COMPOSITIO MATHEMATICA

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Compositio Math. **145** (2009), 993–1000.

doi:10.1112/S0010437X09004138

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Compositio Math. 145 (2009) 993–1000
doi:10.1112/S0010437X09004138

Gorenstein rings through face rings of manifolds

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Abstract

The face ring of a homology manifold (without boundary) modulo a generic system of parameters is studied. Its socle is computed and it is verified that a particular quotient of this ring is Gorenstein. This fact is used to prove that the algebraic g-conjecture for spheres implies all enumerative consequences of its far-reaching generalization (due to Kalai) to manifolds. A special case of Kalai's conjecture is established for homology manifolds that have a codimension-two face whose link contains many vertices.

1. Introduction

In 1980 Stanley proved the necessity of McMullen's conjectured description of f-vectors of boundaries of simplicial convex polytopes [Sta80]. At about the same time, Billera and Lee demonstrated that McMullen's conditions were sufficient [BL81]. Since then, one of the central problems in the field of face numbers of simplicial complexes is the possible extension of McMullen's g-conjecture to other spheres. In its most optimistic form it states that, just as in the case of polytope boundaries, the face ring of a homology sphere modulo a generic system of parameters has a Lefschetz element. We call this the *algebraic g-conjecture*. In the mid 1990s Kalai suggested a far-reaching generalization of this conjecture to all homology manifolds [Nov98, § 7]. It is a remarkable fact that all of the enumerative consequences of Kalai's conjecture are implied by the apparently weaker algebraic g-conjecture. This follows from our main result, Theorem 1.4, that a particular quotient of the face ring of a homology manifold is Gorenstein. We also verify a special case of Kalai's conjecture when the complex has a codimension-two face whose link contains many vertices.

The main objects we will consider are Buchsbaum complexes and more specifically homology manifolds. Historically, Buchsbaum complexes were defined algebraically. Here, we adopt the following theorem of Schenzel [Sch81] as our definition. Let $\mathbf{k}$ be an infinite field of an arbitrary characteristic, let Δ be a simplicial complex, and let $\tilde{H}_i(\Delta)$ be the ith reduced simplicial homology of Δ with coefficients in $\mathbf{k}$. Recall that if τ is a face of Δ, then its link is

$$\operatorname{lk}\tau = \{\sigma \in \Delta \mid \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in \Delta\}.$$

Definition 1.1. A $(d-1)$-dimensional simplicial complex Δ is called *Buchsbaum* (over $\mathbf{k}$) if it is pure and, for every non-empty face $\tau \in \Delta$, $\tilde{H}_i(\operatorname{lk}\tau) = 0$ for all $i < d - |\tau| - 1$.

Received 6 June 2008, accepted in final form 15 January 2009.
2000 Mathematics Subject Classification 13F55, 13H10, 52B05.
Keywords: g-conjectures, socle, homology manifolds and spheres, local cohomology modules.
The first author's research was partially supported by an Alfred P. Sloan Research Fellowship and NSF grant DMS-0500748. The second author's research was partially supported by NSF grant DMS-0600502.

We say that Δ is a *homology manifold* over $\mathbf{k}$ (without boundary) if it is Buchsbaum and in addition $\tilde{H}_{d-|\tau|-1}(\operatorname{lk}\tau) \cong \mathbf{k}$ for all $\emptyset \neq \tau \in \Delta$. A $\mathbf{k}$-*homology sphere* is a complex Δ such that for all $\tau \in \Delta$, including $\tau = \emptyset$,

$$\tilde{H}_i(\operatorname{lk}\tau) \cong \begin{cases} 0 & \text{if } i < d - |\tau| - 1, \\ \mathbf{k} & \text{if } i = d - |\tau| - 1. \end{cases}$$

In particular, a $\mathbf{k}$-homology sphere is a $\mathbf{k}$-homology manifold, and a triangulation of a topological sphere (topological manifold, respectively) is a $\mathbf{k}$-homology sphere ($\mathbf{k}$-homology manifold, respectively) for any field $\mathbf{k}$.

If Δ is a simplicial complex on $[n] := \{1, 2, \ldots, n\}$, then its *face ring* (or the *Stanley–Reisner ring*) is

$$\mathbf{k}[\Delta] := \mathbf{k}[x_1, \ldots, x_n]/I_\Delta, \quad \text{where } I_\Delta = (x_{i_1}x_{i_2}\cdots x_{i_k} : \{i_1 < i_2 < \cdots < i_k\} \notin \Delta).$$

Various combinatorial and topological invariants of Δ are encoded in the algebraic invariants of $\mathbf{k}[\Delta]$ and vice versa. For instance, if Δ is a $(d-1)$-dimensional complex, then the Krull dimension of $\mathbf{k}[\Delta]$, $\dim \mathbf{k}[\Delta]$, is equal to d. In this case, a set of d linear forms $\theta_1, \ldots, \theta_d \in \mathbf{k}[\Delta]$ is called a *linear system of parameters* (l.s.o.p.) if

$$\mathbf{k}(\Delta, \Theta) := \mathbf{k}[\Delta]/(\theta_1, \ldots, \theta_d)$$

has Krull dimension zero (equivalently, $\mathbf{k}(\Delta, \Theta)$ is finite dimensional as a $\mathbf{k}$-vector space). Assuming that $\mathbf{k}$ is infinite, an l.s.o.p. always exists: a generic choice of $\theta_1, \ldots, \theta_d$ does the job. Although the algebraic structure of $\mathbf{k}(\Delta, \Theta)$ generally depends on $\theta_1, \ldots, \theta_d$, in the following we somewhat abuse notation and write $\mathbf{k}(\Delta)$ instead of $\mathbf{k}(\Delta, \Theta)$. Unless it is specifically stated that Θ is generic, our results apply to an arbitrary l.s.o.p.

One invariant that measures how far Δ is from being a homology sphere is the socle of $\mathbf{k}(\Delta)$, $\operatorname{Soc}\mathbf{k}(\Delta)$, where, for a $\mathbf{k}[x_1, \ldots, x_n]$- or $\mathbf{k}[\Delta]$-module M,

$$\operatorname{Soc} M := \{y \in M \mid x_i \cdot y = 0 \text{ for all } i = 1, \ldots, n\}.$$

When Δ is a homology sphere, $\operatorname{Soc}\mathbf{k}(\Delta)$ is a one-dimensional $\mathbf{k}$-space. Since $\mathbf{k}[\Delta]$ is a graded $\mathbf{k}$-algebra for any Δ, the ring $\mathbf{k}(\Delta)$ and its ideal $\operatorname{Soc}\mathbf{k}(\Delta)$ are graded as well. We denote by $\mathbf{k}(\Delta)_i$ and $(\operatorname{Soc}\mathbf{k}(\Delta))_i$ their ith homogeneous components. It is well known (for instance, see [Sta96, Lemma III.2.4(b)]) that for any $(d-1)$-dimensional Δ, $\mathbf{k}(\Delta)_i$, and hence also $(\operatorname{Soc}\mathbf{k}(\Delta))_i$, vanish for all $i > d$. If Δ is a Buchsbaum complex, then, for $i \leq d$, $(\operatorname{Soc}(\mathbf{k}(\Delta))_i$ can be expressed in terms of the local cohomology modules of $\mathbf{k}[\Delta]$ with respect to the irrelevant ideal, $H^j(\mathbf{k}[\Delta])$, as follows; see [NS07, Theorem 2.2].

Theorem 1.2. *Let Δ be a $(d-1)$-dimensional simplicial complex. If Δ is Buchsbaum, then, for all $0 \leq i \leq d$,*

$$(\operatorname{Soc}\mathbf{k}(\Delta))_i \cong \left(\bigoplus_{j=0}^{d-1} \binom{d}{j} H^j(\mathbf{k}[\Delta])_{i-j}\right) \bigoplus \mathcal{S}_{i-d},$$

where $\mathcal{S}$ is a graded submodule of $\operatorname{Soc} H^d(\mathbf{k}[\Delta])$ and rM denotes the direct sum of r copies of M.

While Theorem 1.2 identifies a big chunk of the socle of $\mathbf{k}(\Delta)$, its other part, $\mathcal{S}$, remains a mystery. Here, we solve this mystery in the special case of a connected *orientable* $\mathbf{k}$-homology manifold without boundary, thus verifying [NS07, Conjecture 7.2]. A connected $\mathbf{k}$-homology manifold Δ without boundary is called orientable if $\tilde{H}_{d-1}(\Delta) \cong \mathbf{k}$.

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THEOREM 1.3. *Let Δ be a $(d-1)$-dimensional connected orientable $\mathbf{k}$-homology manifold without boundary. Then $\dim_{\mathbf{k}} \mathcal{S} = \dim_{\mathbf{k}} \mathcal{S}_0 = 1$. In particular, $\dim_{\mathbf{k}} \operatorname{Soc} \mathbf{k}(\Delta)_i = \binom{d}{i}\beta_{i-1}$, where $\beta_{i-1} := \dim_{\mathbf{k}} \tilde{H}_{i-1}(\Delta)$ is the $(i-1)$th reduced Betti number of Δ.*

A graded $\mathbf{k}$-algebra of Krull dimension zero is called *Gorenstein* if its socle is a one-dimensional $\mathbf{k}$-space (see [Sta96, p. 50] for many other equivalent definitions). Let

$$I := \bigoplus_{i=0}^{d-1} \operatorname{Soc} \mathbf{k}(\Delta)_i \quad \text{and} \quad \overline{\mathbf{k}(\Delta)} := \mathbf{k}(\Delta)/I.$$

(Note that the top-dimensional component of the socle is not a part of I.) If Δ is a homology sphere, then $I = 0$ and $\overline{\mathbf{k}(\Delta)} = \mathbf{k}(\Delta)$ is a Gorenstein ring [Sta96, Theorem II.5.1]. What if Δ is a homology manifold other than a sphere? How far is $\mathbf{k}(\Delta)$ from being Gorenstein in this case? The answer (that was conjectured in [NS07, Conjecture 7.3]) turns out to be surprisingly simple.

THEOREM 1.4. *Let Δ be a $(d-1)$-dimensional connected simplicial complex. If Δ is an orientable $\mathbf{k}$-homology manifold without boundary, then $\overline{\mathbf{k}(\Delta)}$ is Gorenstein.*

In [Sch81], Schenzel computed the Hilbert function of $\mathbf{k}(\Delta)$ for a Buchsbaum complex Δ in terms of its face and Betti numbers. It follows from Theorem 1.3, Schenzel's formula, and the Dehn–Sommerville relations [Kle64] that for a connected orientable homology manifold Δ, the Hilbert function of $\overline{\mathbf{k}(\Delta)}$ is symmetric, that is,

$$\dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_i = \dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_{d-i} \quad \text{for all } 0 \le i \le d. \tag{1}$$

As the Hilbert function of a Gorenstein ring of Krull dimension zero is always symmetric [Sta96, p. 50], Theorem 1.4 gives an alternative algebraic proof of (1).

As noted above, when Δ is a $(d-1)$-dimensional homology sphere, $\mathbf{k}(\Delta)$ is a Gorenstein ring, which implies that $\dim_{\mathbf{k}} \mathbf{k}(\Delta)_i = \dim_{\mathbf{k}} \mathbf{k}(\Delta)_{d-i}$. We say that Δ satisfies the *algebraic g-conjecture* if, for sufficiently generic linear forms ω and $\Theta = \{\theta_1, \ldots, \theta_d\}$, and $i \le d/2$, the multiplication

$$\cdot\omega^{d-2i} : \mathbf{k}(\Delta)_i \longrightarrow \mathbf{k}(\Delta)_{d-i}$$

is an isomorphism. At present the only large classes of spheres for which the algebraic g-conjecture is known to hold are polytopal spheres [Sta80] and edge decomposable spheres [Mur, Nev07].

Kalai's far-reaching generalization of the algebraic g-conjecture for homology spheres [Nov98] posits that if Δ is an orientable homology manifold and ω, Θ are sufficiently generic, then

$$\cdot\omega^{d-2i} : \overline{\mathbf{k}(\Delta)}_i \longrightarrow \overline{\mathbf{k}(\Delta)}_{d-i}$$

is still an isomorphism for all $i \le d/2$. Let $h''_i = \dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_i$. Given a system of parameters and ω which satisfy Kalai's conjecture, it is immediate that the multiplication $\cdot\omega : \overline{\mathbf{k}(\Delta)}_i \to \overline{\mathbf{k}(\Delta)}_{i+1}$ is an injection for $i < d/2$. So, $h''_0 \le \cdots \le h''_{\lfloor d/2 \rfloor}$, and examination of $\overline{\mathbf{k}(\Delta)}/(\omega)$ shows that the non-negative integer vector $(h''_0, h''_1 - h''_0, \ldots, h''_{\lfloor d/2 \rfloor} - h''_{\lfloor d/2 \rfloor - 1})$ is an M-vector, i.e. satisfies Macaulay's nonlinear arithmetic conditions (see [Sta96, p. 56]) for the Hilbert series of a homogeneous quotient of a polynomial ring. Applying the same reasoning to $I_{[i]} := \bigoplus_{j=0}^{i} I_j$, which is also an ideal, and $\mathbf{k}(\Delta, i) := \mathbf{k}(\Delta)/I_{[i]}$, instead of I and $\overline{\mathbf{k}(\Delta)} = \mathbf{k}(\Delta)/I$, the second conclusion can be strengthened to $(h''_0, h''_1 - h''_0, \ldots, h''_i - h''_{i-1}, h''_{i+1} - h''_i + \binom{d}{i+1}\beta_i)$ is an M-vector for every $i < \lfloor d/2 \rfloor$. We call these arithmetic restrictions on the h''-vector the *manifold g-conjecture*. In fact, as the following theorem shows (see Theorem 3.2 for a somewhat

https://doi.org/10.1112/S0010437X09004138 Published online by Cambridge University Press

stronger version), these two conclusions follow from the algebraic g-conjecture. (Recall that the links of vertices of a homology manifold are homology spheres.)

Theorem 1.5. *Let Δ be a $(d-1)$-dimensional connected orientable homology manifold. If the links of all vertices of Δ satisfy the algebraic g-conjecture, then, for generically chosen ω and Θ, $\cdot\omega : \overline{\mathbf{k}(\Delta)}_i \longrightarrow \overline{\mathbf{k}(\Delta)}_{i+1}$ is an injection for all $i < \lfloor d/2 \rfloor$ and is a surjection for all $i \geq \lceil d/2 \rceil$.*

For Kalai's conjecture we prove this special case.

Theorem 1.6. *Let Δ be a $(d-1)$-dimensional orientable $\mathbf{k}$-homology manifold with $d \geq 3$. If Δ has a $(d-3)$-dimensional face τ whose link contains all of the vertices of Δ that are not in τ, then, for generic choices of ω and Θ, $\cdot\omega^{d-2} : \overline{\mathbf{k}(\Delta)}_1 \longrightarrow \overline{\mathbf{k}(\Delta)}_{d-1}$ is an isomorphism.*

The condition that the link of τ contains all of the vertices of Δ that are not in τ is equivalent to saying that every vertex of Δ is in the star of τ, $\operatorname{st}\tau := \{\sigma \in \Delta \mid \sigma \cup \tau \in \Delta\}$. This condition is not as restrictive on the homeomorphism type of Δ as one might think: the results of [Swa09, § 5] imply that every connected homology manifold without boundary that has a triangulation always has a triangulation Δ satisfying this condition.

The outline of the paper is as follows. We verify Theorem 1.3 in § 2. The main ingredient in the proof is Gräbe's explicit description of $H^d(\mathbf{k}[\Delta])$ as a $\mathbf{k}[\Delta]$-module in terms of the simplicial (co)homology of the links of faces of Δ and maps between them [Gra84]. Theorem 1.4 is proved in § 3 and is used to explore the relationship between the algebraic g-conjecture and Kalai's conjecture (such as Theorem 1.5). Section 4 is devoted to the proof of Theorem 1.6. The proofs of Theorems 1.4 and 1.6 rely heavily on a result from [Swa09] relating $\mathbf{k}(\operatorname{lk} v)$ (for a vertex v of Δ) to the principal ideal $(x_v) \subset \mathbf{k}(\Delta)$.

2. Socles of homology manifolds

The goal of this section is to verify Theorem 1.3. To do so we analyze $\operatorname{Soc} H^d(\mathbf{k}[\Delta])$ and prove the following.

Theorem 2.1. *If Δ is a connected orientable $(d-1)$-dimensional $\mathbf{k}$-homology manifold without boundary, then $\operatorname{Soc} H^d(\mathbf{k}[\Delta])_i = 0$ for all $i \neq 0$.*

The proof relies on results from [Gra84]. We denote by $|\Delta|$ the geometric realization of Δ. For a face $\tau \in \Delta$, let $\operatorname{cost}\tau := \{\sigma \in \Delta \mid \sigma \not\supset \tau\}$ be the contrastar of τ, let $H^i(\Delta, \operatorname{cost}\tau)$ be the simplicial ith cohomology of a pair (with coefficients in $\mathbf{k}$), and, for $\tau \subset \sigma \in \Delta$, let ι^* be the map $H^i(\Delta, \operatorname{cost}\sigma) \to H^i(\Delta, \operatorname{cost}\tau)$ induced by the inclusion $\iota : \operatorname{cost}\tau \to \operatorname{cost}\sigma$. Also, if $\emptyset \neq \tau \in \Delta$, let $\hat{\tau}$ be the barycenter of τ. Finally, for a vector $U = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, let $s(U) := \{l \mid u_l \neq 0\} \subseteq [n]$ be the support of U, let $\{e_l\}_{l=1}^n$ be the standard basis for $\mathbb{Z}^n$, and let $\mathbb{N}$ denote the set of non-negative integers.

We consider the $\mathbb{Z}^n$-grading of $\mathbf{k}[x_1, \ldots, x_n]$ obtained by declaring x_l to be of degree e_l. This grading refines the usual Z-grading and induces a $\mathbb{Z}^n$-grading of $\mathbf{k}[\Delta]$ and its local cohomology modules. Thus, $H^i(\mathbf{k}[\Delta])_j = \bigoplus H^i(\mathbf{k}[\Delta])_U$, where the sum is over all $U = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ with $\sum u_l = j$, and multiplication by x_l is a linear map from $H^i(\mathbf{k}[\Delta])_U$ to $H^i(\mathbf{k}[\Delta])_{U+e_l}$ for all $U \in \mathbb{Z}^n$.

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THEOREM 2.2 [Gra84]. *The following is an isomorphism of $\mathbb{Z}^n$-graded $\mathbf{k}[\Delta]$-modules:*

$$H^i(\mathbf{k}[\Delta]) \cong \bigoplus_{\substack{-U\in\mathbb{N}^n \\ s(U)\in\Delta}} \mathcal{M}_U^i, \quad \text{where } \mathcal{M}_U^i = H^{i-1}(\Delta, \operatorname{cost} s(U)) \tag{2}$$

and the $\mathbf{k}[\Delta]$-structure on the Uth component of the right-hand side is given by

$$\cdot x_l = \begin{cases} 0\text{-map} & \text{if } l \notin s(U), \\ \text{identity map} & \text{if } l \in s(U) \text{ and } l \in s(U+e_l), \\ \iota^* : H^{i-1}(\Delta, \operatorname{cost} s(U)) \to H^{i-1}(\Delta, \operatorname{cost} s(U+e_l)) & \text{otherwise.} \end{cases}$$

We remark that the isomorphism of (2) on the level of vector spaces (rather than $\mathbf{k}[\Delta]$-modules) is due to Hochster, see [Sta96, § II.4], and that $H^i(\mathbf{k}[\Delta])_0 \cong \mathcal{M}^i_{(0,\dots,0)} = H^{i-1}(\Delta, \emptyset)$.

Proof of Theorem 2.1. In view of Theorem 2.2, if $-U = (-u_1, \dots, -u_n) \in \mathbb{N}^n$ is such that $-u_l \geq 2$ for some $l \in [n]$, then the multiplication $\cdot x_l : H^d(\mathbf{k}[\Delta])_U \to H^d(\mathbf{k}[\Delta])_{U+e_l}$ is an isomorphism, and hence no non-zero element of $H^d(\mathbf{k}[\Delta])_U$ is in the socle. Thus (using Theorem 2.2 once again), to prove that $\operatorname{Soc} H^d(\mathbf{k}[\Delta])_i = 0$ for all $i \neq 0$, it is enough to show that for every $\emptyset \neq \tau \in \Delta$ and $l \in \tau$, the map $\iota^* : H^{d-1}(\Delta, \operatorname{cost} \tau) \to H^{d-1}(\Delta, \operatorname{cost} \sigma)$, where $\sigma = \tau - \{l\}$, is an isomorphism. Assume first that $\sigma \neq \emptyset$. Consider the following diagram.

$$\begin{array}{ccccc} H^{d-1}(|\Delta|) & \xrightarrow{(j^\star)^{-1}} & H^{d-1}(|\Delta|, |\Delta| - \hat{\sigma}) & \xrightarrow{f^\star} & H^{d-1}(\Delta, \operatorname{cost} \sigma) \\ \| & & & & \uparrow{\scriptstyle \iota^\star} \\ H^{d-1}(|\Delta|) & \xrightarrow{(j^\star)^{-1}} & H^{d-1}(|\Delta|, |\Delta| - \hat{\tau}) & \xrightarrow{f^\star} & H^{d-1}(\Delta, \operatorname{cost} \tau) \end{array}$$

The two $f^\star$ maps are induced by inclusion and are isomorphisms by the usual deformation retractions. The homomorphism $j^\star$ is also induced by inclusion. Since Δ is connected and orientable, all of the spaces are one dimensional and $j^\star$ is an isomorphism, so that $(j^\star)^{-1}$ is well defined and is an isomorphism as well. Hence, the compositions $f^\star \circ (j^\star)^{-1}$ are isomorphisms. The naturality of $j^\star$ implies that the diagram is commutative. It follows that $\iota^\star$ is an isomorphism. If $\sigma = \emptyset$, replace $H^{d-1}(|\Delta|, |\Delta| - \hat{\sigma})$ with $H^{d-1}(|\Delta|)$ in the above diagram. The same reasoning applies. □

We are now in a position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Since $\mathcal{S}_i$ is a subspace of $\operatorname{Soc} H^d(\mathbf{k}[\Delta])_i$ (Theorem 1.2), and since the latter space is the zero space whenever $i \neq 0$ (see Theorem 2.1), it follows that $\mathcal{S}_i = 0$ for all $i \neq 0$. For $i = 0$, we have

$$\dim_{\mathbf{k}} \mathcal{S}_0 = \dim_{\mathbf{k}} \operatorname{Soc} \mathbf{k}(\Delta)_d = \dim_{\mathbf{k}} \mathbf{k}(\Delta)_d = \beta_{d-1}(\Delta) = 1.$$

Here, the first step is by Theorem 1.2, the second step is a consequence of $\mathbf{k}(\Delta)_d$ being the last non-vanishing homogeneous component of $\mathbf{k}(\Delta)$, and the third step is by Schenzel's formula [Sch81] (see also [Sta96, Theorem II.8.2]). The 'In particular' part then follows from Theorem 1.2, isomorphism (2), and the standard fact that $H^{i-1}(\Delta, \operatorname{cost} \tau) \cong \tilde{H}_{i-|\tau|-1}(\operatorname{lk} \tau)$, see e.g. [Gra84, Lemma 1.3]. □

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3. Gorenstein property

In this section we verify Theorem 1.4 and use it to discuss connections between various g-conjectures. To prove that $\overline{\mathbf{k}(\Delta)} = \mathbf{k}(\Delta)/I$ is Gorenstein, where Δ is a $(d-1)$-dimensional connected orientable homology manifold and $I = \bigoplus_{i=0}^{d-1} \operatorname{Soc} \mathbf{k}(\Delta)_i$, we have to check that the operation of modding out by I does not introduce new socle elements. This turns out to be a simple application of [Swa09, Proposition 4.24], which we review now.

Let $\Theta = \{\theta_1, \dots, \theta_d\}$ be an l.s.o.p. for $\mathbf{k}[\Delta]$ and let v be a vertex of Δ. Fix a facet $\tau = \{v = v_1, v_2, \dots, v_d\}$ that contains v. By using Gaussian elimination on the $d \times n$ matrix whose (i,j)th entry is the coefficient of x_j in θ_i, we can assume without loss of generality that $\theta_i = x_{v_i} + \sum_{j \notin \tau} \theta_{i,j} x_j$. Denote by θ_i' the linear form obtained from θ_i by removing all summands involving x_j for $\{j\} \notin \operatorname{lk} v$. Then $\Theta' := \{\theta_2', \dots, \theta_d'\}$ can be considered as a subset of $\mathbf{k}[\operatorname{lk} v]_1$. Moreover, it is easy to check, say, using [Sta96, Lemma III.2.4(a)], that Θ' forms an l.s.o.p. for $\mathbf{k}[\operatorname{lk} v]$. The ring $\mathbf{k}(\operatorname{lk} v) := \mathbf{k}[\operatorname{lk} v]/(\Theta')$ has a natural $\mathbf{k}[x_1, \dots, \hat{x_v}, \dots, x_n]$-module structure (if $j \neq v$ is not in the link of v, then multiplication by x_j is the zero map), and defining

$$x_v \cdot y := -\theta_1' \cdot y \quad \text{for } y \in \mathbf{k}(\operatorname{lk} v)$$

extends it to a $\mathbf{k}[x_1, \dots, x_n]$-module structure. Swartz [Swa09, Proposition 4.24] asserts the following.

Theorem 3.1. *Let Δ be an orientable homology manifold. The map*

$$\phi : \mathbf{k}[\operatorname{lk} v]/(\Theta') \to (x_v)\,(\mathbf{k}[\Delta]/(\Theta)) \quad \textit{given by } z \mapsto x_v \cdot z$$

is well defined and is an isomorphism (of degree one) of $\mathbf{k}[x_1, \dots, x_n]$-modules. Its inverse, $x_v \cdot z \mapsto \overline{z}$, is given by replacing each occurrence of x_v in z with $-\theta_1'$ and setting all x_j for $j \neq v$ not in the link of v to zero.

Proof of Theorem 1.4. To prove the theorem, it is enough to show that the socle of $\overline{\mathbf{k}(\Delta)} = \mathbf{k}(\Delta)/I$, where $I = \bigoplus_{j=0}^{d-1} \operatorname{Soc} \mathbf{k}(\Delta)_j$, vanishes in all degrees $j \neq d$. This is clear for $j = d-1$. For $j \leq d-2$, consider any element $y \in \mathbf{k}(\Delta)_j$ such that $x_v \cdot y \in \operatorname{Soc} \mathbf{k}(\Delta)$ for all $v \in [n]$. We have to check that $y \in \operatorname{Soc} \mathbf{k}(\Delta)$. And, indeed, the isomorphism of Theorem 3.1 implies that $\overline{y} := \phi^{-1}(x_v \cdot y) \in \mathbf{k}(\operatorname{lk} v)_j$ is in the socle of $\mathbf{k}(\operatorname{lk} v)$. Since $\operatorname{lk} v$ is a $(d-2)$-dimensional homology sphere, $\mathbf{k}(\operatorname{lk} v)$ is Gorenstein, and hence its socle vanishes in all degrees except the $(d-1)$st one. Therefore, $\overline{y} = 0$, and hence $x_v \cdot y = \phi(\overline{y}) = 0$ in $\mathbf{k}(\Delta)$. Since this happens for all $v \in [n]$, it follows that $y \in \operatorname{Soc} \mathbf{k}(\Delta)$. □

We now turn to the connection between the various g-conjectures. Evidently, Kalai's conjecture implies the algebraic g-conjecture and the manifold g-conjecture. Since homology spheres are homology manifolds, an immediate consequence of the manifold g-conjecture is that all homology spheres satisfy the enumerative restrictions of McMullen's g-conjecture for polytopes. The surprise is that the algebraic g-conjecture implies the manifold g-conjecture. In fact, as the following strengthening of Theorem 1.5 shows, only the algebraic g-conjecture in the middle dimension is needed.

Theorem 3.2. *Let Δ be a $(d-1)$-dimensional connected orientable homology manifold. If, for at least $(n-d)$ of the vertices v of Δ and generically chosen ω and Θ' in $\mathbf{k}[\operatorname{lk} v]_1$, the map $\cdot\omega : \mathbf{k}(\operatorname{lk} v)_{\lfloor (d-1)/2 \rfloor} \to \mathbf{k}(\operatorname{lk} v)_{\lfloor (d-1)/2 \rfloor + 1}$ is surjective, then Δ satisfies the manifold g-conjecture.*

Proof. The condition on the links implies by [Swa09, Theorem 4.26] that for generic choices of ω and Θ in $\mathbf{k}[\Delta]_1$, the map $\cdot\omega : \mathbf{k}(\Delta)_{\lceil d/2 \rceil} \to \mathbf{k}(\Delta)_{\lceil d/2 \rceil + 1}$ is surjective. Hence, the map

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$\cdot\omega : \overline{\mathbf{k}(\Delta)}_{\lceil d/2 \rceil} \to \overline{\mathbf{k}(\Delta)}_{\lceil d/2 \rceil + 1}$ is surjective. Thus, $\overline{\mathbf{k}(\Delta)}/(\omega)_i$ vanishes for $i = \lceil d/2 \rceil + 1$, and hence also for all $i > \lceil d/2 \rceil + 1$, and we infer that $\cdot\omega : \overline{\mathbf{k}(\Delta)}_i \to \overline{\mathbf{k}(\Delta)}_{i+1}$ is surjective for $i \geq \lceil d/2 \rceil$. This in turn yields that the dual map $\cdot\omega : \operatorname{Hom}_{\mathbf{k}}(\overline{\mathbf{k}(\Delta)}_{i+1}, \mathbf{k}) \to \operatorname{Hom}_{\mathbf{k}}(\overline{\mathbf{k}(\Delta)}_i, \mathbf{k})$ is injective for all $i \geq \lceil d/2 \rceil$. Since by Theorem 1.4 $\overline{\mathbf{k}(\Delta)}$ is Gorenstein, $\operatorname{Hom}_{\mathbf{k}}(\overline{\mathbf{k}(\Delta)}_i, \mathbf{k})$ is naturally isomorphic to $\overline{\mathbf{k}(\Delta)}_{d-i}$ (see [Sta96, Theorems I.12.5 and I.12.10]). Therefore, $\cdot\omega : \overline{\mathbf{k}(\Delta)}_j \to \overline{\mathbf{k}(\Delta)}_{j+1}$ is injective for $j < \lfloor d/2 \rfloor$. □

Theorem 3.2 combined with Stanley's g-theorem for polytopes [Sta80], implies that every $(d-1)$-dimensional connected orientable $\mathbb{Q}$-homology manifold all of whose vertex links are polytopal spheres satisfies the manifold g-conjecture.

4. A special case of the g-theorem

In this section we prove Theorem 1.6. As in the proof of Theorem 1.4, we will rely on Theorem 3.1 and the notation introduced there. Since the set of all (ω, Θ) for which $\cdot\omega^{d-2} : \overline{\mathbf{k}(\Delta)}_1 \to \overline{\mathbf{k}(\Delta)}_{d-1}$ is an isomorphism is a Zariski open set (see [Swa09, § 4]), it is enough to find one ω that does the job. Surprisingly, the ω we find is 'very non-generic': as we will see, $w = x_v$, where $v \in \tau$ does the job.

Proof of Theorem 1.6. We use induction on d starting with $d = 3$. In this case, the face τ is simply a vertex, say v. Let $\sigma = \{v = v_1, v_2, v_3\}$ be a facet containing v. Let Θ be a generic l.s.o.p. for $\mathbf{k}[\Delta]$ and, as in § 3, assume that $\theta_i = x_{v_i} + \sum_{j \notin \sigma} \theta_{i,j} x_j$. Since $\dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_1 = \dim_{\mathbf{k}} \overline{\mathbf{k}(\Delta)}_2$ (see (1)) and since $\dim_{\mathbf{k}} \operatorname{Soc} \mathbf{k}(\Delta)_1 = 0$ (e.g. by Theorem 1.3), we will be done if we check that the map $\cdot x_v : \mathbf{k}(\Delta)_1 \to \overline{\mathbf{k}(\Delta)}_2$ is injective. And, indeed, if $x_v \cdot z \in \operatorname{Soc} \mathbf{k}(\Delta)_2$ for some $z \in \mathbf{k}(\Delta)_1$, then, by the isomorphism of Theorem 3.1, $\overline{z} = \phi^{-1}(x_v \cdot z)$ is in $\operatorname{Soc} \mathbf{k}(\operatorname{lk} v)_1$. But, the latter space is the zero space, so $\overline{z} = 0$ in $\mathbf{k}(\operatorname{lk} v)_1$, that is, $\overline{z} \in \operatorname{Span}(\theta'_2, \theta'_3)$. Since $\operatorname{lk} v$ contains all the vertices of Δ except v, it follows that $\theta'_2 = \theta_2$, $\theta'_3 = \theta_3$, and $z - \overline{z}$ is a multiple of θ_1, and hence that $z = 0$ in $\mathbf{k}(\Delta)$.

The proof of the induction step goes along the same lines: let $\tau = \{v_1, \ldots, v_{d-2}\}$, let $\sigma = \tau \cup \{v_{d-1}, v_d\}$ be any facet containing τ, and let Θ be a generic l.s.o.p. for $\mathbf{k}[\Delta]$. We show that, for $v \in \tau$, say $v = v_1$, the map $\cdot x_v^{d-2} : \mathbf{k}(\Delta)_1 \to \overline{\mathbf{k}(\Delta)}_{d-1}$ is an injection. If $x_v^{d-2} \cdot z \in \operatorname{Soc} \mathbf{k}(\Delta)_{d-1}$ for some $z \in \mathbf{k}(\Delta)_1$, then, by Theorem 3.1,

$$x_v^{d-3} \cdot \overline{z} = (-\theta'_1)^{d-3} \cdot \overline{z} \in \operatorname{Soc} \mathbf{k}(\operatorname{lk} v)_{d-2}.$$

However, $\operatorname{lk} v$ is a $(d-2)$-dimensional homology sphere satisfying the same assumptions as Δ: the star of the face $\tau - \{v\}$ contains all the vertices of $\operatorname{lk} v$. Hence, by the inductive hypothesis applied to $\operatorname{lk} v$ with $\omega = -\theta'_1$ and $\Theta' = (\theta'_2, \ldots, \theta'_d)$, the map $\cdot(-\theta'_1)^{d-3} : \mathbf{k}(\operatorname{lk} v)_1 \to \overline{\mathbf{k}(\operatorname{lk} v)}_{d-2}$ is an injection. Thus, $\overline{z} = 0$ in $\mathbf{k}(\operatorname{lk} v)_1$, and so $z = 0$ in $\mathbf{k}(\Delta)_1$. □

Remark. Barnette's lower bound theorem [Kal87] asserts that for every d and n, the f-vector of any $(d-1)$-dimensional connected triangulated manifold with n vertices is minimized componentwise by the f-vector of (the boundary of) a stacked d-polytope with n vertices, $S(d, n)$. There is a conjectural algebraic strengthening of this result based on Kalai's notion of *algebraic shifting*, an operation that was introduced in the mid 1980s as a tool for studying f-numbers of simplicial complexes (see e.g. [BK88] and a survey paper [Kal02]). This conjecture posits that the algebraic shifting of every connected orientable $(d-1)$-dimensional manifold on n vertices contains as a subset $\Delta(S(d, n))$: the algebraic shifting of $S(d, n)$. The complex $\Delta(S(d, n))$ was

https://doi.org/10.1112/S0010437X09004138 Published online by Cambridge University Press

recently computed by Nevo [Nev07, Example 2.18] and independently by Murai [Mur07]. Using their result, together with standard facts on rev-lex generic initial ideals [Eis95, Proposition 15.12] and an algebraic definition of Buchsbaum complexes [Sch81, Definition and Theorem 3.1(ii)], it is easy to show that in the case of symmetric algebraic shifting, a $(d-1)$-dimensional connected orientable $\mathbb{Q}$-homology manifold Δ satisfies the above conjecture if and only if for generically chosen $\theta_1, \ldots, \theta_d$ and ω, the map $\cdot\omega^{d-2} : \overline{\mathbf{k}(\Delta)}_1 \to \overline{\mathbf{k}(\Delta)}_{d-1}$ is a $\mathbb{Q}$-isomorphism. In particular, it follows that this conjecture holds for every $(d-1)$-dimensional orientable $\mathbb{Q}$-homology manifold satisfying the assumptions of Theorem 1.6.

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