{for all } (t,\xi) \in \mathbb{R}_+ \times M_\upsilon\}$, a transition measure specifying the post-jump locations with $Q(x;\{x\}) = 0$. Note that ∂M_υ is the boundary of M_υ and $\partial^* M_\upsilon$ represents those boundary points at which the flow exits from M_υ.

With a convenient choice of the state space $\mathbb{E}$ and parameters $\mathfrak{X}$, λ and Q it is possible to model almost all non-diffusion processes found in the literature. Several important applications were presented in Davis [7,8] . The motion of the PDMP depends on the characteristics $\mathfrak{X}$, λ and Q in the following way. Starting from a point $x = (\upsilon,\xi) \in \mathbb{E}$, we select a jump time T_1 with distribution function $P_x(T_1 > t) = \mathbf{I}_{t<t_*(x)} \exp\{-\int_0^t \lambda(\phi_\upsilon(s,\xi))\, ds\}$, where $t_*(x) = \inf\{t \in \mathbb{R}_+ : \phi_\upsilon(t,\xi) \in \partial^* M_\upsilon\}$. The deterministic variable $t_*(x)$ denotes the time until the set $\partial^*\mathbb{E}$ is reached from a state $x \in \mathbb{E}$. Now select a random variable Z_1 having distribution $Q(\phi_\upsilon(T_1,\xi);\cdot)$. Hence, the trajectory of $X(t)$ for $t \leq T_1$ is

https://doi.org/10.1017/S0269964821000541 Published online by Cambridge University Press

given by

$$X(t)=\begin{cases}(\upsilon,\phi_\upsilon(t,\xi)) & \text{for } t<T_1;\\ Z_1 & \text{for } t=T_1.\end{cases}$$

Starting from $X(T_1)$ we now select the next inter-jump time T_2-T_1 and the post-jump location $X(T_2)$ in a similar way. This gives a piecewise deterministic trajectory $(X(t))$ with jump times $T_1,T_2,\dots$ and post-jump locations $Z_1,Z_2,\dots$. The sequence of random variables $\{Z_n\}$ is the Markov chain associated with the original process $X(t)$, so that previous results on the well-known discrete-time Markov chains were used to investigate the stability and ergodicity of PDMPs, see [4,5,9] . Furthermore, it is assumed that there are only finite many jumps of $X(t)$ in any finite time interval (see assumption 3.1 in Davis [7] or assumption 24.4 in Davis [8]). Thus, if Z_0 is the initial point then the associated Markov chain $\{Z_n,n\geq 0\}$ has the transition measure $\mathcal{P}:\mathbb{E}\cup\partial^*\mathbb{E}\times\mathcal{E}\longrightarrow[0,1]$ given by:

$$\begin{aligned}\mathcal{P}(x;A)=&\int_0^{t_*(x)}Q(\phi_\upsilon(s,\xi);A)\lambda(\phi_\upsilon(s,\xi))\exp(-\Lambda(s,x))\,ds\\&+Q(\phi_\upsilon(t_*(x),\xi);A)\exp(-\Lambda(t_*(x),x)).\end{aligned}$$

where $\Lambda(t,x)=\int_0^t\lambda(\phi_\upsilon(s,\xi))\,ds$ for all $x\in\mathbb{E}$ and $0\leq t\leq t_*(x)$.

Note that $\Lambda(t_*(x),x)=\infty$ whenever $t_*(x)=\infty$.

3. M/G/1 queue with general retrial times as a PDMP

The model considered is an M/G/1 retrial queue. Customers arrive to the system according to a Poisson process with rate λ. Each incoming customer that finds the server busy joins the orbit. Service times are i.i.d. with the same general distribution function F. After a random time in the orbit, each blocked customer repeats his attempt to enter service. Retrial times are i.i.d. with the same general distribution function G. Inter-arrivals, service periods and retrial times are assumed to be mutually independent. The model studied can be represented as a PDMP in the following way. Let $(X(t)=(\upsilon(t),\xi(t));t\in\mathbb{R}_+)$, where $\upsilon(t)=(C(t),N(t))$ and $\xi(t)=(Y(t),\mathbf{R}(t))$, be the PDMP describing the system state at time t. The component $C(t)$ represents the state of the server ($C(t)$ equals 1 or 0 according as the server is busy or free), $N(t)$ is the number of the blocked customers, $Y(t)$ is the residual service time and $\mathbf{R}(t)=(R_1(t),R_2(t),\dots,R_{N(t)}(t))$ where $R_k(t)$ denotes the remaining retrial time of the kth blocked customer for $1\leq k\leq N(t)$. The considered process is defined on the state space $\mathbb{E}=(\mathbb{E}_0\cup\partial^*\mathbb{E}_0)\cup(\mathbb{E}_1\cup\partial^*\mathbb{E}_1)$, where $\mathbb{E}_0=\{(0,n,0,r_1,r_2,\dots,r_n),\ n\in\mathbb{N},\ 0<r_1<r_2<\cdots<r_n\}$, $\partial^*\mathbb{E}_0=\{(0,n,0,0,r_2,\dots,r_n),\ n\in\mathbb{N}^*,\ 0<r_2<\cdots<r_n\}$, $\mathbb{E}_1=\{(1,n,y,r_1,r_2,\dots,r_n),\ n\in\mathbb{N},\ y>0,\ 0<r_1<r_2<\cdots<r_n\}$ and $\partial^*\mathbb{E}_1=\{(1,n,0,r_1,r_2,\dots,r_n),\ n\in\mathbb{N},\ 0<r_1<r_2<\cdots<r_n\}$. This particular formulation of the state space $\mathbb{E}$ allows us to specify the transition measure $Q(x;\cdot)$ of $X(t)$ effortlessly and one can see that it remains suitable to our case where all blocked customers retry to capture the service simultaneously.

Define the flow function ϕ:

$$\phi(t,x)=\begin{cases}(0,n,0,r_1-t,r_2-t,\dots,r_n-t) & \text{for } x\in\mathbb{E}_0;\\ (1,n,y-t,r_1-t,r_2-t,\dots,r_n-t) & \text{for } x\in\mathbb{E}_1.\end{cases}$$

We also give some notations for further use. Let $\mathcal{B}_{(a,b)}$ be the σ-algebra of Borel sets on the interval (a,b) and β_n be the Borel σ-algebra on the set $F^{(n)}=\{(u_1,\dots,u_n)\in(0,\infty)^n:u_1<\cdots<u_n\}$. We define the following function as well for $A\in\beta_n$

$$\mathbf{1}_A(u_1,\dots,u_n)=\begin{cases}1 & \text{if } (u_1,\dots,u_n)\in A,\\ 0 & \text{otherwise.}\end{cases}$$

https://doi.org/10.1017/S0269964821000541 Published online by Cambridge University Press

In the same spirit of Gómez-Corral and López-García [15] and Breuer[1], in order to describe the jumps that can occur more clearly, we are going to introduce three transition measures Q_1, Q_2 and Q_3 which reflect the transitions associated with the arrival process, the service achievement and the removal of a blocked customer from the orbit, respectively.

For states $x \in \mathbb{E}_0$, the transition measure $Q_1(x;\cdot)$ is given by

$$Q_1(x;\{1\}\times\{n\}\times A\times B) = F(A)\mathbf{1}_B(r_1,\ldots,r_n),$$

where $A \in \mathcal{B}_{(0,\infty)}$ and $B \in \beta_n$.

The transition measure $Q_1(x;\cdot)$ captures the transition from state $(0,n,0,r_1,\ldots,r_n)$ to state $(1,n,y,r_1,\ldots,r_n)$ related to a new external incoming customer to the queueing system that joins the idle server immediately.

For states $x \in \partial^*\mathbb{E}_0$, $A \in \mathcal{B}_{(0,\infty)}$ and $B \in \beta_{n-1}$, the transition measure $Q_3(x;\cdot)$ is given by

$$Q_3(x;\{1\}\times\{n-1\}\times A\times B) = F(A)\mathbf{1}_B(r_2,\ldots,r_n).$$

This refers to the transition from state $(0,n,0,0,r_2,\ldots,r_n)$ to state $(1,n-1,y,r'_1,\ldots,r'_{n-1})$ occurring when a blocked customer joins the idle server.

In a similar manner, we determine the transition measures $Q_1(x;\cdot)$ and $Q_2(x;\cdot)$ for states $x \in \mathbb{E}_1$ and $x \in \partial^*\mathbb{E}_1$, respectively. The transition measure $Q_1(x;\cdot)$ is specified as follows:

(i) For $A \in \mathcal{B}_{(0,\infty)}$, $B \in \mathcal{B}_{(0,r_1)}$ and sets $C \in \beta_n$,

$$Q_1(x;\{1\}\times\{n+1\}\times A\times B\times C) = \mathbf{1}_A(y)G(B)\mathbf{1}_C(r_1,\ldots,r_n).$$

(ii) For $A \in \mathcal{B}_{(0,\infty)}$, sets $B \in \beta_k$, $C \in \mathcal{B}_{(r_k,r_{k+1})}$ and sets $D \in \beta_{n-k}$,

$$Q_1(x;\{1\}\times\{n+1\}\times A\times B\times C\times D) = \mathbf{1}_A(y)\mathbf{1}_B(r_1,\ldots,r_k)G(C)\mathbf{1}_D(r_{k+1},\ldots,r_n).$$

(iii) $A \in \mathcal{B}_{(0,\infty)}$, sets $B \in \beta_n$ and $C \in \mathcal{B}_{(r_n,\infty)}$,

$$Q_1(x;\{1\}\times\{n+1\}\times A\times B\times C) = \mathbf{1}_A(y)\mathbf{1}_B(r_1,\ldots,r_n)G(C).$$

The transition measure $Q_1(x;\cdot)$ captures the transition from state $(1,n,y,r_1,\ldots,r_n)$ to state $(1,n+1,y,r'_1,\ldots,r'_n,r'_{n+1})$ resulting when an incoming customer finds the server busy, therefore it joins the orbit. Thus, it becomes a blocked customer and a new remaining retrial time generated from G must be added to the vector $(r_1,\ldots,r_n)$ in its convenient place yielding to a new vector of remaining retrial times $(r'_1,\ldots,r'_n,r'_{n+1})$ with $r'_1 < \cdots < r'_n < r'_{n+1}$.

Finally, the transition measure $Q_2(x;\cdot)$ associated with the transition from state $(1,n,0,r_1,\ldots,r_n)$ to state $(0,n,0,r_1,\ldots,r_n)$, when the server becomes idle, is given by

$$Q_2(x;\{0\}\times\{n\}\times\{0\}\times A) = \mathbf{1}_A(r_1,\ldots,r_n),$$

where $A \in \beta_n$.

Note that, for our process, the jump rate is exactly the arrival rate λ. Hence, we have $\Lambda(t,x) = \int_0^t \lambda(\phi(s,x))\,ds = \lambda t$.

4. The associated martingales

In this section, we will derive the martingales associated with the PDMP that models our retrial queue using its extended generator.

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Theorem 1. *For $0 \le z_1 \le 1$, $0 \le z_2 \le 1$, $\gamma \ge 0$ and $\delta \ge 0$, the function*

$$z_1^{C(t)} z_2^{N(t)} e^{-\gamma Y(t)} e^{-\delta \sum_{k=1}^{N(t)} R_k(t)} e^{\theta_{C(t)}(t)} \tag{4.1}$$

with

$$\theta_{C(t)}(t) = \begin{cases} (\lambda - N(t)\delta)t - \ln\{\lambda z_2 \varphi_R(\delta) + \gamma\} & \\ \quad + \ln\{\lambda z_1 \varphi_S(\delta)[e^{-(\lambda z_2 \varphi_R(\delta)+\gamma)t} - 1] + \lambda z_2 \varphi_R(\delta) + \gamma\} & \textit{for } C(t) = 0; \\ (\lambda - \lambda z_2 \varphi_R(\delta) - N(t)\delta - \gamma)t & \textit{for } C(t) = 1. \end{cases} \tag{4.2}$$

is a martingale for states $x \in \mathbb{E}_{C(t)}$, where

$$\varphi_S(\gamma) = \int_0^\infty e^{-\gamma y}\, dF(y) \quad \text{and} \quad \varphi_R(\delta) = \int_0^\infty e^{-\delta r}\, dG(r).$$

Proof. The infinitesimal generator of the process $X(t)$, acting on a function $f(i, n, y, r_1, \ldots, r_n, t) \in \mathfrak{D}(\mathcal{G})$, is given by

$$\begin{aligned} &\mathcal{G} f(0, n, 0, r_1, r_2, \ldots, r_n, t) \\ &\quad = \lambda \int_0^\infty [f(1, n, y, r_1, r_2, \ldots, r_n, t) - f(0, n, 0, r_1, r_2, \ldots, r_n, t)]\, dF(y) \\ &\qquad - \sum_{k=1}^{n} \frac{\partial}{\partial r_k} f(0, n, 0, r_1, r_2, \ldots, r_n, t) + \frac{\partial}{\partial t} f(0, n, 0, r_1, r_2, \ldots, r_n, t). \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\mathcal{G} f(1, n, y, r_1, r_2, \ldots, r_n, t) \\ &\quad = \lambda \int_0^\infty [f(1, n+1, y, r_1, r_2, \ldots, r_n, r_{n+1}, t) - f(1, n, y, r_1, r_2, \ldots, r_n, t)]\, dG(r_{n+1}) \\ &\qquad - \frac{\partial}{\partial y} f(1, n, y, r_1, r_2, \ldots, r_n, t) - \sum_{k=1}^{n} \frac{\partial}{\partial r_k} f(1, n, y, r_1, r_2, \ldots, r_n, t) \\ &\qquad + \frac{\partial}{\partial t} f(1, n, y, r_1, r_2, \ldots, r_n, t). \end{aligned} \tag{4.4}$$

where $\mathfrak{D}(\mathcal{G})$ is the domain of the generator $\mathcal{G}$ which consists of those functions $f(i, n, y, r_1, \ldots, r_n, t)$ that are differentiable with respect to $y, r_1, \ldots, r_n$ and t for all $i, n, y, r_1, \ldots, r_n, t$, and satisfy the boundary conditions derived from Eq. (5.4) in Davis [7]

$$f(0, n, 0, 0, r_2, \ldots, r_n, t) = \int_0^\infty f(1, n-1, y, r'_1, \ldots, r'_{n-1}, t)\, dF(y); \tag{4.5}$$

$$f(1, n, 0, r_1, \ldots, r_n, t) = f(0, n, 0, r_1, \ldots, r_n, t); \tag{4.6}$$

where $(r'_1, \ldots, r'_{n-1}) = (r_2, \ldots, r_n)$ and verify the integrability conditions

$$E\left\{\left|\int_0^\infty f(1, n, y, r_1, \ldots, r_n, t)\, dF(y) - f(0, n, 0, r_1, \ldots, r_n, t)\right|\right\} < \infty; \tag{4.7}$$

$$E\left\{\left|\int_0^\infty f(1, n+1, y, r_1, \ldots, r_n, r_{n+1}, t)\, dG(r_{n+1}) - f(1, n, y, r_1, \ldots, r_n, t)\right|\right\} < \infty. \tag{4.8}$$

Define now the function

$$f(i,n,y,r_1,r_2,\ldots,r_n,t) = z_1^i z_2^n\, e^{-\gamma y}\, e^{-\delta \sum_{k=1}^{n} r_k}\, e^{\theta_i(t)}, \tag{4.9}$$

where $\theta_i(t)$ is given by Eq. (4.2) for $C(t) = i,\ i \in \{0,1\}$.

By substituting Eq. (4.9) in Eq. (4.3) for $i = 0$, we get

$$\begin{aligned}
\mathcal{G} f(0,n,0,r_1,r_2,\ldots,r_n,t) &= \lambda \int_0^\infty [z_1\, e^{-\gamma y}\, e^{\theta_1(t)-\theta_0(t)} - 1]\, dF(y) + \sum_{k=1}^{n} \delta + \theta_0'(t) \\
&= \lambda z_1\, e^{\theta_1(t)-\theta_0(t)} \varphi_S(\gamma) - \lambda + n\delta + \theta_0'(t) \\
&= \lambda z_1 \frac{(\lambda z_2 \varphi_R(\delta) + \gamma)\, e^{-(\lambda z_2 \varphi_R(\delta)+\gamma)}}{\lambda z_1 \varphi_S(\gamma)[e^{-(\lambda z_2 \varphi_R(\delta)+\gamma)} - 1] + \lambda z_2 \varphi_R(\delta) + \gamma} \varphi_S(\gamma) \\
&\quad - \lambda + n\delta + \lambda - n\delta \\
&\quad - \lambda z_1 \varphi_S(\gamma) \frac{(\lambda z_2 \varphi_R(\delta) + \gamma)\, e^{-(\lambda z_2 \varphi_R(\delta)+\gamma)}}{\lambda z_1 \varphi_S(\gamma)[e^{-(\lambda z_2 \varphi_R(\delta)+\gamma)} - 1] + \lambda z_2 \varphi_R(\delta) + \gamma} \\
&= 0.
\end{aligned}$$

Similarly, we substitute Eq. (4.9) in Eq. (4.4) for $i = 1$.

$$\begin{aligned}
\mathcal{G} f(1,n,y,r_1,r_2,\ldots,r_n,t) &= \lambda \int_0^\infty [z_2\, e^{-\delta r_{n+1} - 1}]\, dG(r_{n+1}) + \gamma + n\delta + \theta_1'(t) \\
&= \lambda z_2 \varphi_R(\delta) - \lambda + \gamma + n\delta + \theta_1'(t) \\
&= \lambda z_2 \varphi_R(\delta) - \lambda + \gamma + n\delta + \lambda - \lambda z_2 \varphi_R(\delta) - n\delta - \gamma \\
&= 0.
\end{aligned}$$

Hence, by the property of the infinitesimal generator, Eq. (4.1) is a martingale for the process $X(t)$. □

Theorem 2. *Let τ_0 and τ_1 be the stopping times that end the server inactivity period (when the process stays in $\mathbb{E}_0$) and the server occupation period (when the process stays in $\mathbb{E}_1$), respectively. The processes*

$$\begin{aligned}
&f(C(t),N(t),Y(t),\mathbf{R}(t),t) - f(0,N(0),0,\mathbf{R}(0),0) \\
&\quad - \int_0^t \mathcal{G} f(0,N(s),0,\mathbf{R}(s),s)\, ds \quad \text{for } t \in [0,\tau_0]
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&f(C(t),N(t),Y(t),\mathbf{R}(t),t) - f(1,N(0),Y(0),\mathbf{R}(0),0) \\
&\quad - \int_0^t \mathcal{G} f(1,N(s),Y(s),\mathbf{R}(s),s)\, ds \quad \text{for } t \in [0,\tau_1]
\end{aligned} \tag{4.11}$$

are martingales for any function $f \in \mathfrak{D}(\mathcal{G})$.

Proof. Define the following process

$$M^f(t) = f(X(t)) - f(X(0)) - \int_0^t \mathcal{G} f(X(s))\, ds$$

where $\mathcal{G}$ is the infinitesimal generator of the PDMP $X(t)$ and f is a measurable function satisfying (i)–(iii) in Theorem 5.5 of Davis [7]. Hence, according to Proposition 14.13 in Davis [8], $M^f(t)$ is a martingale.

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Using the generators Eqs. (4.3) and (4.4), the result follows immediately. □

Note that according to the treatment of our model as a PDMP, stopping times τ_0 and τ_1 are given by the random variables $\min(A(t), R_1(t))$ and $Y(t)$, respectively. The component $A(t)$ refers to the inter-arrival time which is exponentially distributed with parameter λ.

5. Mean number of customers in the orbit

In this section, we will derive the expected number of customers blocked in the orbit during the periods of inactivity and occupation of the server separately. To this end, we will mainly use the result of Theorem 2. In fact, authors in Dassios and Zhao [6] have shown that this method based on martingales allows to calculate any moment of $N(t)$ without taking into account the stability condition (see Theorems 3.6 and 3.8).

Theorem 3. *The conditional expectation of the number of blocked customers $N(t)$ given $N(0) = n_0$ and $Y(0) = y_0$ (when $Y(t) \in \mathbb{E}_1 \cup \partial^*\mathbb{E}_1$) is given by:*

$$E[N(t)|N(0) = n_0] = \begin{cases} n_0 + \lambda t & \text{for } t \in [0, \tau_0]; \\ n_0 + \lambda t + \dfrac{1}{\mu_1}(y_0 - t) - 1 & \text{for } t \in [0, \tau_1]. \end{cases}$$

where $\mu_1 = \int_0^\infty y \, dF(y)$.

Proof. By setting $f(i, n, y, r_1, \ldots, r_n, t) = y + n\mu_1$ and verifying the conditions Eqs. (4.5)–(4.8), we have

$$\mathcal{G} f(0, n, 0, r_1, r_2, \ldots, r_n, t) = \lambda\mu_1 \quad \text{and} \quad \mathcal{G} f(1, n, y, r_1, r_2, \ldots, r_n, t) = \lambda\mu_1 - 1.$$

According to Theorem 2, $Y(t) + N(t)\mu_1 - n_0\mu_1 - \int_0^t \lambda\mu_1 \, ds$ and $Y(t) + N(t)\mu_1 - y_0 - n_0\mu_1 - \int_0^t (\lambda\mu_1 - 1)\, ds$ are martingales. Hence, for $t \in [0, \tau_0]$

$$E[N(t)|N(0) = n_0] = \frac{1}{\mu_1} E[Y(t)] + \lambda t + n_0 \tag{5.1}$$

and for $t \in [0, \tau_1]$

$$E[N(t)|N(0) = n_0, Y(0) = y_0] = n_0 + \lambda t + \frac{1}{\mu_1}(y_0 - t - E[Y(t)]). \tag{5.2}$$

Besides, we have

$$E[Y(t)] = \begin{cases} 0 & \text{for } t \in [0, \tau_0]; \\ \mu_1 & \text{for } t \in [0, \tau_1]. \end{cases}$$

By substituting $E[Y(t)]$ in Eqs. (5.1) and (5.2), the result follows. □

6. Conclusion

The M/G/1 retrial queue with classical retrial policy, where each blocked customer in the orbit retries for service, and general retrial times has been modeled by aPDMP. Using the extended generator of the PDMP, we have derived the associated martingales capturing the dynamics of the retrial queue. These results have been exploited to find the conditional expected number of customers in the orbit in a transient regime. The approach of modeling with PDMPs can be applied to other retrial policies such as the constant retrial policy and the control policy. In further works, we will investigate the stationary regime of the considered model through the PDMP framework.

Acknowledgments. The second author acknowledges the financial support of the Directorate General for Scientific Research and Technological Development (DGRSDT) of Algeria for the Research Laboratory in Intelligent Informatics, Mathematics and Applications (RIIMA).

Competing interests. The authors declare no conflict of interest.

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Cite this article: Meziani S and Kernane T (2023). Extended generator and associated martingales for M/G/1 retrial queue with classical retrial policy and general retrial times. *Probability in the Engineering and Informational Sciences* **37**, 206–213. https://doi.org/10.1017/S0269964821000541