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Noncommutative Symmetric Bessel Functions

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Abstract. The consideration of tensor products of 0-Hecke algebra modules leads to natural analogs of the Bessel J-functions in the algebra of noncommutative symmetric functions. This provides a simple explanation of various combinatorial properties of Bessel functions.

1 Introduction

It is known that the theory of noncommutative symmetric functions and quasi-symmetric functions is related to 0-Hecke algebras in the same way as ordinary symmetric functions are related to symmetric groups. Thus, one may expect that natural questions about representations of 0-Hecke algebras lead to the introduction of interesting families of noncommutative symmetric functions. By interesting, one may mean "noncommutative analogs" of the Frobenius characteristics of representations of symmetric groups based on combinatorial objects, which may themselves give back various identities for the ordinary, exponential, or q-exponential generating functions of these objects. This amounts to specializing the complete noncommutative symmetric functions $S_n(A)$ to $h_n(X)$, 1, $\frac{1}{n!}$ or $\frac{1}{(q)_n}$, respectively.

Examples of this situation can be found in [18], where the analysis of the representation of $H_n(0)$ on parking functions leads naturally to the combinatorics of the noncommutative Lagrange inversion formula, and to the introduction of noncommutative analogs of various special functions, such as the Abel polynomials, the Lambert binomial series or the Eisenstein exponential, and allows one to recover in a straightforward and unified way a number of enumerative formulas.

The present paper addresses the following question. The 0-Hecke algebra is the algebra of a monoid, hence admits a natural coproduct for which the monoid elements are grouplike. This allows one to define the tensor product of 0-Hecke modules, which induces on quasi-symmetric functions an analog of the internal product of symmetric functions. What are the properties of this operation, and of the dual coproduct on noncommutative symmetric functions?

It turns out that the second part of the question is the most interesting. Basically, the answer is: *the dual coproduct governs the combinatorics of Bessel functions*. Indeed, making this more explicit leads to the introduction of noncommutative analogs $\mathbf{J}_n(A, B)$ of the J-functions of integer index, of which a few basic properties are readily established. Then the above mentioned specializations (and other more complicated ones) give back various classical enumerative formulas.

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2 Background

2.1 Notations

Our notations for noncommutative symmetric functions are as in [12, 14]. The Hopf algebra of noncommutative symmetric functions is denoted by **Sym**, or by **Sym**(A) if we consider the realization in terms of an auxiliary alphabet. Bases of the homogeneous component $\mathbf{Sym}_n$ are labelled by compositions $I = (i_1, \dots, i_r)$ of n. The noncommutative complete and elementary functions are denoted by S_n and Λ_n, and the notation S^I means $S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by R_I. The notation $I \vDash n$ means that I is a composition of n. The conjugate composition is denoted by $I^\sim$, the mirror image composition by $\bar{I}$. The *descent set* of I is $\mathrm{Des}(I) = \{i_1, i_1 + i_2, \dots, i_1 + \cdots + i_{r-1}\}$.

The graded dual of **Sym** is *QSym* (quasi-symmetric functions). The dual basis of (S^I) is (M_I) (monomial), and that of (R_I) is (F_I).

We denote by **FQSym** the algebra of free quasi-symmetric functions [8].

The *Hecke algebra* $H_n(q)$ ($q \in \mathbb{C}$) associated with the symmetric group $\mathfrak{S}_n$ is the $\mathbb{C}$-algebra generated by $n-1$ elements $T_1, \dots, T_{n-1}$ satisfying the braid relations and $(T_i - 1)(T_i + q) = 0$. We are interested in the case $q = 0$, whose representation theory can be described in terms of quasi-symmetric functions and noncommutative symmetric functions [8, 15].

The Hopf structures on **Sym** and *QSym* allow one to extend the λ-ring notation of ordinary symmetric functions (see [14, 16] for background on the original commutative version). If A and X are totally ordered sets of noncommuting and commuting variables respectively, the noncommutative symmetric functions of XA are defined by

$$\sigma_t(XA) = \sum_{n \geq 0} t^n S_n(XA) = \prod_{x \in X}^{\rightarrow} \sigma_{tx}(A) = \sum_I t^{|I|} M_I(X) S^I(A)\,.$$

Thanks to the commutative image homomorphism $\mathbf{Sym} \to Sym$, noncommutative symmetric functions can be evaluated on any element x of a λ-ring, $S_n(x)$ being $S^n(x)$, the n-th symmetric power. Recall that x is said to be *of rank one* (resp. *binomial*) if $\sigma_t(x) = (1 - tx)^{-1}$ (resp. $\sigma_t(x) = (1-t)^{-x}$). The scalar $x = 1$ is the only element having both properties. We usually consider that our auxiliary variable t is of rank one, so that $\sigma_t(A) = \sigma_1(tA)$.

The argument A of the noncommutative symmetric functions can be a *virtual alphabet*. This means that being algebraically independent, the S_n can be specialized to any sequence $\alpha_n \in \mathcal{A}$ of elements of any associative algebra $\mathcal{A}$. Writing $\alpha_n = S_n(A)$ defines all the symmetric functions of A, and allows one to use the powerful notations $F(nA)$, $F((1-q)A)$, etc., for more or less complicated transformations of the specialized functions.

The (commutative) specializations $A = \mathbb{E}$, defined by $S_n(\mathbb{E}) = \frac{1}{n!}$ and $A = \frac{1}{1-q}$, for which

$$S_n\Big(\frac{1}{1-q}\Big) = \frac{1}{(q)_n} = \prod_{i=1}^{n} \frac{1}{1-q^i}$$

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are of special importance.

2.2 Noncommutative Analogs of Special Functions

Since the discovery by D. André of a combinatorial interpretation of tangent and secant numbers, several classical generating functions have been lifted to the algebra of symmetric functions, and more recently, to noncommutative symmetric functions. The general idea is as follows. Given the exponential generating function

$$f(t) = \sum_{n\geq 0} c_n \frac{t^n}{n!}$$

of a combinatorial sequence $c_n \in \mathbb{N}$, one looks for a noncommutative symmetric function $F(A)$ such that $F(t\mathbb{E}) = f(t)$. The noncommutative analog is interesting when $F_n(A)$ can be directly interpreted as the formal sum of the combinatorial objects counted by c_n under the embedding of **Sym** into some larger algebra. For example, in the case of tangent and secant numbers, the series

$$\Big(\sum_{n\geq 0}(-1)^n S_{2n}(A)\Big)^{-1}\Big(1+\sum_{n\geq 0}(-1)^n S_{2n+1}(A)\Big)$$

becomes the formal sum of the alternating permutations (shapes (2^n) and (2^n1)) under the embedding $S_n \mapsto \mathbf{F}_{12\ldots n}$ of **Sym** in **FQSym** [12]. One can also find the noncommutative Eulerian polynomials [12], and analogs of the Abel polynomials and of the Lambert and Eisenstein functions [18].

In general, F_n turns out to be the characteristic of some projective 0-Hecke module. Projective modules are always specializations of generic modules, thus also representations of the symmetric group, whose Frobenius characteristics are then the commutative images $F_n(X)$. In general, setting $X = \frac{t}{1-q}$ gives back an interesting q-analog of $f(t)$.

In this note, we shall show that the consideration of 0-Hecke modules obtained from a natural notion of tensor products leads immediately to noncommutative analogs of the Bessel J (or I) functions. Here, we need two alphabets A and B, and we are led to the combinatorics of bi-exponential generating functions.

3 Tensor Products of 0-Hecke Modules

3.1 The 0-Hecke Algebra as a Monoid Algebra

The 0-Hecke algebra $H_n(0)$ is the algebra $\mathbb{C}[\Pi_n]$, where the monoid Π_n is generated by elements $\pi_1,\ldots,\pi_{n-1}$ ($\pi_i = -T_i$) satisfying the braid relations

$$\pi_i\pi_j = \pi_j\pi_i \quad |i-j|>1$$

$$\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$$

and the idempotency condition $\pi_i^2 = \pi_i$. This algebra has 2^{n-1} simple modules S_I, which are one-dimensional and labelled by compositions I of n. The action of $H_n(0)$

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on $S_I = \mathbb{C}\cdot v$ is given by

$$\pi_i v = \begin{cases} v & \text{if } i \in \mathrm{Des}(I), \\ 0 & \text{otherwise.} \end{cases}$$

There is a canonical coproduct on $H_n(0)$ defined by $\delta_\wedge \pi = \pi \otimes \pi$ for $\pi \in \Pi_n$. Hence, tensor products of $H_n(0)$-modules can be defined, and it is obvious from the definition of the simple module S_I that, for H, K, I compositions of n, $S_H \otimes S_K = S_I$ where $\mathrm{Des}(I) = \mathrm{Des}(H) \cap \mathrm{Des}(K)$. This induces an internal product $\wedge$ on $QSym_n = G_0(H_n(0))$, similar to the internal product of symmetric functions, such that

$$F_H \wedge F_K = F_I$$

where $I = H \wedge K$, that is, $\mathrm{Des}(I) = \mathrm{Des}(H) \cap \mathrm{Des}(K)$. By duality, this defines a coproduct on $\mathbf{Sym}_n$, given by

$$\gamma_\wedge R_I = \sum_{\mathrm{Des}(I)=\mathrm{Des}(H)\cap\mathrm{Des}(K)} R_H \otimes R_K\,.$$

3.2 Another Interpretation of the Coproduct

There is a canonical involution ι on $H_n(0)$, defined by $\iota(\pi_i) = \bar{\pi}_i = 1 - \pi_i$, so that we can regard $H_n(0)$ as $\mathbb{C}[\bar{\Pi}_n]$ as well. Hence, we have another tensor product, defined from the coproduct $\delta_\vee \bar{\pi} = \bar{\pi} \otimes \bar{\pi}$ for $\pi \in \Pi_n$, which induces a second internal product $\vee$ on $QSym$, $F_H \vee F_K = F_I$ where $I = H \vee K$, that is, $\mathrm{Des}(I) = \mathrm{Des}(H) \cup \mathrm{Des}(K)$. It is of course sufficient to study one of them. Indeed, it can be shown that the involution ι maps the simple module S_I and the indecomposable projective module P_I to $S_{\bar{I}^\sim}$ and $P_{\bar{I}^\sim}$, respectively.

However, it is interesting to observe that this second product appears in another guise in [8], in the process of calculating a basis of primitive elements of **FQSym**. Let us recall this construction. For a composition $I = (i_1, \dots, i_r)$, we denote by p_I the endomorphism of **FQSym** defined by $p_I(\mathbf{F}_\sigma) = \mathbf{F}_{\alpha_1}\mathbf{F}_{\alpha_2}\cdots\mathbf{F}_{\alpha_r}$, where $\alpha_j = \mathrm{Std}(u_j)$, and $\sigma = u_1 u_2 \cdots u_r$ is the factorization of the word σ into r factors of respective lengths $|u_j| = i_j$. This is the special case $q = 0$ of the q-convolution defined in [8]: $p_I = p_{i_1} \odot_0 \cdots \odot_0 p_{i_r}$. It is proved in [8] that the p_I are mutually commuting projectors, and more precisely that

$$p_I \circ p_J = \begin{cases} 0 & \text{if } |I| \neq |J|. \\ p_{I\vee J} & \text{otherwise.} \end{cases}$$

Hence, $j\colon F_I \mapsto p_I$ defines an embedding of $(QSym, \vee)$ in the composition algebra of graded endomorphisms of **FQSym**. Moreover,

$$\pi = \sum_{|I|\geq 1} (-1)^{l(I)-1} p_I,$$

which is a projector onto the primitive Lie algebra of **FQSym**, is the image of the primitive element $\sum_n M_n$ of $QSym$ under j, and it is easy to see that more generally, for any $f \in QSym$ $(j \otimes j)(\Delta_{QSym} f) = \Delta_{\mathbf{FQSym}} \circ j(f)$. However, j does not map the usual (external) product of $QSym$ to the ordinary convolution of endomorphisms. It is nevertheless interesting to pull back the 0-convolution to $QSym$, by defining

$$F_I \odot_0 F_J = F_{I \cdot J},$$

where $I \cdot J$ means as usual concatenation of the compositions. Then we have a splitting formula

$$(f_1 \odot_0 f_2 \odot_0 \cdots \odot_0 f_r) \vee g = \mu_0[(f_1 \otimes \cdots \otimes f_r) \vee_r \Delta^r_{QSym}(g)]$$

analogous to the one satisfied in **Sym**.

3.3 The Main Result

Identifying as usual a tensor product $F \otimes G$ with $F(A)G(B)$, where A and B are two mutually commuting alphabets, we have

$$\sigma_1(XA) \wedge \sigma_1(XB) = \sum_K F_K(X) \gamma_\wedge(R_K) = \gamma_\wedge \sigma_1(XA),$$

which may be compared with the following identity relating the internal product $*$ of **Sym** and its dual coproduct $\delta F = F(XY)$ on $QSym$:

$$\sigma_1(XA) * \sigma_1(YA) = \sigma_1(XYA) = \delta\sigma_1(XA).$$

Theorem 3.1 *The coproduct $\gamma_\wedge$ is a morphism for the ordinary (outer) product of noncommutative symmetric functions, that is, for $F, G \in$ **Sym**, $\gamma_\wedge(FG) = \gamma_\wedge(F)\gamma_\wedge(G)$. In particular, it is completely determined by the images of the elementary functions, $\gamma_\wedge \Lambda_n = \Lambda_n \otimes \Lambda_n$, which implies the combinatorial inversion formula*

$$\Big(\sum_{n \geq 0} (-1)^n \Lambda_n \otimes \Lambda_n\Big)^{-1} = \sum_{\mathrm{Des}(H) \cap \mathrm{Des}(K) = \varnothing} R_H \otimes R_K.$$

Proof This is equivalent to Theorem 4.1 below. ∎

As we will see, this simple identity has many interesting enumerative corollaries. Applying the involution ω on the second factor gives the inverse of

$$\Big(\sum_{n \geq 0} (-1)^n \Lambda_n \otimes S_n\Big)^{-1} = \sum_{\mathrm{Des}(H) \cap \mathrm{Des}(K) = \varnothing} R_H \otimes R_{K^\sim}.$$

The right-hand side of this equality occurs in [13], where it is interpreted as the decomposition of a certain algebra $\mathrm{H}\mathfrak{S}_n$ as a bimodule over itself. The inverse of the left-hand side legitimately can be considered as a noncommutative analog of the

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Bessel function J_0, as when we specialize both sides to $x\mathbb{E}$, we recover $J_0(2x)$. Moreover, specializing A to $x/(1-q)$ gives a classical q-analog of J_0, and the other ones are obtained by simple transformations. This first step being granted, it is not difficult to guess the correct definition of the noncommutative analogues of the other J_ν. This will be done in the forthcoming section.

4 Noncommutative Bessel Functions

Let A and B be two mutually commuting alphabets. Recall that $\sigma_z = \sum S_n z^n$ and $\lambda_z = \sum \Lambda_n z^n = 1/\sigma_{-z}$. The noncommutative Bessel functions $\mathbf{J}_n(A,B)$ are defined by the generating series $\sum_{n\in\mathbb{Z}} z^n \mathbf{J}_n(A,B) = \lambda_{-1/z}(A)\sigma_z(B)$, that is,

$$\mathbf{J}_n(A,B) = \sum_{m\geq 0}(-1)^m \Lambda_{m-n}(A) S_m(B).$$

For $A = B = x\mathbb{E}$, this is the usual Bessel function $J_n(2x)$. In particular,

$$\mathbf{J}_0(A,B) = \sum_{m\geq 0}(-1)^m \Lambda_m(A) S_m(B)$$

can be regarded as $\lambda_{-1}(\mathbb{J})$, for the virtual alphabet $\mathbb{J} = (A,B)$ such that

$$\Lambda_n(\mathbb{J}) = \Lambda_n(A) S_n(B).$$

This defines an embedding of algebras

$$\begin{aligned} j\colon \mathbf{Sym} &\longrightarrow \mathbf{Sym}(A,B) = \mathbf{Sym}\otimes\mathbf{Sym} \\ \Lambda_n(A) &\mapsto \Lambda_n(\mathbb{J}) = \Lambda_n \otimes S_n. \end{aligned}$$

It is not difficult to describe the image of the ribbon basis under this embedding. We need the following piece of notation. For two compositions I and J of the same integer n, we define the composition $K = I\backslash J$ of n by the condition

$$\mathrm{Des}(K) = \mathrm{Des}(I)\setminus \mathrm{Des}(J) \quad \text{(set difference)}.$$

Then we can state the following.

Theorem 4.1 *The image of R_K by j is $R_K(\mathbb{J}) = \sum_{I\backslash J=K} R_I(A)R_J(B)$.*

Proof The formula is true for $K = (1^n)$ by definition. The general case follows by induction on $l(K^\sim)$, the number of columns of the ribbon diagram of K. Indeed, it suffices to prove that $R_K(\mathbb{J})R_{1^m}(\mathbb{J}) = R_{K\cdot 1^m}(\mathbb{J}) + R_{K\triangleright 1^m}(\mathbb{J})$, where as usual, for $K = (k_1,\dots,k_r)$ and $L = (l_1,\dots,l_s)$, $K\cdot L = (k_1,\dots,k_r,l_1,\dots,l_s)$ and $K\triangleright L = (k_1,\dots,k_{r-1},k_r+l_1,l2,\dots,l_s)$. This follows easily from the usual multiplication rule of ribbon functions. ∎

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Corollary 4.2 ([3]) *Let a_n be defined by*

$$\frac{1}{J_0(2\sqrt{t})} = \sum_{n\geq 0} a_n \frac{t^n}{(n!)^2}.$$

Then a_n is equal to the number of pairs of permutations $(\sigma,\tau) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $\mathrm{Des}(\sigma) \subseteq \mathrm{Des}(\tau)$.

Indeed, $\lambda_{-1}(A)^{-1} = \sigma_1(A)$ and it is well known that $n!R_I(\mathbb{E})$ is the number of permutations with descent composition I.

Let $\overleftarrow{\partial}$ be the linear operator on **Sym** (acting on the right) defined by

$$S^{(i_1,\dots,i_r)}\overleftarrow{\partial} = S^{(i_1,\dots,i_r-1)}. \tag{4.1}$$

It has the following properties (see [14, Proposition 9.1]): $(FG)\overleftarrow{\partial} = F\cdot(G\overleftarrow{\partial})+(F\overleftarrow{\partial})\cdot G_0$, where G_0 denotes the constant term of G, and

$$R_I\overleftarrow{\partial} = \begin{cases} R_{i_1,\dots,i_r-1} & \text{if } i_r > 1,\\ 0 & \text{if } i_r = 1.\end{cases}$$

In particular, if $G_0 = 0$, $(1-G)^{-1}\overleftarrow{\partial} = (1-G)^{-1}(G\overleftarrow{\partial})$. Let us apply this with $\overleftarrow{\partial} = \overleftarrow{\partial}_B$ acting only on $\mathbf{Sym}(B)$ to

$$\mathbf{J}_0(A,B)^{-1} = \Bigl(1 - \sum_{n\geq 1}(-1)^{n-1}\Lambda_n(B)S_n(A)\Bigr)^{-1} = \sum_I S^I(A)R_I(B).$$

We obtain $\mathbf{J}_0(A,B)^{-1}\mathbf{J}_{-1}(A,B) = \sum_I S^I(A)(R_I\overleftarrow{\partial}(B))$.

Corollary 4.3 ([3]) *The coefficient c_n in*

$$\frac{J_1(2x)}{J_0(2x)} = \sum_{n\geq 1} c_n \frac{x^{2n-1}}{(n-1)!n!}$$

is equal to the number of pairs of permutations $(\alpha,\beta) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(\beta)$ *and* $\beta(n) = n$.

4.1 Bessel-Carlitz Functors

In the same way as series of symmetric functions with coefficients in $\mathbb{N}$ on the Schur basis correspond to polynomial or analytic functors (see [17]), series of noncommutative symmetric functions with coefficients in $\mathbb{N}$ on the ribbon basis correspond to functors (actually, on the category of filtered vector spaces), and yield in particular analytic functors in the usual sense if one forgets about filtrations. As we shall see, the functors associated to noncommutative Bessel functions produce quadratic algebras from pairs of vector spaces.

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Let $\mathbf{F}$ be the functor which associates with a pair of vector spaces (V, W) the graded subalgebra of the exterior algebra $\Lambda(V \oplus W)$

$$\mathbf{F}(V, W) = \bigoplus_{n \geq 0} \Lambda_n(V) \otimes \Lambda_n(W).$$

This is a quadratic algebra (see [19]). If (v_i), (w_j) are bases of V and of W, the relations are as follows. For $i < k$ and $j < l$,

$$\begin{bmatrix} i\,k \\ j\,l \end{bmatrix} + \begin{bmatrix} k\,i \\ j\,l \end{bmatrix} = 0, \quad \begin{bmatrix} i\,k \\ j\,l \end{bmatrix} + \begin{bmatrix} i\,k \\ l\,j \end{bmatrix} = 0, \quad \begin{bmatrix} i\,i \\ j\,l \end{bmatrix} = 0, \quad \begin{bmatrix} i\,k \\ j\,j \end{bmatrix} = 0,$$

where $\begin{bmatrix} i \\ k \end{bmatrix} = v_i \otimes w_k$.

Hence, the Koszul dual $\mathbf{G}(V, W) = \mathbf{F}(V, W)^!$ is the quadratic algebra on $V^* \otimes W^*$ presented by

$$\begin{bmatrix} i\,k \\ j\,l \end{bmatrix} = \begin{bmatrix} k\,i \\ j\,l \end{bmatrix} = \begin{bmatrix} i\,k \\ l\,j \end{bmatrix} \quad \text{for } i < k \text{ and } j < l.$$

The combinatorial investigation of Bessel functions has been initiated by Carlitz [2]. Hence, the polynomial bi-functors defined by $\mathbf{F}$ and $\mathbf{G}$ can appropriately be called Bessel–Carlitz functors. One or two occurrences of Λ can be replaced by S in the definition of $\mathbf{F}$. In the mixed case $\Lambda \otimes S$, the best interpretation is probably as functors defined on super (*i.e.,* $\mathbb{Z}_2$-graded) vector spaces $V = V_0 \oplus V_1$. It is likely that these functors will play a role in the representation theory of degenerate versions quantum supergroups extending the quantum groups investigated in [15].

5 The θ-Specialization

This section is devoted to the interpretation of a few formulas [4, 10, 11] in terms of noncommutative symmetric functions.

5.1 Carlitz–Koszul Duality and θ-Alphabets

Let $\theta \subseteq A \times A$ be any binary relation. We denote by $\overline{\theta}$ the complement of θ in $A \times A$ and set

$$X = X(A;\theta) = \{w = a_1 \cdots a_n \in A^* \mid a_1\theta a_2\theta \cdots \theta a_n\}, \quad Y = Y(A;\theta) = X(A;\overline{\theta}),$$

where we write $a\theta b$ for $(a, b) \in \theta$. Note that the empty word 1 and the letters belong to both X and Y.

The θ-specialization $\mathbf{Sym}(A;\theta)$ is then defined by specifying the elementary symmetric functions $\Lambda_n(A;\theta) = \sum_{w \in X \cap A^n} w$. The following basic lemma, implicit in [4], generalizes the case $\theta = \{(a, b) \mid a > b\}$.

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Lemma 5.1 **(Carlitz–Koszul duality for alphabets)** *The complete symmetric functions $S_n(A;\theta)$ are given by $S_n(A;\theta) = \Lambda_n(A;\overline{\theta})$. More generally, if one denotes by $\theta\mathrm{Adj}(w) = \{i \mid a_i\theta a_{i+1}\}$ the θ-adjacency set of $w = a_1a_2\cdots a_n$, and by $C_\theta(w)$ the associated composition of n, one has $R_I(A;\theta) = \sum_{C_\theta(w)=I} w$.*

Proof We need to prove that $\sum_{k=0}^{n}(-1)^k\Lambda_k(A,\theta)\Lambda_{n-k}(A,\bar{\theta}) = 0$ for $n > 0$. Let $w = uv$ be such that $u \in \Lambda_k(A,\theta)$ and $v \in \Lambda_{n-k}(A,\bar{\theta})$. Then if $\mathrm{last}(u)\theta\,\mathrm{first}(v)$, w appears in $\Lambda_{k+1}(A,\theta)\Lambda_{n-k-1}(A,\bar{\theta})$, and similarly, if $\mathrm{last}(u)\bar{\theta}\,\mathrm{first}(v)$, then w appears in $\Lambda_{k-1}(A,\theta)\Lambda_{n-k+1}(A,\bar{\theta})$. Moreover, w cannot appear in any other product, so that its coefficient in the sum is 0. ∎

5.2 The θ-Eulerian Polynomials

Recall from [12] that the noncommutative Eulerian polynomials

$$\mathbf{A}_n(t;A) = \sum_{I\vDash n} t^{l(I)}R_I(A)$$

admit the generating function

$$\mathcal{A}(t;A) = \sum_{n\geq 0}\mathbf{A}_n(t;A) = \frac{1-t}{1-t\sigma_{1-t}(A)}$$

(see [7] for the commutative version of this identity), and since

$$l(C_\theta(w)) = \theta\mathrm{adj}(w) + 1,$$

we have immediately

$$\sum_{w\in A^*} t^{\theta\mathrm{adj}(w)+1}w = \frac{1-t}{1-t\sigma_{1-t}(A;\theta)},$$

where $\theta\mathrm{adj}(w) = |tAdj(w)|$. Note that $\theta\mathrm{adj}(w) + \overline{\theta}\,\mathrm{adj}(w) = n-1$. Replacing θ by $\overline{\theta}$, A by $t^{-1}A$, then t by t^{-1}, and simplifying by $(1-t)$ the resulting expression, we obtain

$$\sum_{w\in A^*} t^{\theta\mathrm{adj}(w)}w = \frac{1}{1-\sum_{w\in X(A;\theta)^+}(t-1)^{l(w)-1}w}, \tag{5.1}$$

which is [11, Theorem 2].

For a letter $c \in A$, denote by $\overleftarrow{\partial}_c$ the linear operator defined by

$$w\overleftarrow{\partial}_c = \begin{cases} u & \text{if } w = uc \text{ for some } u,\\ 0 & \text{otherwise.}\end{cases}$$

Then as in (4.1), for any series F without constant term,

$$(1-F)^{-1}\overleftarrow{\partial}_c = (1-F)^{-1}\cdot(F\overleftarrow{\partial}_c).$$

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The same is true for the operators $D_C = \sum_{c\in C} \overleftarrow{\partial}_c \cdot c$ where C is a subset of A. Applying this to (5.1), we obtain, setting for short $X = X(A;\theta)$,

$$\sum_{w\in A^*C} t^{\theta\mathrm{adj}(w)} w = \frac{-\sum_{w\in XC}(t-1)^{l(w)-1}w}{1-\sum_{w\in X^+}(t-1)^{l(w)-1}w},$$

which is [11, Theorem 3].

5.3 The θ-Major Index

If one defines the θ-Major index by $\theta\mathrm{maj}(w) = \sum_{i\in\theta\mathrm{Adj}(w)} i$ one has clearly from [14, (125)],

$$\sum_{w\in A^n} q^{\theta\mathrm{maj}(w)} w = \sum_{I\vDash n} q^{\mathrm{maj}(I)} R_I(A;\theta) = (q)_n S_n\Big(\frac{A}{1-q};\theta\Big),$$

where as usual

$$\sigma_z\Big(\frac{A}{1-q};\theta\Big) = \overrightarrow{\prod_{n\geq 0}} \sigma_{zq^n}(A;\theta).$$

6 Double Eulerian Polynomials and Bessel Functions

The noncommutative Bessel function $\mathbf{J}_0(A,B)$ can now be properly interpreted as a generating series of θ elementary symmetric functions, if we interpret $\mathbb{J}$ as the product alphabet $A\times B$, endowed with the relation $(a,b)\theta(a',b') \Leftrightarrow a > a'$ and $b \leq b'$. As is customary, we denote words over $A\times B$ by biwords

$$w = [u,v] = \begin{bmatrix} u \\ v \end{bmatrix} \quad u\in A^n,\ v\in B^n.$$

Observing that

$$\theta\mathrm{Adj}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \mathrm{Des}(u)\cap\overline{\mathrm{Des}(v)} = \mathrm{Des}(u)\setminus\mathrm{Des}(v),$$

we can now write the following.

Theorem 6.1 *One has*

$$\text{(6.1)}\qquad \sum_{w=(u,v)\in(A\times B)^*} t^{\theta\mathrm{adj}(w)} z^{l(w)} w = \frac{1-t}{\mathbf{J}_0((1-t)z;A,B)-t}$$
$$= \sum_K z^{|K|} t^{l(K)-1} R_K(A,B;\theta)$$

where from now on we shall use the notation

$$\mathbf{J}_0(x;A,B) = \lambda_{-x}(\mathbb{J}) = \lambda_{-x}(A,B;\theta).$$

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The coefficient of z^n is the n-th double θ-Eulerian polynomial, denoted by $\mathbf{A}_n(t;A,B;\theta)$. Setting $A=B=\mathbb{E}$, we recover the enumeration of pairs of permutations $(\alpha,\beta)\in\mathfrak{S}_n\times\mathfrak{S}_n$ by the cardinality of $\mathrm{Des}(\alpha)\cap\overline{\mathrm{Des}(\beta)}$ (*cf.* [3]).

7 The Fédou–Rawlings Polynomials

By considering simultaneously the specializations of (6.1) to all positive q- and p-integers, $A_i=[i+1]_q$ and $B_j=[j+1]_p$, one arrives at the five parameter generalizations of the double Eulerian polynomials introduced by Fédou and Rawlings [11].

For $w\in A^n$, where A is the infinite chain $A=\{a_1<a_2<\cdots\}$, let q^w be the image of w by the multiplicative homomorphism $a_i\mapsto q^{i-1}$. For a composition I of n, write $R_I(A)=\sum_{C(\sigma)=I}\sum_{\mathrm{Std}(w)=\sigma}w$, where $C(\sigma)$ is the descent composition of σ and $\mathrm{Std}(w)$ is the standardized word of w. Taking into account the identity

$$\sum_{\mathrm{Std}(w)=\sigma}x^{\max(w)}q^w=\frac{x^{\mathrm{des}(\sigma^{-1})}q^{\mathrm{coimaj}(\sigma)}}{(xq;q)_n}$$

where $\mathrm{coimaj}(\sigma)$ denotes the co-major index of σ^{-1},

$$\mathrm{coimaj}(\sigma)=\sum_{d\in\mathrm{Des}(\sigma^{-1})}(n-d),$$

(indeed, it is easily checked that the minimal word v for the lexicographic order such that $\mathrm{Std}(v)=\sigma$ satisfies $q^v=q^{\mathrm{coimaj}(\sigma)}$), we find

$$\sum_{i\geq 0}x^iR_I(1,q,\ldots,q^i)=\frac{1}{1-x}\sum_{C(w)=I}x^{\max(w)}q^w=\frac{1}{(x;q)_{n+1}}\sum_{C(\sigma)=I}x^{\mathrm{des}(\sigma^{-1})}q^{\mathrm{coimaj}(\sigma)}$$

so that, from Theorem 6.1 we recover the double generating series of [11]

$$\begin{aligned}(7.1)\qquad &\sum_{i,j\geq 0}x^iy^j\frac{1-t}{\mathbf{J}_0((1-t)z;A_i,B_j)-t}\\ &=\sum_{n\geq 0}\frac{z^n}{(x;q)_{n+1}(y;p)_{n+1}}\sum_{\alpha,\beta\in\mathfrak{S}_n}t^{\mathrm{desris}(\alpha,\beta)}x^{\mathrm{des}(\alpha^{-1})}y^{\mathrm{des}(\beta^{-1})}q^{\mathrm{coimaj}(\alpha)}p^{\mathrm{coimaj}(\beta)},\end{aligned}$$

where $\mathrm{desris}(\alpha,\beta)=|\,\mathrm{Des}(\alpha)\setminus\mathrm{Des}(\beta)|$.

The second generating series of [11] is recovered in the same way. If we denote by b_j the greatest letter of B_j, then, on the one hand, $S_n(B_j)\overleftarrow{\partial}_{b_j}=S_{n-1}(B_j)$. On the other hand,

$$\begin{aligned}\sum_{j\geq 0}y^jR_J(B_j)\,\overleftarrow{\partial}_{b_j}\cdot b_j&=\frac{1}{1-y}\sum_{\substack{C(\sigma)=J\\ \max(v)=\mathrm{last}(v)}}y^{max(v)}v\\ &=\frac{1}{1-y}\sum_{\substack{C(\sigma)=J\\ \sigma(n)=n}}\sum_{\mathrm{Std}(v)=\sigma}y^{max(v)}v,\end{aligned}$$

so that, applying the operator $\overleftarrow{\partial}_{b_j} \cdot b_j$ to the coefficient of $x^i y^j$ in (7.1), we obtain

$$\sum_K z^{|K|} \sum_{I \setminus J = K} t^{l(K)-1} \sum_{i,j\geq 0} x^i y^j R_I(A_i) R_J(B_j) \cdot \overleftarrow{\partial}_{b_j} \cdot b_j$$
$$= \sum_{i,j\geq 0} x^i y^j \Big(1 - \sum_{n\geq 1} z^n (t-1)^{n-1} \Lambda_n(A_i) S_n(B_j) \Big)^{-1} \cdot \overleftarrow{\partial}_{b_j} \cdot b_j$$
$$= \sum_{i,j\geq 0} x^i y^j \frac{\Big(-\sum_{n\geq 1} z^n (t-1)^{n-1} \Lambda_n(A_i) S_{n-1}(B_j) b_j \Big) (1-t)}{\mathbf{J}_0(z(1-t); A_i, B_j) - t}$$
$$= \sum_{i,j\geq 0} x^i y^j \frac{\mathbf{J}_{-1}((1-t)z; A_i, B_j) b_j}{\mathbf{J}_0(z(1-t); A_i, B_j) - t}.$$

Specializing $A_i = [i+1]_q$, $B_j = [j+1]_p$, this becomes, in the notation of [11],

$$\sum_{i,j\geq 0} x^i (py)^j \frac{J_1^{(i,j)}((1-t)z; q, p)}{J_0^{(i,j)}((1-t)z; q, p) - t}$$
$$= \sum_{n\geq 0} \frac{z^n}{(x;q)_{n+1} (y;p)_n} \sum_{\substack{\alpha,\beta \in \mathfrak{S}_n \\ \beta(n)=n}} t^{\mathrm{desris}(\alpha,\beta)} x^{\mathrm{des}(\alpha^{-1})} y^{\mathrm{des}(\beta^{-1})} q^{\mathrm{coimaj}(\alpha)} p^{\mathrm{coimaj}(\beta)}$$

which is equivalent to [11, (3)]. Here, $J_\nu^{(i,j)}(z; q, p) := (-1)^\nu \mathbf{J}_\nu(z[i+1]_q, [j+1]_p)$. The other results of [11, §8] can be rederived in the same way by changing the specializations of (A_i, B_j) to $([i]_q, [j+1]_p)$, $([i+1]_q, -[j+1]_p)$, and $(-[i+1]_q, -[j+1]_p)$.

8 Heaps of Segments and Polyominos

Bessel functions and their multiparameter analogs play a crucial role in the enumerative theory of polyominos [1,6]. Elegant combinatorial proofs of such enumerative results can be achieved by means of Viennot's theory of heaps of segments [1,20]. As we shall see, this can also be conveniently formulated in terms of θ-noncommutative symmetric functions.

8.1 Staircase Polyominos

A *parallelogram* (or *staircase*) *polyomino* P, which is also the same as a connected skew Young diagram, can be encoded as a biword

$$w = a_{i_1 j_1} \cdots a_{i_n j_n} = \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix},$$

where j_k is the height of the k-th column C_k, and i_k is the number of common rows between C_k and C_{k+1} (with a conventional value $i_n = 1$ for the last column). For

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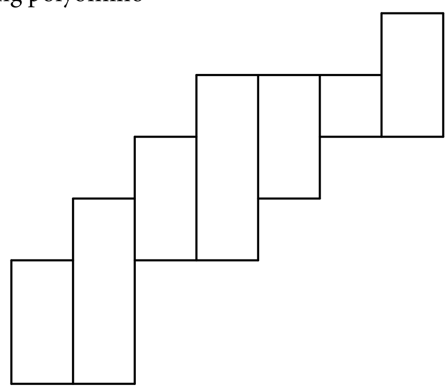
example, the following polyomino



is encoded by the biword

$$\begin{bmatrix} 2122111 \\ 2323212 \end{bmatrix}$$

The biwords corresponding to polyominos are the words over the alphabet

$$A = \{a_{ij} \mid i \le j\}$$

satisfying the θ-adjacency conditions

$$a_{i_k j_k}\, \theta\, a_{i_{k+1} j_{k+1}} \Longleftrightarrow i_k \le j_{k+1} \tag{8.1}$$

and the ending condition

$$i_n = 1. \tag{8.2}$$

Hence, the generating series (by length) of all biwords satisfying (8.1) is

$$\lambda_t(A,\theta) = [\lambda_{-t}(A,\bar{\theta})]^{-1} = \Bigl(1 - \sum_{n\ge 1}(-1)^{n-1} t^n \sum_{i_k > j_{k+1}} \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix}\Bigr)^{-1},$$

and restriction of the series to the biwords satisfying (8.2) is achieved as above by applying the operator $D = \sum_{j\ge 1} \overleftarrow{\partial}_{a_{1j}} a_{ij}$, which yields the following.

Theorem 8.1 *The generating series of the biwords satisfying* (8.1) *and* (8.2) *is*

$$\Bigl(1 - \sum_{n\ge 1}(-1)^{n-1} t^n \sum_{i_k > j_{k+1}} \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix}\Bigr)^{-1} \sum_{n\ge 1}(-1)^{n-1} t^n \sum_{i_k > j_{k+1}; i_n = 1} \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix}.$$

Again, it acquires the structure J_1/J_0 once A is specialized to $a_{ij} = xy^{j-i}q^j$, which gives the generating series by width, height, and area.

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8.2 Comparison with Viennot's Formalism

This can, of course, be interpreted in terms of heaps of segments. A *segment* is an interval $[i, j]$ of $\mathbb{N}^*$. To each segment, we associate a variable

$$a_{ij} = \begin{bmatrix} i \\ j \end{bmatrix},$$

in our $A = \{a_{ij} \mid i \leq j\}$. The *monoid of heaps* is the quotient of the free monoid A^* by the commutation relations $a_{ij}a_{kl} \equiv a_{kl}a_{ij}$ if $j < k$ which means that the segments do not overlap and can be vertically slid independently of each other.

The first basic lemma of Viennot's theory [20] (which is also a special case of the Cartier–Foata formula for the Moebius functions of free partially commutative monoids [5]) amounts to the calculation of $S_n(A, \theta)$ for the relation defined by

$$a_{ij}\, \theta\, a_{kl} \Longleftrightarrow i \leq l.$$

Indeed, with this choice, $\Lambda_n(A, \bar{\theta})$ is the formal sum of trivial heaps (products of mutually commuting segments arranged in decreasing order), and $S_n(A, \theta) = \Lambda_n(A, \theta)$ is the sum of all biwords

$$w = a_{i_1 j_1} \cdots a_{i_n j_n} = \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix}$$

such that $i_k \leq j_{k+1}$ for all k, those encoding polyominos. This is precisely Viennot's lemma. Hence, the discussion in Section 8.1 proves in particular that each heap, or, equivalently, each element of $A^*/\equiv$ has a unique representative of this form.

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