BULL. AUSTRAL. MATH. SOC. 16A32, 16A38, 16A51
VOL. 37 (1988) [161–164]

RINGS IN WHICH EVERY ELEMENT IS THE SUM OF TWO IDEMPOTENTS

YASUYUKI HIRANO AND HISAO TOMINAGA

Let R be a ring with prime radical P. The main theorems of this paper are (1) The following conditions are equivalent: 1) R is a commutative ring in which every element is the sum of two idempotents; 2) R is a ring in which every element is the sum of two commuting idempotents; 3) R satisfies the identity $x^3 = x$. (2) If R is a PI-ring in which every element is the sum of two idempotents, then R/P satisfies the identity $x^3 = x$. (3) Let R be a semi-perfect ring in which every element is the sum of two idempotents. If ${}_RR_R$ is quasi-projective, then R is a finite direct sum of copies of $GF(2)$ and/or $GF(3)$.

Throughout, R will represent a ring with prime radical P. A Boolean ring is defined as a ring in which every element is an idempotent. As a generalisation of Boolean rings, we consider the class of rings in which every element is the sum of two idempotents. We begin with an example which shows that such a ring need not be Boolean or even commutative.

Example. Let $A(\neq 0)$ and B be Boolean rings, and $W(\neq 0)$ a B-A-bimodule. Assume, furthermore, that W is s-unital as a right A-module, that is, for any w in W, there exists an element e in A such that $we = w$. Then every element of the non-commutative ring $R = \begin{pmatrix} A & 0 \\ W & B \end{pmatrix}$ is the sum of two idempotents. In fact, $\begin{pmatrix} a & 0 \\ w & b \end{pmatrix} = \begin{pmatrix} e & 0 \\ w & 0 \end{pmatrix} + \begin{pmatrix} a-e & 0 \\ 0 & b \end{pmatrix}$, where e is an element of A such that $we = w$.

Our present objective is to prove the following theorems.

THEOREM 1. *The following conditions are equivalent:*

1) *R is a commutative ring in which every element is the sum of two idempotents.*
2) *R is a ring in which every element is the sum of two commuting idempotents.*
3) *R satisfies the identity $x^3 = x$.*

Received 20 March 1987
Dedicated to Professor Adil Yaqub on his 60th birthday

THEOREM 2. *Let R be a PI-ring in which every element is the sum of two idempotents. Then R/P satisfies the identity $x^3 = x$.*

THEOREM 3. *Let R be a semi-perfect ring in which every element is the sum of two idempotents. If ${}_RR_R$ is quasi-projective, then R is a finite direct sum of copies of $GF(2)$ and/or $GF(3)$.*

In preparation for proving our theorems, we state four lemmas.

LEMMA 1. *Let $R(\neq 0)$ be a ring in which every element is the sum of two idempotents. If R contains no non-trivial idempotents, then R is either $GF(2)$ or $GF(3)$.*

PROOF: Since 0 and 1 are the only idempotents of R, we have either $R = \{0, 1\}$ or $R = \{0, 1, 2\}$. Thus R is either $GF(2)$ or $GF(3)$. ∎

LEMMA 2. *Let a be an element of R with $a^2 = 0$.*

(1) *If $a = e + f$ for idempotents e, f then $4a = 0$.*

(2) *If $a = e + f$ for commuting idempotents e, f then $a = 2e$ and $4e = 0$.*

PROOF: (1) Obviously,

$$\begin{aligned} 0 &= a^3 - 2a^2 \\ &= a + 2(ef + fe) + efe + fef - 2(a + ef + fe) \\ &= efe + fef - a. \end{aligned}$$

Hence $0 = ea^2e + fa^2f = a + 3(efe + fef) = 4a$.

(2) Since $0 = a^2 = a + 2ef$, we get $a = -2ef$, and so $0 = a(f - e) = f - e$. Hence $a = 2e$ and $4e = a^2 = 0$. ∎

LEMMA 3. *Let R be a ring with 1, and n a positive integer greater than 1. Then the $n \times n$ full matrix ring $M_n(R)$ over R contains an element which cannot be written as the sum of two idempotents.*

PROOF: We write $M_n(R) = \sum_{i,j=1}^{n} Re_{ij}$, where e_{ij} are matrix units. Suppose, to the contrary, that every element of $M_n(R)$ is the sum of two idempotents. Then, by Lemma 2(1), $4e_{12} = 0$ and so $4R = 0$. Consider the element $a = e_{11} + e_{12} + e_{21}$, and choose idempotents $e = \sum r_{ij}e_{ij}$ and f such that $a = e + f$. Since $a - e = f = f^2 = a^2 - ae - ea + e$, we get $a^2 = a + ae + ea - 2e$. Comparing the coefficients of e_{11}, e_{12} and e_{21} on both sides, we get $1 = r_{12} + r_{21}$, $0 = r_{11} + r_{22} - r_{12}$ and $0 = r_{11} + r_{22} - r_{21}$, and therefore $1 = 2r_{12}$. Then $4R = 0$ implies that $1 = 4r_{12}^2 = 0$, which is a contradiction. ∎

https://doi.org/10.1017/S000497270002668X Published online by Cambridge University Press

LEMMA 4. *Let R be a prime ring in which every element is the sum of two idempotents. If $R \neq Z$, the centre of R, then* char $R = 2$ *and Z is either 0 or $GF(2)$.*

PROOF: First, we claim that R cannot be reduced. Actually, if R is reduced, then every idempotent is central, and so $R = Z$ by hypothesis, a contradiction. Hence R has a non-zero element a with $a^2 = 0$. By Lemma 2 (1), we conclude that char $R = 2$. Now, let z be an arbitrary element of Z. By hypothesis, we can write $z = e + f$ for idempotents e, f in R. Then it is easily observed that $ef = fe$. Since char $R = 2$, we obtain that $z^2 = e + f + 2ef = e + f = z$. Since R is prime, this implies that z is either 0 or 1. This completes the proof. ∎

PROOF OF THEOREM 1: 1) $\Longrightarrow$ 3). It is well-known that R is a subdirect sum of subdirectly irreducible rings R_λ. Since, by Lemma 1, each R_λ is either $GF(2)$ or $GF(3)$, R satisfies the identity $x^3 = x$.

3) $\Longrightarrow$ 2). As is well-known, R is a commutative ring. Replacing x by $2x$ in $x^3 = x$, we obtain $6x = 0$. Further, replacing x by $x^2 - x$ in $x^3 = x$, we obtain $3x^2 = 3x$. By making use of these, we see easily that $\left(-2x^2\right)^2 = 4x^4 = -2x^4 = -2x^2$ and $\left(x + 2x^2\right)^2 = x^2 + 4x^3 + 4x^4 = x^2 + 4x + 4x^2 = x + 2x^2 + 3\left(x - x^2\right) + 6x^2 = x + 2x^2$. Hence x is the sum of the idempotents $-2x^2$ and $x + 2x^2$.

2) $\Longrightarrow$ 1). Let a be an element of R with $a^2 = 0$. Then, by virtue of Lemma 2 (2), there exists an idempotent e such that $a = 2e$ and $4e = 0$. Now, $-e = f + g$ with some commuting idempotents f, g. Then $e = (-e)^2 = -e + 2fg$, so $2e = 2fg = 2efg$. Noting that $fe = ef$, we see easily that $a = 2e = 2efg = 2ef(-e - f) = -4ef = 0$. Hence R is a reduced ring. As is well-known, every idempotent of the reduced ring R is central, and so R is commutative. ∎

COROLLARY 1. *Let R be a semiprime ring. If R has the property that every element is the sum of two idempotents, then the centre Z of R has the same property.*

PROOF: Since R is semiprime, R is a subdirect sum of prime rings $R_\lambda (\lambda \in \Lambda)$. By Lemmas 1 and 4, the centre Z_λ of R_λ is 0, or $GF(2)$, or $GF(3)$. Now we may regard Z as a subring of the direct product $\prod_{\lambda \in \Lambda} Z_\lambda$. Hence Z satisfies the identify $x^3 = x$. Then, by Theorem 1, every element of Z is the sum of two idempotents in Z. ∎

PROOF OF THEOREM 2: In view of Lemma 1, it suffices to show that every prime factor ring of R is commutative. Suppose, to the contrary, that a prime factor ring R' of R is not commutative. By [**3**, Corollary 1], the ring $Q(R')$ of central quotients of R' is a full matrix ring over a division ring. Then, by Lemma 4, we have that $R' = Q(R')$. Now, Lemmas 1 and 3 force a contradiction that R' is either $GF(2)$ or $GF(3)$. ∎

COROLLARY 2. *Let R be an Azumaya Z-algebra in which every element is the* *sum of two idempotents. Then R satisfies the identity $x^3 = x$.*

PROOF: By [**1**, Lemma II.3.1], Z is a Z-direct summand of R, say $R = Z \oplus T$. Then $P = (P \cap Z) \oplus (P \cap Z)T$ by [**1**, Corollary II.3.7]. As is well-known (see, for example, [**1**, Theorem II.3.4]), R is a finitely generated Z-module, and therefore R is a PI-algebra. Hence, by Theorem 2, R/P is commutative. Then, by [**1**, Proposition II.1.11], we obtain $(P \cap Z)T = T$. Since $P \cap Z$ is a nil ideal of Z, and T is a finitely generated Z-module, we conclude that $T = 0$, and hence $R = Z$. Now, by Theorem 1, R satisfies the identity $x^3 = x$. ∎

PROOF OF THEOREM 3: By [**2**, Theorem 4.6], R is the finite direct sum of full matrix rings over local rings. Hence, by Lemmas 1 and 3, R is the finite direct sum of copies of $GF(2)$ and/or $GF(3)$. ∎

Remark. As is shown in [**5**] (see also [**4**]), the following conditions are equivalent:

1) There exists an involution * of R such that $xx^\star x = x^\star$ for all x in R;
2) R is an anti-inverse ring, that is, every element x in R has an anti-inverse $x^\star$; $xx^\star x = x^\star$ and $x^\star x x^\star = x$;
3) For each element x of R there exists $x^\star$ in R such that $x^2 x^\star = x^\star$ and $x^{\star 2} x = x$;
4) R is a (dense) subdirect sum of fields isomorphic to $GF(2)$ or $GF(3)$
5) R satisfies the identity $x^3 = x$.

REFERENCES

[**1**] F. DeMeyer and E. Ingraham, 'Separable algebras over commutative rings', in *Lecture Note In Math.* **181** (Springer-Verlag, Berlin-Heidelberg-New York, 1971).

[**2**] Y. Miyashita, 'Quasi-projective modules, perfect modules, and a theorem for modular lattices', *J. Fac. Sci. Hokkaido Univ. Ser I* **19** (1966), 86–110.

[**3**] L. Rowen, 'Some results on the center of a ring with polynomial identity', *Bull. Amer. Math. Soc.* **79** (1973), 219–223.

[**4**] B.M. Schein, 'O-rings and LA-rings', *Izv. Vysš. Učeb. Zaved. Matematika* **2(51)** (1966), 111-122. Amer. Math. Soc. Transl. (2) **96** (1970), 137–152. .

[**5**] H. Tominaga, 'On a theorem of M. Yamada', *Proc. 9th Sympos. Semigroups and Related Topics* (1985), 40–42.

Department of Mathematics
Okayama University
Okayama 700
JAPAN