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Uniqueness and stability of equilibrium states for random non-uniformly expanding maps

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Abstract. We consider a robust class of random non-uniformly expanding local homeomorphisms and Hölder continuous potentials with small variation. For each element of this class we develop the thermodynamical formalism and prove the existence and uniqueness of equilibrium states among non-uniformly expanding measures. Moreover, we show that these equilibrium states and the random topological pressure vary continuously in this setting.

Key words: random dynamical systems, stability, thermodynamical formalism
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1. *Introduction*

The thermodynamic formalism, developed by Sinai, Ruelle and Bowen in the 1970s and 1990s, is a part of ergodic theory that came into existence through the application of techniques and results from statistical mechanics in the realm of smooth dynamics. One of its main goals is to describe the statistical behavior of a dynamical system via invariant measures, called *equilibrium states*, that maximize the free energy of the system.

In the classical setting, an *equilibrium state* associated to a continuous transformation $T : M \to M$ defined on a compact metric space M and a continuous potential $\phi : M \to \mathbb{R}$ is an invariant probability measure $\mu_{T,\phi}$ characterized by the following variational principle:

$$P_T(\phi) = h_{\mu_{T,\phi}}(T) + \int \phi \, d\mu_{T,\phi} = \sup_{\mu \in \mathcal{M}_T(M)} \left\{ h_\mu(T) + \int \phi \, d\mu \right\}$$

where $P_T(\phi)$ is the topological pressure, $h_\mu(T)$ denotes the entropy and the supremum is taken over all invariant probability measures.

This theory was initiated by the pioneering work of Sinai [27] who proved the existence and uniqueness of equilibrium states for Anosov diffeomorphisms and Hölder continuous potentials. In subsequent works Bowen [11] and Ruelle [25] extended the results of Sinai to uniformly hyperbolic systems and Hölder continuous potentials. Since then, important contributions for this theory in the deterministic case have been given by several authors (see, for example, [13, 21, 26, 29]).

In the context of random dynamical systems, the study of equilibrium states is still quite far from being well understood, despite some advances in the area. Briefly, a random dynamical system is a skew-product $F(w, x) = (\theta(w), f_w(x))$ where the randomness is modeled by an invertible transformation θ preserving an ergodic measure $\mathbb{P}$. We are interested in understanding the dynamics of compositions

$$f_w^n := f_{\theta^{n-1}(w)} \circ \cdots \circ f_{\theta(w)} \circ f_w.$$

As in the deterministic case, the random topological pressure of the system is the supremum of the entropy plus the integration of the potential among all invariant probability measures whose marginal is $\mathbb{P}$. We refer the reader to [6] for the background to and a treatment of this topic.

Having established a variational principle for random maps, it is natural to ask what kinds of random dynamical systems and potentials we can develop the theory of equilibrium states. In [16] Kifer proved the existence and uniqueness of equilibrium states for random uniformly expanding maps associated to Hölder continuous potentials. In [18] Liu extended this result for uniformly hyperbolic random systems. Later, the thermodynamical formalism was developed by Kifer [15] for random expansion in average transformations, and by Mayer, Skorulski and Urbanski [20] for distance expanding random mappings. In the context of random countable Markov shifts, the thermodynamic formalism was proved by Denker, Kifer and Stadlbauer in [14]. The existence of equilibrium states with positive Lyapunov exponents was proved by Arbieto, Matheus and Oliveira [5] for certain non-uniformly expanding maps and continuous potentials with low variation. In [10] Bilbao and Oliveira obtained uniqueness of maximizing entropy measures in this context. Recently, Stadlbauer, Suzuki and Varandas [28] developed the thermodynamical formalism for a wide class of random maps with non-uniform expansion and differentiable potentials at high temperature.

In this work we develop the thermodynamical formalism for a robust class of random non-uniformly expanding local homeomorphisms associated to Hölder continuous potentials with small variation. First, we prove the existence of an invariant measure absolutely continuous with respect to the leading eigenmeasures of the dual transfer operators. This invariant measure is indeed an equilibrium state for the random dynamical system and it is unique in the setting of non-uniformly expanding measures. Moreover, we show that the random topological pressure is the integral of the leading eigenvalues of the transfer operators. As an application of our techniques, we extend the results obtained in [5, 9] for Hölder continuous potentials with small variation.

Finally, we study the persistence of the equilibrium state under small perturbations of the system. In the context of Sinai–Ruelle–Bowen measures, the continuous dependence with respect to the dynamics was obtained by Alves and Viana [4] for maps with non-uniform expansion. Such continuity was also proved by Baladi [7] and Young [30] for random perturbations of uniformly hyperbolic systems and by Alves and Araújo in [1] for random perturbations of non-uniformly expanding maps. More generally, the continuity of the equilibrium state was proved by Castro and Varandas [12] for a class of non-uniformly expanding maps and potentials with small variation. This property was also obtained by Alves, Ramos and Siqueira [3] for non-uniformly hyperbolic systems and hyperbolic potentials. Here, we deal with a family of random non-uniformly expanding maps and potentials with small variation. We prove that the non-uniformly expanding equilibrium state as well as the random topological pressure vary continuously within this family.

We organize this paper as follows. In §2 we present our setting and state the main results. In §3 we introduce basic definitions such as random topological pressure and projective metrics. In §4 we recall the definition of reference measures and prove some properties that will be useful throughout the work. In §5 we use the projective metric approach to obtain the thermodynamical formalism. In §6 we prove the existence and uniqueness of equilibrium states among non-uniformly expanding measures. In §7 we show the continuous dependence of these equilibrium states and the topological pressure as functions of the random dynamics and the potential. In the final section we describe some applications of our results.

2. *Setting and main results*

Let M be a compact and connected manifold with distance d and Ω the space of local homeomorphisms defined on M. Consider a Polish space X (that is, a separable complete metric space) and an invertible measurable map $\theta : X \to X$ preserving an ergodic Borel measure $\mathbb{P}$ of X. We recall that a *random dynamical system* is a continuous map $f : X \to \Omega$ given by $w \mapsto f_w \in \Omega$ where $(w, x) \mapsto f_w(x)$ is measurable. For every $n \geq 0$ we define

$$f_w^0 := Id, \quad f_w^n := f_{\theta^{n-1}(w)} \circ \cdots \circ f_{\theta(w)} \circ f_w, \quad f_w^{-n} = (f_w^n)^{-1}.$$

The skew-product generated by the maps f_w is the measurable transformation

$$F : X \times M \to X \times M;\ F(w, x) = (\theta(w), f_w(x)).$$

In particular, $F^n(w, x) = (\theta^n(w), f_w^n(x))$ for every $n \in \mathbb{Z}$.

Let $\mathcal{M}_{\mathbb{P}}(X \times M)$ be the space of probability measures on $X \times M$ such that the marginal on Ω is $\mathbb{P}$. Denote by $\mathcal{M}_{\mathbb{P}}(F) \subset \mathcal{M}_{\mathbb{P}}(X \times M)$ the set of F-invariant measures. Note that, by Rokhlin's disintegration theorem [24], for every $\mu \in \mathcal{M}_{\mathbb{P}}(F)$ there exists a system of sample measures $\{\mu_w\}_{w \in X}$ of μ such that

$$d\mu(w, x) = d\mu_w(x)\, d\mathbb{P}(w).$$

We say that an F-invariant measure μ is *ergodic* if (F, μ) is ergodic.

2.1. *Hypothesis about the generating maps.*

For each $w \in X$, let $f_w : M \to M$ be a local homeomorphism satisfying the following requirement: there exist $\delta_w > \delta > 0$ and a

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continuous function $L_w : M \to \mathbb{R}_+$ such that for every $x \in M$ we can find a neighborhood U_x where $f_w : U_x \to B_{\theta(w)}(f_w(x), \delta_w)$ is invertible and

$$d(f_w^{-1}(y), f_w^{-1}(z)) \leq L_w(x)d(y, z) \quad \text{for all } y, z \in f_w(U_x) = B_{\theta(w)}(f_w(x), \delta_w).$$

As f_w is a local homeomorphism defined on a compact metric space, we have that the number of preimages $\deg(f_w) : \# f_w^{-1}(x)$ is constant for all $x \in M$. We assume that $\deg(F) = \sup_w \deg(f_w) < \infty$.

Suppose that there are an open region $\mathcal{A}_w \subset M$, constants $\sigma_w > 1$ and $L_w \geq 1$ close enough to 1 such that the following conditions hold.

(I) $L_w(x) \leq L_w$ for every $x \in \mathcal{A}_w$ and $L_w(x) < \sigma_w^{-1}$ for every $x \in \mathcal{A}_w^c = M \setminus \mathcal{A}_w$.

(II) There exists a finite covering $\mathcal{U}_w$ of M, by open domains of injectivity for f_w, such that $\mathcal{A}_w$ can be covered by $q_w < \deg(f_w)$ elements of $\mathcal{U}_w$.

(III) For every $\varepsilon > 0$ we can find some positive integer $\tilde{n} = \tilde{n}(w, \varepsilon)$ satisfying $f^{\tilde{n}}_{\theta^j(w)}(B_{\theta^j(w)}(f_w^j(x), \varepsilon)) = M$ for any $j \geq 0$.

We observe that the continuous function $L_w(\cdot)$ associates the Lipschitz constant of the inverse branches. This property and the assumption $\delta_w > \delta > 0$ imply the uniform openness $B_{\theta(w)}(f_w(x), \delta) \subset f_w(U_x)$, and thus for every $(w, x) \in X \times M$ there exists a unique continuous inverse branch of f_w defined on $B_{\theta(w)}(f_w(x), \delta)$ sending $f_w(x)$ to x. Conditions (I) and (II) mean that expanding and contracting behavior may exist in M, but at least one preimage is required for every point in the expanding region. Condition (III) means that the skew-product F is *topologically exact*.

Next, we present the setting of potentials that will be considered. For $\alpha > 0$, consider the space $C^\alpha(M)$ of Hölder continuous functions $\varphi : M \to \mathbb{R}$ endowed with the seminorm

$$|\varphi|_\alpha = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\alpha}$$

and the norm

$$\|\varphi\|_\alpha = \|\varphi\|_\infty + |\varphi|_\alpha,$$

where $\|\cdot\|_\infty$ stands for the sup norm. Denote by $\mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M))$ the space of all measurable functions $\phi : X \times M \to \mathbb{R}$ such that for all $w \in X$, the fiber potential $\phi_w : M \to \mathbb{R}$ defined by $\phi_w(x) := \phi(w, x)$ is Hölder continuous and $\|\phi\|_1 = \int_X \|\phi_w\|_\infty \, d\mathbb{P}(w) < +\infty$. For $\phi \in \mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M))$ we assume the existence of some positive $\varepsilon_\phi > 0$ satisfying, for all $w \in X$,

$$\sup \phi_w - \inf \phi_w + \varepsilon_\phi < \log \deg f_w - \log q_w \quad \text{and} \quad |e^{\phi_w}|_\alpha < \varepsilon_\phi e^{\inf \phi_w}. \tag{IV}$$

Notice that all potentials $\phi \in \mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M))$ in a neighborhood of zero satisfy condition (IV). In the literature this class of potentials is referred to as *small variation*.

Let $p_w := \deg f_w - q_w$. The choice of ε_ϕ and L_w must satisfy, for each $w \in X$,

$$\gamma_w := e^{\varepsilon_\phi}\left[\frac{p_w \sigma_w^{-\alpha} + q_w L_w^\alpha (1 + (L_w - 1)^\alpha)}{\deg(f_w)}\right] + \varepsilon_\phi L_w^\alpha \left[1 + m(\operatorname{diam} M)^\alpha\right] \leq \gamma < 1. \tag{V}$$

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2.2. *Statement of results.* Consider the space $C^0(M)$ of real continuous functions $\psi : M \to \mathbb{R}$ endowed with the uniform convergence norm. Given $w \in X$, let $f_w : M \to M$ be the dynamics and $\phi_w : M \to \mathbb{R}$ be the potential on the fiber. The *Ruelle–Perron–Frobenius operator* or simply *transfer operator* associated to (f_w, ϕ_w) is the linear operator $\mathcal{L}_w : C^0(M) \to C^0(M)$ defined by

$$\mathcal{L}_w(\psi)(x) = \sum_{y \in f_w^{-1}(x)} e^{\phi_w(y)} \psi(y).$$

Its dual operator $\mathcal{L}_w^* : [C^0(M)]^* \to [C^0(M)]^*$ acts on the space of Borel measures as follows:

$$\int \psi \, d\mathcal{L}_w^*(\rho_{\theta(w)}) = \int \mathcal{L}_w(\psi) \, d\rho_{\theta(w)}.$$

In our first result we describe the thermodynamic formalism for random non-uniformly expanding maps.

THEOREM A. *Consider a random dynamical system $F : X \times M \to X \times M$ satisfying conditions (I), (II) and (III). For any potential $\phi : X \times M \to \mathbb{R}$ satisfying (IV) and (V) the following assertions hold.*

(1) *There exists a unique measurable family of probabilities $\{\nu_w\}_{w\in X}$ such that*

$$\mathcal{L}_w^* \nu_{\theta(w)} = \lambda_w \nu_w \quad \textit{where } \lambda_w = \nu_{\theta(w)}(\mathcal{L}_w(1)), \textit{ for almost every } w \in X.$$

(3) *There exists a unique measurable family of Hölder continuous functions $\{h_w\}_{w\in X}$ bounded away from zero and infinity such that*

$$\mathcal{L}_w h_w = \lambda_w h_{\theta(w)} \quad \textit{and} \quad \nu_w(h_w) = 1 \quad \textit{for almost every } w \in X.$$

(3) *The probability measure $\mu := \{\mu_w\}_{w\in X}$ where $\mu_w := h_w \nu_w$ is F-invariant.*

We also derive that the F-invariant family $\{\mu_w\}_{w\in X}$ obtained in the last theorem has an exponential decay of correlations for Hölder continuous observables.

THEOREM B. *There exists $0 < \tau < 1$ such that for any $\varphi \in L^1(\mu_{\theta^n(w)})$ and $\psi \in C^\alpha(M)$ there exists a positive constant $K(\varphi, \psi)$ satisfying*

$$\left| \int (\varphi \circ f_w^n) \psi \, d\mu_w - \int \varphi \, d\mu_{\theta^n(w)} \int \psi \, d\mu_w \right| \le K(\varphi, \psi) \tau^n,$$

for all $n \ge 1$.

The weak hyperbolicity property of the generating maps allows us to prove that the F-invariant measure given by Theorem A is indeed an equilibrium state for the random dynamical system. Moreover, it is unique if we consider only the measures whose pressure is located on the expanding region. We specify the setting as follows.

Suppose that there exists $c > 0$ such that for $\mathbb{P}$-almost every $w \in X$ we can find $\tilde{L}_w$ close enough to 1 and $\tilde{\sigma}_w > 1$ satisfying for every $j \ge 0$ that

$$L_{\theta^j(w)} \le \tilde{L}_w, \quad \tilde{\sigma}_w \le \sigma_{\theta^j(w)} \quad \text{and} \quad \tilde{L}_w^\rho \tilde{\sigma}_w^{-(1-\rho)} < e^{-2c} < 1, \tag{VI}$$

where ρ is given by Lemma 4.2.

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We say that a subset H of $X \times M$ is *non-uniformly expanding* if there exists some positive constant $c > 0$ such that

$$H := \left\{(w, x) \in X \times M; \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log L_{\theta^j(w)}(f_w^j(x)) \leqslant -2c < 0\right\}. \qquad (\star)$$

A probability measure η, not necessarily invariant, is called *non-uniformly expanding* with exponent c if $\eta(H) = 1$. Condition (VI) above will be used to prove that the F-invariant measure given by Theorem A is non-uniformly expanding.

Our next result states uniqueness of equilibrium states for random dynamical systems among non-uniformly expanding measures.

THEOREM C. *Let $F : X \times M \to X \times M$ be a random dynamical system and $\phi : X \times M \to \mathbb{R}$ be a potential function satisfying conditions (I)–(VI). There exists only one F-invariant non-uniformly expanding measure $\mu_{F,\phi} \in \mathcal{M}_{\mathbb{P}}(F)$ maximizing the variational principle*

$$P_{F|_\theta}(\phi) = \int \log \lambda_w \, d\mathbb{P}(w) = h_{\mu_{F,\phi}}(F|\theta) + \int \phi \, d\mu_{F,\phi} = \sup \left\{h_\mu(F|\theta) + \int \phi \, d\mu\right\}$$

where the supremum is taken in the set $\mathcal{M}_{\mathbb{P}}(F)$. Thus, $\mu_{F,\phi}$ is the unique non-uniformly expanding equilibrium state of $(F|\theta, \phi)$.

Once we have proved uniqueness of equilibrium states, we will investigate its persistence under small perturbations of the random system and the potential.

As defined above, consider the space $\mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M))$ of integrable potentials and let $\mathcal{D} \subset \Omega$ be the space of C^1 local diffeomorphisms defined on M. We shall consider the product topology on $\mathcal{D} \times \mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M))$. We fix an invertible transformation $\theta : X \to X$ preserving an ergodic measure $\mathbb{P}$ and consider the family $\mathcal{S}$ of skew-products generated by maps of $\mathcal{D}$,

$$F : X \times M \to X \times M; \; F(w, x) = (\theta(w), f_w(x)),$$

where $(w, x) \mapsto f_w \in \mathcal{D}$ is measurable. Now we define the family

$$\mathcal{H} = \{(F, \phi) \in \mathcal{S} \times \mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M)); \, (F, \phi) \text{ satisfying conditions (I)–(VI)}\}.$$

By Theorem C, each $(F, \phi) \in \mathcal{H}$ has only one non-uniformly expanding equilibrium state. Our last main result establishes the continuity in the weak star topology of such equilibria within this family; this property is called *equilibrium stability*. We also prove the continuity of the random topological pressure in this setting.

THEOREM D. *The non-uniformly expanding equilibrium state and the topological pressure vary continuously on $\mathcal{H}$.*

We point out that we are fixing an invertible transformation $\theta : X \to X$ preserving an ergodic measure $\mathbb{P}$. However, the proof of Theorem D remains true if we vary θ in the space of continuous functions.

3. *Preliminaries*

In this section we state some basic definitions and results about random dynamical systems that will be used throughout the text. We also recall the notion of hyperbolic times and projective metrics.

3.1. *Entropy and topological pressure.*

We start with the definition of entropy and topological pressure for random transformations. The reader can consult more results and properties in Kifer [15] and Liu [18].

Let $\mu \in \mathcal{M}_{\mathbb{P}}(F)$ be an F-invariant measure. Given a finite measurable partition ξ of M, we set

$$h_\mu(F|\theta;\xi) := \lim_{n\to+\infty} \frac{1}{n}\int_X H_{\mu_w}\left(\bigvee_{j=0}^{n-1} f_w^{-j}(\xi)\right) d\mathbb{P}(w)$$

where $H_\nu(\xi) = -\sum_{P\in\xi} \nu(P)\log\nu(P)$ for a finite partition ξ and μ_w is the sample measure of μ. The *entropy* of $(F|_\theta, \mu)$ is

$$h_\mu(F|\theta) := \sup_\xi\{h_\mu(F|\theta;\xi)\}$$

where the supremum is taken over all finite measurable partitions of M.

Denote by $\mathbb{L}^1_{\mathbb{P}}(X, C^0(M))$ the space of all measurable functions $\phi : X \times M \to \mathbb{R}$ such that $\phi_w : M \to \mathbb{R}$ defined by $\phi_w(x) := \phi(w, x)$ is continuous for all $w \in X$ and $\|\phi\|_1 = \int_X \|\phi_w\|_\infty \, d\mathbb{P}(w) < +\infty$.

Fix $w \in X$. Given $\varepsilon > 0$ and an integer $n \geq 1$, we say that a subset $F_n \subseteq M$ is (w, n, ε)-separated if for every two distinct points $y, z \in F_n$ there exists some $j \in \{0, 1, \ldots, n-1\}$ such that $d(f_w^j(y), f_w^j(z)) > \varepsilon$.

For $\phi \in \mathbb{L}^1_{\mathbb{P}}(X, C^0(M))$, $\varepsilon > 0$ and $n \geq 1$ we consider

$$P_{F|\theta}(\phi)(w,n,\varepsilon) = \sup\left\{\sum_{y\in F_n} e^{S_n\phi(w,y)};\ F_n \text{ is a } (w,n,\varepsilon)\text{-separated set}\right\}$$

where $S_n\phi(w, y) := \sum_{j=0}^{n-1} \phi_{\theta^j(w)}(f_w^j(y))$.

The *random topological pressure* of ϕ relative to θ is defined by

$$P_{F|\theta}(\phi) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\int_X \log P_{F|\theta}(\phi)(w,n,\varepsilon)\, d\mathbb{P}(w).$$

Thus, the following pressure map is well defined:

$$\begin{array}{rccc} P_{F|\theta}: & \mathbb{L}^1_{\mathbb{P}}(X, C^0(M)) & \longrightarrow & \mathbb{R}\cup\{\infty\} \\ & \phi & \longmapsto & P_{F|\theta}(\phi). \end{array}$$

In particular, the topological entropy of F relative to θ is $h_{\mathrm{top}}(F|_\theta) = P_{F|\theta}(0)$.

The topological pressure and the entropy are related by the well-known variational principle. The reader can see a proof of this result in [18].

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THEOREM 3.1. (Variational principle) *Assume that* $(X, \mathbb{P})$ *is a Lebesgue space. Then for any* $\phi \in \mathbb{L}^1_{\mathbb{P}}(X, C^0(M))$ *we have*

$$P_{F|\theta}(\phi) = \sup_{\mu \in \mathcal{M}_{\mathbb{P}}(F)} \bigg(h_\mu(F|\theta) + \int \phi \, d\mu \bigg). \tag{1}$$

Moreover, when $\mathbb{P}$ *is ergodic, we can consider the supremum over ergodic measures.*

Motivated by the variational principle, we say that an F-invariant measure $\mu \in \mathcal{M}_{\mathbb{P}}(F)$ is an *equilibrium state* for $(F|_\theta, \phi)$ relative to θ if the supremum (1) is attained by μ, that is,

$$P_{F|\theta}(\phi) = h_\mu(F|\theta) + \int \phi \, d\mu.$$

Next we define the random topological pressure via open coverings and via dynamic balls. We take as reference the deterministic case where this approach is characteristic in dimension theory. We refer the reader to [22, 28] for more details.

Let $w \in X$ and denote by $\mathcal{U}$ a finite open cover of M. Denote by $\mathcal{S}_n(\mathcal{U})$ the set of all strings $\mathbf{U} = \{U_{i_0}, \ldots, U_{i_{n-1}}; U_{i_j} \in \mathcal{U}\}$ of length $n = n(\mathbf{U})$ and put $\mathcal{S} = \mathcal{S}(\mathcal{U}) = \bigcup_{n \geq 0} \mathcal{S}_n(\mathcal{U})$. Given a string $\mathbf{U} = \{U_{i_0}, \ldots, U_{i_{n-1}}\} \in \mathcal{S}(\mathcal{U})$, we consider the cylinder

$$X_w = X_w(\mathbf{U}) := \{x \in M;\ f_w^j(x) \in U_{i_j} \text{ for } j = 0, \ldots, n(\mathbf{U}) - 1\}.$$

Let $\mathcal{F}_{(N,w)}$ be the collection of all cylinders of depth at least N, that is,

$$\mathcal{F}_{(N,w)} = \mathcal{F}_{(N,w)}(\mathcal{U}) = \{X_w(\mathbf{U}); \mathbf{U} \in \mathcal{S}_n(\mathcal{U}) \text{ for } n \geq N\}.$$

For $\beta \in \mathbb{R}$ and $\phi \in \mathbb{L}^1_{\mathbb{P}}(X, C^0(M))$ let

$$m_\beta(w, \phi, F|\theta, \mathcal{U}, N) = \inf_{\mathcal{F}} \bigg\{ \sum_{X_w \in \mathcal{F}_{(N,w)}} e^{-\beta n(\mathbf{U}) + S_{n(\mathbf{U})}\phi(X_w)} \bigg\}, \tag{2}$$

where $S_{n(\mathbf{U})}\phi(X_w) = \sup_{y \in X_w} \sum_{j=0}^{n(\mathbf{U})-1} \phi_{\theta^j(w)}(f_w^j(y))$ and the infimum is taken over all finite families $\mathcal{F}$ of $\mathcal{F}_{(N,w)}$ in order that (2) is measurable in w (see, for example, §9 of [28]). As N goes to infinity we define

$$m_\beta(w, \phi, F|\theta, \mathcal{U}) = \lim_{N \to \infty} m_\beta(w, \phi, F|\theta, \mathcal{U}, N).$$

The existence of the limit above is guaranteed by the function $m_\beta(w, \phi, F|\theta, \mathcal{U}, N)$ to be increasing with N. Taking the infimum over β we call

$$P_{F|\theta}(w, \phi, \mathcal{U}) = \inf\{\beta : m_\beta(w, \phi, F|\theta, \mathcal{U}) = 0\}.$$

Let $|\mathcal{U}| = \max\{\text{diam} U_i; U_i \subset \mathcal{U}\}$ be the diameter of the cover $\mathcal{U}$ and consider

$$P_{F|\theta}(w, \phi) = \lim_{|\mathcal{U}| \to 0} P_{F|\theta}(w, \phi, \mathcal{U}).$$

In [22, Theorem 11.1] it was showed that this quantity is well defined and does not depend on the cover $\mathcal{U}$. Moreover, since all quantities defined above are measurable functions of $w \in X$ (see, for example, §9 of [28]), we can define the *random topological pressure* of

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$(F|\theta, \phi)$ as

$$P_{F|\theta}(\phi) = \int P_{F|\theta}(w, \phi)\, d\mathbb{P}(w).$$

In the following we present another way to define the random topological pressure. We fix $w \in X$ and $\varepsilon > 0$. For $n \in \mathbb{N}$, $x \in M$, let $B_w(x, n, \varepsilon)$ be the dynamic ball

$$B_w(x, n, \varepsilon) := \{y \in M : d(f_w^j(x), f_w^j(y)) < \varepsilon, \text{for } 0 \leq j \leq n\}.$$

We denote by $G_{(N,w)}$ the collection of dynamic balls:

$$G_{(N,w)} := \{B_w(x, n, \varepsilon) : x \in M \text{ and } n \geq N\}.$$

Let U_w be a finite or countable family of $G_{(N,w)}$ which covers M. For every $\beta \in \mathbb{R}$ and $\phi \in \mathbb{L}^1_{\mathbb{P}}(X, C^0(M))$ let

$$m_\beta(w, \phi, F|\theta, \varepsilon, N) = \inf_{U_w \subset G_{(N,w)}} \left\{ \sum_{B_w(x,n,\varepsilon) \in U_w} e^{-\beta n + S_n\phi(B_w(x,n,\varepsilon))} \right\},$$

where $S_n\phi(B_w(x, n, \varepsilon)) = \sup_{y \in B_w(x,n,\varepsilon)} \sum_{j=0}^{n-1} \phi_{\theta^j(w)}(f_w^j(y))$. When N goes to infinity we consider

$$m_\beta(w, \phi, F|\theta, \varepsilon) = \lim_{N \to \infty} m_\beta(w, \phi, F|\theta, \varepsilon, N).$$

Taking the infimum over β, we define

$$P_{F|\theta}(w, \phi, \varepsilon) = \inf\{\beta : m_\beta(w, \phi, F|\theta, \varepsilon) = 0\}.$$

Since $P_{F|\theta}(w, \phi, \varepsilon)$ is decreasing on ε we can take the limit

$$P_{F|\theta}(w, \phi) = \lim_{\varepsilon \to 0} P_{F|\theta}(w, \phi, \varepsilon).$$

Now if we consider a finite open cover $\mathcal{U}$ of M with Lebesgue number $\varepsilon(\mathcal{U})$ we have

$$B_w\big(x, n(\mathbf{U}), \tfrac{1}{2}\varepsilon\big) \subset X_w(\mathbf{U}) \subset B_w(x, n(\mathbf{U}), 2|\mathcal{U}|)$$

which implies that

$$P_{F|\theta}(w, \phi) = \lim_{\varepsilon \to 0} P_{F|\theta}(w, \phi, \varepsilon) = \lim_{|\mathcal{U}| \to 0} P_{F|\theta}(w, \phi, \mathcal{U}).$$

Therefore, the definitions of random topological pressure via coverings and via dynamic balls coincide.

3.2. *Hyperbolic times.* In order to explore the non-uniform expansion of the set H we need the notion of hyperbolic times. The reader can obtain more details of this concept in [2, 5]. In our context, the function $L_w(\cdot)$ plays the role of the derivative $\|Df_w(\cdot)^{-1}\|$.

Definition 3.1. Let $w \in X$ and $L_w : X \to \mathbb{R}$ be as in §2. We say that $n \in \mathbb{N}$ is a c-hyperbolic time for $(w, x) \in X \times M$ if

$$\prod_{j=n-k}^{n-1} L_{\theta^j(w)}(f_w^j(x)) \leqslant e^{-ck} \quad \text{for every } 1 \leqslant k \leqslant n. \tag{3}$$

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It is a well-known fact that if η is a non-uniformly expanding measure with exponent c then η-almost every point $(w, x) \in H$ has infinitely many c-hyperbolic times. A proof of this result can be found in [2].

The uniform domination required in condition (VI) will allow us to prove that the F-invariant measure μ given by Theorem A is non-uniformly expanding. Thus, we will conclude that μ-almost every point $(w, x) \in X \times M$ has infinitely many hyperbolic times.

LEMMA 3.1. *Given $c > 0$, there exists $\tilde{\delta} = \tilde{\delta}(c) > 0$ such that, for $\mathbb{P}$-almost every $w \in X$, if n is a c-hyperbolic time of (w, x) then the dynamical ball $B_w(x, n, \tilde{\delta})$ around x is mapped homeomorphically onto the ball $B_{\theta^n(w)}(f_w^n(x), \tilde{\delta})$. Moreover, for $z \in B_w(x, n, \tilde{\delta})$ and $f_w^n(z) \in B_{\theta^n(w)}(f_w^n(x), \tilde{\delta})$, we have*

$$d(f_w^{n-k}(z), f_w^{n-k}(x)) \leq e^{-ck/2} d(f_w^n(z), f_w^n(x)),$$

for each $1 \leq k \leq n$.

We point out that the proof of the lemma above is analogous to the proof of Lemma 5.5 in [5] since in the definition of hyperbolic times we just replace the function $\|Df_w(\cdot)^{-1}\|$ by the Lipschitz function of the inverse branches $L_w(\cdot)$.

Let $\mathcal{B}$ be the Borel σ-algebra of M. We say that ξ is a *μ-generating partition* if

$$\bigvee_{j=0}^{+\infty} f_w^{-j}(\xi) \equiv_\mu \mathcal{B} \quad \text{for } \mathbb{P}\text{-almost every } w \in X.$$

The next result states that every non-uniformly expanding measure admits a generating partition. See a proof of this in [5].

LEMMA 3.2. *Given a non-uniformly expanding measure η with exponent $c > 0$, consider $\tilde{\delta} = \tilde{\delta}(c) > 0$ as in Lemma 3.1. Then any measurable partition $\mathcal{P}$ of M with diameter less than $\tilde{\delta}$ is an η-generating partition.*

3.3. *Projective metrics.* To finish this section we present the definition of projective metrics associated to convex cones. This theory was introduced by Birkhoff [10] and provides an interesting way to obtain spectral properties of the transfer operator (see, for instance, [8, 19]).

Consider a Banach space V. We say that a subset $C \subset V \setminus \{0\}$ is a *cone* in V if $C \cap (-C) = \{0\}$ and $\lambda \cdot v \in C$ for all $v \in C$, $\lambda > 0$. Moreover, a cone C is *convex* if $v, w \in C$ and $\lambda, \eta > 0$ and we have $\lambda \cdot v + \eta \cdot w \in C$. The closure of a cone C, denoted by $\bar{C}$, is the set

$$\bar{C} := \{w \in V \mid \text{there are } v \in C \text{ and } \lambda_n \to 0 \text{ such that } (w + \lambda_n v) \in C \text{ for all } n \geq 1\}.$$

We say that a cone C is *closed* if $\bar{C} = C \cup \{0\}$.

Consider a closed convex cone C. Given $v, w \in C$, define

$$A(v, w) = \sup\{t > 0 : w - tv \in C\} \quad \text{and} \quad B(v, w) = \inf\{s > 0 : sv - w \in C\},$$

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where by convention $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$. It is straightforward to check that $A(v, w)$ is finite, $B(v, w)$ is positive and $A(v, w) \leq B(v, w)$ for all $v, w \in C$. We set

$$\Theta(v, w) = \log\left(\frac{B(v, w)}{A(v, w)}\right).$$

From the properties of A and B it follows that $\Theta(v, w)$ is well defined and takes values in $[0, +\infty]$. Notice that $\Theta(v, w) = 0$ if and only if $v = tw$ for some $t > 0$. Therefore, Θ defines a pseudo-metric in the cone C, and so it induces a metric on a projective quotient space of C. This metric is called the *projective metric of* C.

It is easy to verify that the projective metric depends monotonically on the cone: if $C_1 \subset C_2$ are two convex cones in V, then $\Theta_2(v, w) \leq \Theta_1(v, w)$ for any $v, w \in C_1$, where Θ_1 and Θ_2 are the projective metrics in C_1 and C_2, respectively.

In particular, if V_1, V_2 are complete vector spaces and $L : V_1 \to V_2$ is a linear operator such that $L(C_1) \subset C_2$ for C_1, C_2 convex cones in V_1, V_2 respectively, then $\Theta_2(L(v), L(w)) \leq \Theta_1(v, w)$ for any $v, w \in C_1$, where Θ_1 and Θ_2 are the projective metrics in C_1 and C_2, respectively. The next result states that L will be a strict contraction if $L(C_1)$ has finite diameter in C_2.

THEOREM 3.2. *Let C_1 and C_2 be closed convex cones in the Banach spaces V_1 and V_2, respectively. If $L : V_1 \to V_2$ is a linear operator such that $L(C_1) \subset C_2$ and $\Delta = \operatorname{diam}_{\Theta_2}(L(C_1)) < \infty$, then*

$$\Theta_2(L(\varphi), L(\psi)) \leq (1 - e^{-\Delta}) \cdot \Theta_1(\varphi, \psi) \quad \textit{for all } \varphi, \psi \in C_1.$$

In this work we will restrict our attention to cones of locally Hölder continuous observables. We prove that, applying the last result, the transfer operator is a contraction in this setting.

We fix $\delta > 0$ as in §2 and we say that a function $\varphi : M \to \mathbb{R}$ is (C, α)-Hölder continuous in balls of radius δ if for some constant $C > 0$ it follows that

$$|\varphi(x) - \varphi(y)| \leq C d(x, y)^\alpha \quad \text{for all } y \in B(x, \delta).$$

Denote by $|\varphi|_{\alpha,\delta}$ the smallest Hölder constant of φ in balls of radius $\delta > 0$.

The next lemma states that every locally Hölder continuous function defined on a compact and connected metric space is Hölder continuous.

LEMMA 3.3. *Let M be a compact and connected metric space. Given $\delta > 0$, there exists $m \geqslant 1$ (depending only on δ) such that if $\varphi : M \to \mathbb{R}$ is (C, α)-Hölder continuous in balls of radius δ then it is (Cm, α)-Hölder continuous.*

Proof. The compactness allows us to cover M with N balls of radius δ where N depends only on δ. Moreover, since M is connected, given $x, y \in M$ there are $z_0 = x, z_1 \ldots z_{N+1} = y$ satisfying $d(z_i, z_{i+1}) \leq \delta$ and $d(z_i, z_{i+1}) \leq d(x, y)$ for all

$i = 0, \ldots, N$. Since φ is (C, α)-Hölder continuous in balls of radius δ we have that

$$|\varphi(x) - \varphi(y)| \leq \sum_{i=0}^{N} |\varphi(z_i) - \varphi(z_{i+1})| \leq \sum_{i=0}^{N} Cd(z_i, z_{i+1})^\alpha \leq C(N+1)d(x, y)^\alpha$$

which implies that φ is $(C \cdot m, \alpha)$-Hölder continuous for $m = N + 1$. $\square$

Notice that the same argument used in the lemma above gives an estimate for the Hölder constant of φ in balls of radius $(1 + r)\delta$ for $0 < r \leq 1$. Indeed, let $r \in [0, 1]$ and $x, y \in M$ with $d(x, y) < (1 + r)\delta$. Since M is connected there exists $z \in M$ such that $d(x, z) = \delta$ and $d(z, y) < rd(x, z)$. Thus,

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq |\varphi(x) - \varphi(z)| + |\varphi(z) - \varphi(y)| \\ &\leq Cd(x, z)^\alpha + Cd(z, y)^\alpha \leq C(1 + r^\alpha)d(x, y)^\alpha. \end{aligned}$$

Therefore, we conclude that if $\varphi : M \to \mathbb{R}$ is (C, α)-Hölder continuous in balls of radius δ then φ is $(C(1 + r^\alpha), \alpha)$-Hölder continuous in balls of radius $(1 + r)\delta$ for each $0 < r \leq 1$.

For each $k > 0$ we consider the convex cone of locally Hölder continuous observables defined on M by

$$C_\delta^k = \left\{\varphi : M \to \mathbb{R} : \varphi > 0 \text{ and } \frac{|\varphi|_{\alpha,\delta}}{\inf \varphi} \leq k\right\}. \tag{4}$$

It follows by definition that $C_\delta^{k_1} \subset C_\delta^{k_2}$, if $k_1 \leq k_2$.

From Lemma 3.3 and from the definition of $|\varphi|_{\alpha,\delta}$ we have that

$$\sup \varphi - \inf \varphi \leq |\varphi|_{\alpha,\delta} \cdot m \cdot d(x, y)^\alpha \leq (\inf \varphi \cdot k) \cdot m \cdot (\operatorname{diam} M)^\alpha, \tag{5}$$

and thus $\sup \varphi \leq \inf \varphi \cdot (1 + m(\operatorname{diam} M)^\alpha k)$ for any $\varphi \in C_\delta^k$.

In the cone C_δ^k of locally Hölder continuous observables we can give a more explicit expression for the projective metric. We refer the reader to [12] for its proof.

LEMMA 3.4. *The projective metric Θ_k in the cone C_δ^k is given by*

$$\Theta_k(\varphi, \psi) = \log\left(\frac{B_k(\varphi, \psi)}{A_k(\varphi, \psi)}\right),$$

where

$$A_k(\varphi, \psi) := \inf_{d(x,y)<\delta, z\in M} \frac{k|x - y|^\alpha \psi(z) - (\psi(x) - \psi(y))}{k|x - y|^\alpha \varphi(z) - (\varphi(x) - \varphi(y))}$$

and

$$B_k(\varphi, \psi) := \sup_{d(x,y)<\delta, z\in M} \frac{k|x - y|^\alpha \psi(z) - (\psi(x) - \psi(y))}{k|x - y|^\alpha \varphi(z) - (\varphi(x) - \varphi(y))}.$$

In particular, we have that

$$A_k(\varphi, \psi) \leq \inf_{x\in M}\left\{\frac{\varphi(x)}{\psi(x)}\right\} \quad \textit{and} \quad B_k(\varphi, \psi) \geq \sup_{x\in M}\left\{\frac{\varphi(x)}{\psi(x)}\right\}.$$

From the expression for the projective metric in the cone C_δ^k one can prove that its diameter is finite for k large enough; see [12].

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PROPOSITION 3.1. *For $0 < \gamma < 1$, the cone $C_\delta^{\gamma k}$ has finite diameter in C_δ^k.*

4. *Reference measure*

For $w \in X$, let $f_w : M \to M$ be the fiber dynamics and $\phi_w : M \to \mathbb{R}$ be the potential. Let $\mathcal{L}_w : C^0(M) \to C^0(M)$ be the transfer operator associated to (f_w, ϕ_w), defined by

$$\mathcal{L}_w(\psi)(x) = \sum_{y \in f_w^{-1}(x)} e^{\phi_w(y)} \psi(y).$$

Consider also its dual operator $\mathcal{L}_w^* : [C^0(M)]^* \to [C^0(M)]^*$ which satisfies

$$\int \psi \, d\mathcal{L}_w^*(\rho_{\theta(w)}) = \int \mathcal{L}_w(\psi) \, d\rho_{\theta(w)}.$$

We say that a probability measure $\nu_w \in \mathcal{M}^1(M)$ is a *reference measure* associated to $\lambda_w \in \mathbb{R}$ if ν_w satisfies

$$\mathcal{L}_w^*(\nu_{\theta(w)}) = \lambda_w \nu_w.$$

As in the deterministic case, by applying the Schauder–Tychonoff fixed point theorem, it is straightforward to prove the existence of a system of reference measures $\{\nu_w\}_{w \in X}$ where ν_w is associated to λ_w given by

$$\lambda_w = \mathcal{L}_w^* \nu_{\theta(w)}(1) = \nu_{\theta(w)}(\mathcal{L}_w(1)) \tag{6}$$

for $\mathbb{P}$-almost every $w \in X$. See [20] for details. In what follows we derive some properties of the reference measure.

The *Jacobian* of a measure η with respect to f is a measurable function $J_\eta f$ such that

$$\eta(f(A)) = \int_A J_\eta f \, d\eta,$$

for any measurable set A where $f|_A$ is injective.

LEMMA 4.1. *The Jacobian of ν_w with respect to f_w is given by $J_{\nu_w} f_w = \lambda_w e^{-\phi_w}$. Moreover, ν_w is an open measure. In particular,* $\operatorname{supp}(\nu_w) = M$.

Proof. Let $A \subset M$ be a measurable set such that $f_w|_A$ is injective. Notice that, for any bounded sequence $\{\zeta_n\} \in C^0(M)$ which converges to the characteristic function $\mathcal{X}_A$ of A, we have

$$\begin{aligned}\int_M \lambda_w e^{-\phi_w} \zeta_n \, d\nu_w &= \int_M e^{-\phi_w} \zeta_n \, d(\mathcal{L}_w^* \nu_{\theta(w)}) = \int_M \mathcal{L}_w(e^{-\phi_w} \zeta_n)(y) \, d\nu_{\theta(w)}(y) \\ &= \int_M \sum_{f_w(z)=y} \zeta_n(z) \, d\nu_{\theta(w)}(y) = \int_M \sum_{f_w(z)=y} \zeta_n(f_w^{-1}(y)) \, d\nu_{\theta(w)}(y).\end{aligned}$$

Since $\int_M \sum_{f_w(z)=y} \zeta_n(f_w^{-1}(y)) \, d\nu_{\theta(w)}(y)$ converges to $\int_M \mathcal{X}_A(f_w^{-1}(y)) \, d\nu_{\theta(w)}(y)$ and $\int_M \mathcal{X}_A(f_w^{-1}(y)) \, d\nu_{\theta(w)}(y) = \int_M \mathcal{X}_{f_w(A)} \, d\nu_{\theta(w)} = \nu_{\theta(w)}(f_w(A))$ we conclude that

$$\nu_{\theta(w)}(f_w(A)) = \int_A \lambda_w e^{-\phi_w} \, d\nu_w.$$

Moreover, by induction, we obtain for every $n \in \mathbb{N}$ that

$$\nu_{\theta^n(w)}(f_w^n(A)) = \int_A \lambda_w^n e^{-S_n\phi_w}\, d\nu_w, \tag{7}$$

where $\lambda_w^n = \lambda_{\theta^{n-1}(w)}\lambda_{\theta^{n-2}(w)} \cdots \lambda_{\theta(w)}\lambda_w$.

Now we prove that ν_w is an open measure. By contradiction, suppose the existence of some non-empty open set $U_w \subset M$ such that $\nu_w(U_w) = 0$. By the exactness assumption, we can take $\tilde{n} \in \mathbb{N}$ such that $f_w^{\tilde{n}}(U_w) = M$. Partitioning U_w into mensurable subsets $U_{w,1} \ldots U_{w,k}$ where $f_w^{\tilde{n}}|_{U_{w,j}}$ is injective for $j = 1 \ldots k$, we have

$$\nu_{\theta^{\tilde{n}}(w)}(M) \leq \sum_{j=1}^{k} \nu_{\theta^{\tilde{n}}(w)}(f_w^{\tilde{n}}(U_{w,j})) = \sum_{j=1}^{k} \int_{U_{w,j}} J_{\nu_w} f_w^{\tilde{n}}\, d\nu_w = 0$$

which is a contradiction. This completes the proof. $\square$

In the next proposition we show that the family $\{\nu_w\}_w$ satisfies a Gibbs property at hyperbolic times.

PROPOSITION 4.1. *Let n be a hyperbolic time for (w, x). For every $0 < \varepsilon \leq \tilde{\delta}$ there exist $K_\varepsilon(w) > 0$ and $0 < \gamma_\varepsilon(\theta^n(w)) \leq 1$ such that, for all $y \in B_w(x, n, \varepsilon)$,*

$$\gamma_\varepsilon(\theta^n(w))K_\varepsilon(w)^{-1} \leq \frac{\nu_w(B_w(x, n, \varepsilon))}{\exp(S_n\phi_w(y) - \log \lambda_w^n)} \leq K_\varepsilon(w)$$

where $S_n\phi_w(y) = \sum_{j=0}^{n-1} \phi_{\theta^j(w)}(f_w^j(y))$ and $\lambda_w^n = \lambda_w\lambda_{\theta(w)} \cdots \lambda_{\theta^{n-1}(w)}$.

Proof. Fix $0 < \varepsilon \leq \tilde{\delta}$. From Lemma 3.1 and condition (IV) we get

$$\begin{aligned} |S_n\phi_w(z) - S_n\phi_w(y)| &\leq \sum_{k=0}^{n-1} |\phi_{\theta^{n-k}(w)}(f_w^{n-k}(z)) - \phi_{\theta^{n-k}(w)}(f_w^{n-k}(y))| \\ &\leq \sum_{k=0}^{n-1} |\phi_{\theta^{n-k}(w)}|_\alpha e^{-ck/2} d(f_w^n(z), f_w^n(y)) \\ &\leq \varepsilon \sum_{k=0}^{\infty} |\phi_{\theta^k(w)}|_\alpha e^{-ck/2} \leq K_\varepsilon(w) \end{aligned}$$

for every $z, y \in B_w(x, n, \varepsilon)$. By once again applying Lemma 3.1 we know that f_w^n maps homeomorphically $B_w(x, n, \varepsilon)$ into the ball $B_{\theta^n(w)}(f_w^n(x), \varepsilon)$. Hence, since the Jacobian of ν_w is bounded away from zero and infinity we can write

$$0 < \gamma_\varepsilon(\theta^n(w)) \leq \nu_{\theta^n(w)}(f_w^n(B_w(x, n, \varepsilon))) = \int_{B_w(x,n,\varepsilon)} \lambda_w^n e^{-S_n\phi_w(z)}\, d\nu_w \leq 1$$

where $\gamma_\varepsilon(\theta^n(w))$ depends only on the radius ε of the ball $B_{\theta^n(w)}(f_w^n(x), \varepsilon)$. Therefore, for every $y \in B_w(x, n, \varepsilon)$ it follows that

$$\begin{aligned} \gamma_\varepsilon(\theta^n(w)) \leq \int_{B_w(x,n,\varepsilon)} \lambda_w^n e^{-S_n\phi_w(z)}\, d\nu_w &= \int_{B_w(x,n,\varepsilon)} \lambda_w^n e^{-S_n\phi_w(y)} \left(\frac{\lambda_w^n e^{-S_n\phi_w(z)}}{\lambda_w^n e^{-S_n\phi_w(y)}}\right) d\nu_w \\ &\leq K_\varepsilon(w) e^{-S_n\phi_w(y) + \log \lambda_w^n} \nu_w(B_w(x, n, \varepsilon)). \end{aligned}$$

Applying the same argument, we have that

$$e^{-S_n\phi_w(y)+\log\lambda_w^n}\nu_w(B_w(x,n,\varepsilon)) \le K_\varepsilon(w)\int_{B_w(x,n,\varepsilon)}\lambda_w^n e^{-S_n\phi_w(y)}\left(\frac{\lambda_w^n e^{-S_n\phi_w(z)}}{\lambda_w^n e^{-S_n\phi_w(y)}}\right)d\nu_w$$

which completes the proof. □

Remark 4.1. It is possible to obtain a lower bound for $\gamma_\varepsilon(\theta^n(w))$. Indeed, from hypothesis we may find $\tilde{n}=\tilde{n}(w,\varepsilon)$ such that $f^{\tilde{n}}_{\theta^n(w)}(B_{\theta^n(w)}(f^n_w(x),\varepsilon))=M$ and by definition of Jacobian it follows that

$$\begin{aligned}1 &= \nu_{\theta^{n+\tilde{n}}(w)}(f^{\tilde{n}}_{\theta^n(w)}(B_{\theta^n(w)}(f^n_w(x),\varepsilon)))\\ &\le \int_{B_{\theta^n(w)}(f^n_w(x),\varepsilon)}\lambda^{\tilde{n}}_{\theta^n(w)}e^{-S_{\tilde{n}}\phi_{\theta^n(w)}}\,d\nu_{\theta^n(w)}\\ &\le \lambda^{\tilde{n}}_{\theta^n(w)}e^{-\tilde{n}\inf\phi_{\theta^n(w)}}\nu_{\theta^n(w)}(B_{\theta^n(w)}(f^n_w(x),\varepsilon)).\end{aligned}$$

Thus, $e^{\tilde{n}\inf\phi_{\theta^n(w)}-\log\lambda^{\tilde{n}}_{\theta^n(w)}}\le\gamma_\varepsilon(\theta^n(w))$. Since $\tilde{n}$ depends only on $w\in X$ and $\varepsilon>0$ we conclude that $\gamma_\varepsilon(\theta^n(w))$ is uniformly bounded.

Consider $c>0$ given by condition (VI). Given $w\in X$, let $H_w\subset M$ be the subset of M such that (w,x) has infinitely many hyperbolic times, that is,

$$H_w := \left\{x\in M;\ \limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log L_{\theta^j(w)}(f^j_w(x))\leqslant -2c<0\right\}.$$

We next prove that $\nu_w(H_w)=1$ for $\mathbb{P}$-almost every $w\in X$.

Recall that we fix $\varepsilon_\phi>0$ small satisfying $\varepsilon_\phi<\inf_w(\log(\deg f_w)-\log q_w)$. In view of (IV) we may find $0<\varepsilon_0<\varepsilon_\phi$ such that

$$\sup\phi_w-\inf\phi_w+\varepsilon_0<\log(\deg f_w)-\log q_w\quad\text{for all } w\in X. \tag{8}$$

Let $\mathcal{P}$ be a partition of M with cardinality $\#\mathcal{P}=k$. We suppose without loss of generality that the set $\mathcal{A}_w$ is contained in the first q_w elements of $\mathcal{P}$ for all $w\in X$. Consider the numbers

$$\bar{p}_w=k-q_w,\quad \hat{q}=\sup_{w\in X}q_w,\quad \bar{q}=\inf_{w\in X}q_w\quad\text{and}\quad \hat{p}=\sup_{w\in X}\bar{p}_w.$$

These numbers are well defined since we assume that $\deg(F)=\sup_w\deg(f_w)<\infty$.

For $\rho\in(0,1)$ and $n\in\mathbb{N}$ let $I(\rho,n)$ be the set of itinerates

$$I(\rho,n)=\{(i_w\ldots i_{\theta^{n-1}(w)})\in\{1\ldots k\}^n;\ \#\{0\le j\le n-1: i_{\theta^j(w)}\le q_{\theta^j(w)}\}>\rho n\}$$

and consider

$$C_\rho := \limsup_n\frac{1}{n}\log\#I(\rho,n).$$

LEMMA 4.2. [29, Lemma 3.1] *Given $\varepsilon>0$, there exists $\rho_0\in(0,1)$ such that $C_\rho<\log\hat{q}+\varepsilon$ for every $\rho\in(\rho_0,1)$.*

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Proof. Notice that $\#I(\rho, n) \le \sum_{k=[\rho n]}^{n} \binom{n}{k} q_w q_{\theta(w)} \cdots q_{\theta^{k-1}(w)} p_w p_{\theta(w)} \cdots p_{\theta^{n-(k-1)}(w)}$. By applying Stirling's formula we have

$$\sum_{k=[\rho n]}^{n} \binom{n}{k} = \frac{n}{2}\binom{n}{[\rho n]} \le C_1 \exp\,(2t(1-\rho)n) \quad \text{for } \rho > \frac{1}{2}.$$

Thus, there exist C_1 and $t > 0$ such that $\#I(\rho, n) \le C_1 \exp\,(2t(1-\rho)n)\ \hat{q}^n \hat{p}^{(1-\rho)n}$. Taking the limit as n goes to infinity, we have

$$C_\rho = \limsup_n \frac{1}{n} \log \#I(\rho, n) \le \log \hat{q} + \varepsilon$$

for any ρ close enough to 1. $\square$

From this lemma we can fix $\rho < 1$ such that

$$C_\rho < \log \hat{q} + \frac{\varepsilon_0}{4}.$$

Recalling equation (8) and the definition of λ_w in (6), we have that

$$\lambda_w \ge \deg f_w e^{\inf \phi_w} \ge e^{\log(\deg f_w)+\sup \phi_w - \log(\deg f_w)+\log q_w+\varepsilon_0} = e^{(\log q_w+\sup \phi_w+\varepsilon_0)}.$$

Now, using Lemma 4.1, we obtain that

$$J_{\nu_w} f_w = \lambda_w e^{-\phi_w} \ge e^{(\sup \phi_w+\log q_w+\varepsilon_0-\phi_w)} \ge e^{\log q_w+\varepsilon_0} > q_w. \tag{9}$$

PROPOSITION 4.2. *We have $\nu_w(H_w) = 1$ for almost every $w \in X$.*

Proof. Given $n \in \mathbb{N}$, denote by $B_w(n)$ the set of points $x \in M$ whose frequency of visits to $\{\mathcal{A}_{\theta^j(w)}\}_{0\le j\le n-1}$ up to time n is at least ρ, that is,

$$B_w(n) = \bigg\{x \in M \,\bigg|\, \frac{1}{n}\#\{0 \le j \le n-1 : f_w^j(x) \in A_{\theta^j(w)}\} \ge \rho\bigg\}.$$

Let $\mathcal{P}^{(n)}$ be the partition $\bigvee_{j=0}^{n-1}(f_w^j)^{-1}\mathcal{P}$. We cover $B_w(n)$ by elements of $\mathcal{P}^{(n)}$ and since f_w^n is injective on every $P \in \mathcal{P}^{(n)}$, we may use (9) to obtain

$$\begin{aligned} 1 \ge \nu_{\theta^n(w)}(f_w^n(P)) &= \int_P J_{\nu_w}(f_w^n)\, d\nu_w = \int_P \prod_{j=0}^{n-1} J_{\nu_{\theta^j(w)}} f_{\theta^j(w)}\, d\nu_w \\ &\ge \prod_{j=0}^{n-1} e^{(\log q_{\theta^j(w)}+\varepsilon_0)} \nu_w(P) \ge e^{(\log \bar{q}+\varepsilon_0)n} \nu_w(P). \end{aligned}$$

Thus,

$$\nu_w(P) \le e^{-(\log \bar{q}+\varepsilon_0)n}.$$

Since we can assume $\hat{q} < \bar{q} e^{\varepsilon_0/2}$ it follows that

$$\nu_w(B_w(n)) \le \#I(\rho, n) e^{-(\log \bar{q}+\varepsilon_0)n} \quad \le e^{(\log \hat{q}+\varepsilon_0/4)n} e^{-(\log \bar{q}+\varepsilon_0)n} \le e^{(\log \hat{q}/\bar{q}-\varepsilon_0/2)n}.$$

Hence, the measure $\nu_w(B_w(n))$ decreases exponentially fast as n goes to infinity. Applying the Borel–Cantelli lemma, we conclude that ν_w-almost every point belongs to $B_w(n)$ for at most finitely many values of n. Then, in view of (VI) we obtain for n large enough that

$$\sum_{j=0}^{n-1} \log L_{\theta^j(w)}(f_w^j(x)) \leq \rho \log \tilde{L}_w + (1-\rho) \log \tilde{\sigma}_w^{-1} \leq -2c < 0$$

which proves that ν_w-almost every point has infinitely many hyperbolic times. □

Notice that from the last proposition, and recalling that ν_w is an open measure, we conclude that H_w is dense in M.

5. *Transfer operator*

Here we prove Theorems A and B. We use the projective metric approach to show that the transfer operator is a contraction in some cone of locally Hölder continuous functions. This contraction implies the existence of the invariant family $\{h_w\}_w$ uniformly bounded away from zero and infinity. Recalling the reference measure ν_w constructed in the previous section, we define the probability measure $\mu_w := h_w \nu_w$. From the exponential approximation of functions in the cone to the family $\{h_w\}_w$ we derive that μ_w has an exponential decay of correlations.

5.1. *Invariant family.*

For the construction of the invariant family $\{h_w\}_w$ we follow the ideas of Castro and Varandas [12].

Let $\delta > 0$ be as in §2 and consider for each $k > 0$ the cone of locally Hölder continuous functions

$$C_\delta^k(w) = \left\{ \varphi_w : M \to \mathbb{R} : \varphi_w > 0 \text{ and } \frac{|\varphi_w|_{\alpha,\delta}}{\inf \varphi_w} \leq k \right\}. \tag{10}$$

Since the cone does not depend on w, we denote this by C_δ^k. The next proposition shows its invariance by the transfer operator.

PROPOSITION 5.1. *For every $w \in X$, there exists $0 < \gamma_w < 1$ such that*

$$\mathcal{L}_w(C_\delta^k) \subset C_\delta^{\gamma_w k} \subset C_\delta^k$$

for some positive constant k large enough.

Proof. For any $\varphi \in C_\delta^k$ we will show that

$$\frac{|\mathcal{L}_w(\varphi)|_{\alpha,\delta}}{\inf \mathcal{L}_w(\varphi)} \leq \gamma_w k \quad \text{for some } 0 < \gamma_w < 1.$$

Given $y, z \in M$ satisfying $d(y,z) < \delta$, we denote by y_j, z_j, $1 \leq j \leq \deg(f_w)$, the respective preimages under f_w. Note that for any continuous function φ we have

$$\mathcal{L}_w(\varphi)(y) = \sum_{j=1}^{\deg(f_w)} e^{\phi_w(y_j)} \varphi(y_j) \geq \deg(f_w) e^{\inf \phi_w} \inf \varphi. \tag{11}$$

From the definition of $\mathcal{L}_w$ and the constant $|\mathcal{L}_w(\varphi)|_{\alpha,\delta}$ we obtain

$$\frac{|\mathcal{L}_w(\varphi)|_{\alpha,\delta}}{\inf \mathcal{L}_w(\varphi)} = \sup_{d(y,z)<\delta} \frac{|\mathcal{L}_w(\varphi(y)) - \mathcal{L}_w(\varphi(z)|}{\inf \mathcal{L}_w(\varphi)\, d(y,z)^\alpha}$$
$$\leq \sup_{d(y,z)<\delta} \sum_{j=1}^{\deg(f_w)} \frac{|e^{\phi_w(y_j)}\varphi(y_j) - e^{\phi_w(z_j)}\varphi(z_j)|}{\inf \mathcal{L}_w(\varphi)\, d(y,z)^\alpha}.$$

By equation (11) the last inequality is less than or equal to

$$\sup_{d(y,z)<\delta} \sum_{j=1}^{\deg(f_w)} \bigg(\frac{e^{\sup \phi_w}|\varphi(y_j) - \varphi(z_j)|}{\deg(f_w) e^{\inf \phi_w} \inf \varphi\, d(y,z)^\alpha} + \frac{\sup \varphi |e^{\phi_w(y_j)} - e^{\phi_w(z_j)}|}{\deg(f_w) e^{\inf \phi_w} \inf \varphi\, d(y,z)^\alpha} \bigg). \tag{12}$$

Recall that, by hypothesis, we can take some $x \in M$ and neighborhood $U_x \subset M$ such that $y, z \in B_{\theta(w)}(f_w(x), \delta)$ and so the inverse branches satisfy

$$d(f_w^{-1}(y), f_w^{-1}(z)) \leq L_w(x)\, d(y,z).$$

Moreover, we are assuming that every point has q_w preimages in the region $\mathcal{A}$ and p_w preimages in the expanding region $M \setminus \mathcal{A}$ where $L_w \leq \sigma_w^{-1}$. Since, by Lemma 3.3, φ is $((1 + (L_w - 1)^\alpha)|\varphi|_{\alpha,\delta}, \alpha)$-Hölder continuous in balls of radius $L_w\delta$ we conclude that the sum (12) is bounded from above by

$$\sum_{j=1}^{q_w} \bigg(\frac{e^{\sup \phi_w}(1 + (L_w - 1)^\alpha)|\varphi|_{\alpha,\delta} L_w^\alpha d(y,z)^\alpha}{\deg(f_w) e^{\inf \phi_w} \inf \varphi\, d(y,z)^\alpha} + \frac{\sup \varphi |e^{\phi_w}|_\alpha L_w^\alpha d(y,z)^\alpha}{\deg(f_w) e^{\inf \phi_w} \inf \varphi\, d(y,z)^\alpha} \bigg)$$
$$+ \sum_{j=1}^{p_w} \bigg(\frac{e^{\sup \phi_w}|\varphi|_{\alpha,\delta} \sigma_w^{-\alpha} d(y,z)^\alpha}{\deg(f_w) e^{\inf \phi_w} \inf \varphi d(y,z)^\alpha} + \frac{\sup \varphi |e^{\phi_w}|_\alpha L_w^\alpha d(y,z)^\alpha}{\deg(f_w) e^{\inf \phi_w} \inf \varphi\, d(y,z)^\alpha} \bigg).$$

And this expression is equal to

$$\frac{e^{\sup \phi_w}[p_w \sigma_w^{-\alpha} + q_w L_w^\alpha(1 + (L_w - 1)^\alpha)]\, |\varphi|_{\alpha,\delta}}{\deg(f_w) e^{\inf \phi_w} \inf \varphi} + \frac{\sup \varphi |e^{\phi_w}|_\alpha L_w^\alpha}{e^{\inf \phi_w} \inf \varphi}.$$

Using inequality (5), the definition of cone and condition (IV), it follows that the sum above is less than or equal to

$$\bigg[e^{\varepsilon_\phi} \bigg[\frac{p_w \sigma_w^{-\alpha} + q_w L_w^\alpha (1 + (L_w - 1)^\alpha)}{\deg(f_w)} \bigg] + \varepsilon_\phi L_w^\alpha [1 + m(\operatorname{diam} M)^\alpha] \bigg] k.$$

By hypothesis (condition (V)), there exists some positive constant $0 < \gamma_w < 1$ such that the previous sum is bounded from above by $\gamma_w k$. This finishes the proof. $\square$

From the last proposition we have the invariance of the cone C_δ^k. Since this cone has finite diameter, according to Proposition 3.1, we can apply Theorem 3.2 to conclude the next result.

PROPOSITION 5.2. *For every $w \in X$ the operator $\mathcal{L}_w$ is a contraction in the cone C_δ^k, that is, writing $\Delta_w = \operatorname{diam}_{\Theta_k}(C_\delta^{\gamma_w k}) > 0$, it follows that*

$$\Theta_k(\mathcal{L}_w(\varphi), \mathcal{L}_w(\psi)) \leq (1 - e^{-\Delta_w}) \cdot \Theta_k(\varphi, \psi) \quad \textit{for all } \varphi, \psi \in C_\delta^k.$$

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Since we assume in condition (V) the existence of $\gamma \in (0, 1)$ such that $\gamma_w \leq \gamma$ for all $w \in X$ we conclude that

$$\Theta_k(\mathcal{L}_w(\varphi), \mathcal{L}_w(\psi)) \leq (1 - e^{-\Delta}) \cdot \Theta_k(\varphi, \psi) \quad \text{for all } \varphi, \psi \in C^k_\delta \text{ and } w \in X$$

where $\Delta = \sup_w(\Delta_w) \leq \mathrm{diam}_{\Theta_k}(C^{\gamma k}_\delta)$.

Let $\{\nu_w\}_w$ be the family of reference measures and $\lambda_w = \nu_{\theta(w)}(\mathcal{L}_w(1))$. The contraction in the cone allows us to prove the existence of the family $\{h_w\}_w$ invariant by the transfer operator.

PROPOSITION 5.3. *For almost $w \in X$ there exists a Hölder continuous function $h_w : M \to \mathbb{R}$ bounded away from zero and infinity satisfying $\mathcal{L}_w h_w = \lambda_w h_{\theta(w)}$.*

Proof. Consider the normalized operator $\hat{\mathcal{L}}_w := \lambda_w^{-1}\mathcal{L}_w$ and define the sequence $(\varphi_n)_n$ by $\varphi_n := \hat{\mathcal{L}}^n_{\theta^{-n}(w)}(\mathbf{1})$ where

$$\hat{\mathcal{L}}^n_{\theta^{-n}(w)} := \hat{\mathcal{L}}_{\theta^{-1}(w)} \circ \hat{\mathcal{L}}_{\theta^{-2}(w)} \circ \cdots \circ \hat{\mathcal{L}}_{\theta^{-(n-1)}(w)} \circ \hat{\mathcal{L}}_{\theta^{-n}(w)}$$

for each $n \geq 0$. By definition of conformal measure we have

$$\int \varphi_n \, d\nu_w = \int \hat{\mathcal{L}}^n_{\theta^{-n}(w)}(\mathbf{1}) \, d\nu_w = \int \mathbf{1} \, d(\hat{\mathcal{L}}^*)^n_{\theta^{-n}(w)}\nu_w = \int \mathbf{1} \, d\nu_{\theta^{-n}(w)} = 1.$$

Hence, each term φ_n satisfies $\sup \varphi_n \geq 1$ and $\inf \varphi_n \leq 1$. Since $\mathbf{1} \in C^k_\delta$ and C^k_δ is invariant, it follows that $\varphi_n \in C^k_\delta$ and so, applying inequality (5), we obtain that the sequence $(\varphi_n)_n$ is uniformly bounded away from zero and infinity by

$$\frac{1}{R} \leq \inf \varphi_n \leq 1 \leq \sup \varphi_n \leq R.$$

where $R = (1 + mk\mathrm{diam}(M)^\alpha)$. Moreover, as φ_n is C-Hölder continuous in balls of radius δ, by Lemma 3.3 we obtain that φ_n is a Cm-Hölder continuous function.

Next we prove that $(\varphi_n)_n$ is a Cauchy sequence in the C^0-norm. From Proposition 5.2, for every $m, l \geq n$ the projective metric satisfies

$$\Theta_k(\varphi_m, \varphi_l) = \Theta_k(\hat{\mathcal{L}}^m_{\theta^{-m}(w)}(\mathbf{1}), \hat{\mathcal{L}}^l_{\theta^{-l}(w)}(\mathbf{1})) \leq \Delta\tau^n \quad \text{where } \tau := 1 - e^{-\Delta}.$$

Recalling the expression for the projective metric $\Theta_k(\varphi_m, \varphi_l) = \log(B_k(\varphi_m, \varphi_l)/A_k(\varphi_m, \varphi_l))$, we apply Lemma 3.4 to obtain

$$e^{-\Delta\tau^n} \leq A_k(\varphi_m, \varphi_l) \leq \inf \frac{\varphi_m}{\varphi_l} \leq 1 \leq \sup \frac{\varphi_m}{\varphi_l} \leq B_k(\varphi_m, \varphi_l) \leq e^{\Delta\tau^n}.$$

Thus, for all $m, l \geq n$, we have

$$\|\varphi_m - \varphi_l\|_\infty \leq \|\varphi_l\|_\infty \left\| \frac{\varphi_m}{\varphi_l} - 1 \right\|_\infty \leq R(e^{\Delta\tau^n} - 1) \leq \tilde{R}\tau^n$$

which proves that $(\varphi_n)_n$ is a Cauchy sequence. Hence, $(\varphi_n)_n$ converges uniformly to a function $h_w : M \to \mathbb{R}$ in the cone C^k_δ satisfying $\int h_w \, d\nu_w = 1$ for almost every $w \in X$. In particular, this function is Hölder continuous and uniformly bounded away from zero and infinity. To complete the proof of the proposition, we will show that $\mathcal{L}_w h_w = \lambda_w h_{\theta(w)}$.

Consider the sequence

$$\tilde{\varphi}_{n,w} := \frac{1}{n}\sum_{j=0}^{n-1}\varphi_j = \frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}^j_{\theta^{-j}(w)}(\mathbf{1}).$$

By what we have proved above, $(\tilde{\varphi}_{n,w})$ converges uniformly to h_w for almost every $w \in X$. From the continuity of $\mathcal{L}_w$ we obtain

$$\begin{aligned}\hat{\mathcal{L}}_w(h_w) = \lim_{n\to+\infty}\hat{\mathcal{L}}_w(\tilde{\varphi}_{n,w}) &= \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}_w(\hat{\mathcal{L}}^j_{\theta^{-j}(w)}(\mathbf{1}))\\ &= \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}^j_{\theta^{-j}(\theta(w))}(\mathbf{1}) + \frac{1}{n}(\hat{\mathcal{L}}^n_{\theta^{-n}(\theta(w))}(\mathbf{1}) - \mathbf{1}).\end{aligned}$$

Since $\hat{\mathcal{L}}^n_{\theta^{-n}(\theta(w))}(\mathbf{1})$ is uniformly bounded we conclude that $\hat{\mathcal{L}}_w(h_w) = h_{\theta(w)}$. □

From the proof of the previous proposition we conclude that the family $\{h_w\}_{w\in X}$ is uniquely determined. Moreover, every h_w satisfies

$$\frac{1}{R} \le \inf h_w \le 1 \le \sup h_w \le R.$$

where $R = (1 + mk\,\mathrm{diam}(M)^\alpha)$.

5.2. *Measurability.* In the previous subsection we proved the existence of a family $\{h_w\}_w$ invariant under the action of the transfer operator. Here we prove that this family is measurable as well as the family $\{\nu_w\}_w$. Moreover, defining the probability measure $\mu_w := h_w\nu_w$, we also prove that μ_w has an exponential decay of correlations and that the family $\{\mu_w\}_w$ is F-invariant.

The next proposition states an exponential approximation of functions in the cone to the invariant family $\{h_w\}_w$. This is the main ingredient in the proof of the exponential decay of correlations.

PROPOSITION 5.4. *For almost $w \in X$ there exist constants $K > 0$ and $0 < \tau < 1$ such that for every $\varphi \in C^k_\delta$ satisfying $\int \varphi\, d\nu_w = 1$ we have that*

$$\|\hat{\mathcal{L}}^n_w(\varphi) - \hat{\mathcal{L}}^n_w(h_w)\|_\infty \le K\tau^n \quad \textit{for all } n \ge 1,$$

where $\hat{\mathcal{L}}_w = \lambda_w^{-1}\mathcal{L}_w$ is the normalized operator.

Proof. Given $\varphi \in C^k_\delta$ satisfying $\int \varphi\, d\nu_w = 1$, we have for every $n \ge 1$,

$$\int \hat{\mathcal{L}}^n_w(\varphi)\, d\nu_{\theta^n(w)} = \int \varphi\, d(\mathcal{L}^*_w)^n(\nu_{\theta^n(w)}) = \int \varphi\, d\nu_w = 1.$$

Since $h_w \in C^k_\delta$ also satisfies $\int h_w\, d\nu_w = 1$, we derive for every $n \ge 1$ that

$$\inf \frac{\hat{\mathcal{L}}^n_w(\varphi)}{\hat{\mathcal{L}}^n_w(h_w)} \le 1 \le \sup \frac{\hat{\mathcal{L}}^n_w(\varphi)}{\hat{\mathcal{L}}^n_w(h_w)}.$$

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Recalling that $\hat{\mathcal{L}}_w^n(h_w) = h_{\theta^n(w)}$ for all $n \geq 1$, we apply the same projective metric argument used in the proof of Proposition 5.3 to obtain

$$\|\hat{\mathcal{L}}_w^n(\varphi) - \hat{\mathcal{L}}_w^n(h_w)\|_\infty \leq \|h_{\theta^n(w)}\|_\infty \left\| \frac{\hat{\mathcal{L}}_w^n(\varphi)}{\hat{\mathcal{L}}_w^n(h_w)} - 1 \right\|_\infty \leq R(e^{\Delta\tau^n} - 1) \leq K\tau^n. \quad \square$$

Let μ_w be the probability measure defined by $\mu_w := h_w \nu_w$. From the last proposition we derive the proof of Theorem B.

THEOREM B. *For almost every $w \in X$, the probability measure μ_w has exponential decay of correlations for Hölder continuous observables: there exists $0 < \tau < 1$ such that for any $\varphi \in L^1(\mu_{\theta^n(w)})$ and $\psi \in C^\alpha(M)$ there exists a positive constant $K(\varphi, \psi)$ satisfying for all $n \geq 1$ that*

$$\left| \int (\varphi \circ f_w^n)\psi \, d\mu_w - \int \varphi \, d\mu_{\theta^n(w)} \int \psi \, d\mu_w \right| \leq K(\varphi, \psi)\tau^n.$$

Proof. Given $\varphi \in L^1(\mu_{\theta^n(w)})$ and $\psi \in C^\alpha(M)$, we suppose without loss of generality that $\int \psi \, d\mu_w = 1$. Let $\hat{\mathcal{L}}_w = \lambda_w^{-1}\mathcal{L}_w$ be the normalized operator. As a first case we consider $\psi \cdot h_w$ in the cone C_δ^k for k large enough. Recalling that $\mu_w = h_w \nu_w$ and that $\mathcal{L}_w^* \nu_{\theta(w)} = \lambda_w \nu_w$, we have

$$\begin{aligned}
&\left| \int (\varphi \circ f_w^n)\psi \, d\mu_w - \int \varphi \, d\mu_{\theta^n(w)} \int \psi \, d\mu_w \right| \\
&\quad = \left| \int (\varphi \circ f_w^n)\psi \cdot h_w \, d\nu_w - \int \varphi \cdot h_{\theta^n(w)} \, d\nu_{\theta^n(w)} \right| \\
&\quad = \left| \int \varphi \cdot \hat{\mathcal{L}}_w^n(\psi \cdot h_w) \, d\nu_{\theta^n(w)} - \int \varphi \cdot h_{\theta^n(w)} \, d\nu_{\theta^n(w)} \right| \\
&\quad \leq \|\varphi\|_1 \|\hat{\mathcal{L}}_w^n(\psi \cdot h_w) - \hat{\mathcal{L}}_w^n(h_w)\|_\infty.
\end{aligned}$$

Since $\psi \cdot h_w \in C_\delta^k$ and $\int \psi \cdot h_w \, d\nu_w = \int \psi \, d\mu_w = 1$ we can apply Proposition 5.4 to conclude the existence of constants $K > 0$ and $0 < \tau < 1$ such that

$$\|\hat{\mathcal{L}}_w^n(\psi \cdot h_w) - \hat{\mathcal{L}}_w^n(h_w)\|_\infty \leq K\tau^n \quad \text{for every } n \geq 1.$$

For the general case we write $\psi \cdot h_w = g$ where

$$g = g^+ - g^-; \, g^\pm = \tfrac{1}{2}(|g| \pm \psi) + C \quad \text{and} \quad C = k^{-1}|\psi \cdot h_w|_{\alpha,\delta}.$$

Therefore, $g^\pm \in C_\delta^k$. From the previous estimates on $g^\pm$ and by linearity the proposition holds. $\square$

Now we state the measurability of the families $\{\nu_w\}_w$ and $\{h_w\}_w$. We start by observing that for almost every $w \in X$ and every continuous function $g \in C^0(M)$ we have

$$\frac{1}{h_{\theta^n(w)}} \hat{\mathcal{L}}_w^n(g \cdot h_w) \longrightarrow \int g \cdot h_w \, d\nu_w \quad \text{when } n \to \infty, \tag{13}$$

where $\hat{\mathcal{L}}$ is the normalized operator $\hat{\mathcal{L}}_w = \lambda_w^{-1}\mathcal{L}_w$. Indeed, we can suppose that the function g_w is Hölder continuous because any continuous function is approximated by such functions. Moreover, following the proof of Theorem 5.2, we can just consider the case $g \cdot h_w \in C_\delta^k$ for k large enough. We have that

$$\begin{aligned}\left\| \frac{1}{h_{\theta^n(w)}}\, \hat{\mathcal{L}}_w^n(gh_w) - \int gh_w\, d\nu_w \right\|_\infty &\le \left\| \frac{1}{h_{\theta^n(w)}} \right\| \left\| \hat{\mathcal{L}}_w^n(gh_w) - \int gh_w\, d\nu_w \cdot h_{\theta^n(w)} \right\| \\ &\le R\|gh_w\|_\infty \left\| \hat{\mathcal{L}}_w^n\left(\frac{gh_w}{\int gh_w\, d\nu_w} \right) - \hat{\mathcal{L}}_w^n(h_w) \right\|_\infty .\end{aligned}$$

By Proposition 5.4 the convergence in (13) follows. We use this on the next result.

LEMMA 5.1. *Let $\lambda_w = \nu_{\theta(w)}(\mathcal{L}_w(1))$. The family $\{\nu_w\}$ is uniquely determined by*

$$\mathcal{L}_w^* \nu_{\theta(w)} = \lambda_w \nu_w.$$

Moreover, the map $w \mapsto \nu_w(g_w)$ is measurable for any $g \in \mathbb{L}_{\mathbb{P}}^1(X, C^0(M))$.

Proof. Fix $w \in X$ and let (x_n) be a sequence of points in M. Define the probability

$$\nu_{w,n} = \frac{(\mathcal{L}_w^n)^* \delta_{x_n}}{\mathcal{L}_w^n 1(x_n)}.$$

Since ν_w satisfies the condition $\mathcal{L}_w^* \nu_{\theta(w)} = \lambda_w \nu_w$ we can apply the convergence (13) to conclude that for almost every $w \in X$ and any continuous function g_w we have

$$\begin{aligned}\lim_{n\to\infty} \nu_{w,n}(g_w) = \lim_{n\to\infty} \frac{(\mathcal{L}_w^n)^* \delta_{x_n}(g_w)}{\mathcal{L}_w^n 1(x_n)} &= \lim_{n\to\infty} \frac{\mathcal{L}_w^n g_w(x_n)}{\mathcal{L}_w^n 1(x_n)} \\ &= \lim_{n\to\infty} \frac{\mathcal{L}_w^n\left(\dfrac{g_w}{h_w} \cdot h_w \right)(x_n)}{\mathcal{L}_w^n\left(\dfrac{1}{h_w} \cdot h_w \right)(x_n)} = \nu_w(g_w).\end{aligned}$$

The convergence of $\nu_{w,n} \xrightarrow{w*} \nu_w$ therefore follows. Since the sequence (x_n) was arbitrary the uniqueness of the family $\{\nu_w\}$ is proved. Moreover, the equality

$$\lim_{n\to\infty} \frac{\|\mathcal{L}_w^n g_w\|_\infty}{\|\mathcal{L}_w^n 1\|_\infty} = \nu_w(g_w)$$

implies the measurability of $w \mapsto \nu_w(g_w)$ since the transfer operator is measurable. □

The lemma above enables us to define the probability measure ν on the Borel sets of $X \times M$ by

$$\nu(g) = \int_X \int_M g_w\, d\nu_w\, d\mathbb{P}(w).$$

Let $c > 0$ be given by condition (VI) and let $H \subset X \times M$ be the non-uniformly expanding set defined in ($\star$). As in §4, for $w \in X$ consider

$$H_w := \left\{x \in M; \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log L_{\theta^j(w)}(f_w^j(x)) \leqslant -2c < 0\right\}.$$

By Proposition 4.2 for almost every $w \in X$ we have $\nu_w(H_w) = 1$. Thus, we conclude that $\nu(H) = 1$, that is, ν is a non-uniformly expanding measure.

Notice that since $\lambda_w = \nu_{\theta(w)}(\mathcal{L}_w(1))$ we have that the map $w \mapsto \lambda_w \in \mathbb{R}$ is measurable. Recalling that for almost every $w \in X$ the function h_w is given by

$$h_w = \lim_{n\to\infty} \hat{\mathcal{L}}^n_{\theta^{-n}(w)} 1,$$

we deduce the measurability of the map $(w, y) \mapsto h_w(y)$ from the measurability of λ_w and the transfer operator. From this result the probability measure $\mu_{F,\phi} \in \mathcal{M}_{\mathbb{P}}(X \times M)$ is well defined by the formula

$$\mu_{F,\phi}(g) = \int_X \int_M g_w \cdot h_w \, d\nu_w \, d\mathbb{P}(w).$$

In order to prove the F-invariance of $\mu_{F,\phi}$ we first observe that

$$\begin{aligned}\mu_w(g_{\theta(w)} \circ f_w) &= \int g_{\theta(w)} \circ f_w \cdot h_w \, d\nu_w \\ &= \int \hat{\mathcal{L}}_w(g_{\theta(w)} \circ f_w \cdot h_w) \, d\nu_{\theta(w)} \\ &= \int \frac{g_{\theta(w)} \cdot \hat{\mathcal{L}}_w(h_w)}{h_{\theta(w)}} \, d\mu_{\theta(w)} = \mu_{\theta(w)}(g_{\theta(w)}).\end{aligned}$$

Therefore, for every integrable function we get

$$\int g \circ F \, d\mu_{F,\phi} = \iint g_{\theta(w)} \circ f_w(x) \, d\mu_w d\mathbb{P}(w) = \iint g_{\theta(w)} \, d\mu_{\theta(w)} \, d\mathbb{P}(w) = \int g \, d\mu_{F,\phi}.$$

This finishes the proof of Theorem A.

6. *Equilibrium states*

In this section we prove that the measure $\mu_{F,\phi}$ constructed above is an equilibrium state for $(F|_\theta, \phi)$. Moreover, we show that any ergodic non-uniformly expanding equilibrium state has disintegration absolutely continuous with respect to the system of reference measures. From this we derive the uniqueness.

As a first step, in the next proposition we obtain an upper bound for the topological pressure of the random dynamical system.

PROPOSITION 6.1. *For any potential ϕ satisfying condition (IV) we have that*

$$P_{F|\theta}(\phi) \leq \int_X \log \lambda_w \, d\mathbb{P}(w).$$

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Proof. Fix $w \in X$ such that H_w is not empty (see definition in §4) and let $\varepsilon > 0$ be small. Since every point $x \in H_w$ has infinitely many hyperbolic times, for $N > 1$ large enough we have

$$H_w \subset \bigcup_{n \geq N} \bigcup_{x \in H_n} B_w(x, n, \varepsilon),$$

where $H_n = H_n(w)$ denotes the set of points that have n as a hyperbolic time. From Lemma 3.1 each $f^n(B_w(x, n, \varepsilon))$ is the ball $B_{\theta^n(w)}(f_w^n(x), \varepsilon)$ in M, thus by applying the Besicovitch covering lemma it is straightforward to check that there exists a countable family $F_n \subset H_n$ such that every point $x \in H_n$ is covered by at most $d = d(\dim(M))$ dynamical balls $B_w(x, n, \varepsilon)$ with $x \in F_n$. Therefore,

$$\mathcal{F}_N = \{B_w(x, n, \varepsilon) : x \in F_n \text{ and } n \geq N\}$$

is a countable open covering of H_w by dynamic balls with diameter less than $\varepsilon > 0$. Recalling that H_w is dense on M and the closure of $B_w(x, n, \varepsilon)$ is a subset of $B_w(x, n, 2\varepsilon)$, we have that

$$M = \overline{H}_w \subset \bigcup_{B_w(x,n,\varepsilon) \in \mathcal{F}_N} \overline{B_w(x, n, \varepsilon)} \subset \bigcup_{B_w(x,n,\varepsilon) \in \mathcal{F}_N} B_w(x, n, 2\varepsilon).$$

Hence,

$$\mathcal{G}_N = \{B_w(x, n, 2\varepsilon) : B_w(x, n, \varepsilon) \in \mathcal{F}_N\}$$

is a countable open covering of M by dynamic balls with diameter less than 2ε.

Let $\beta > \int \log \lambda_w \, d\mathbb{P}(w)$. By the definition of topological pressure given in §3 and by applying Lemma 4.1 to each element in $\mathcal{G}_N$ we obtain

$$\begin{aligned}
m_\beta(w, \phi, F|_\theta, 2\varepsilon, N) &\leq \sum_{B_w(x,n,2\varepsilon) \in \mathcal{G}_N} e^{-\beta n + S_n \phi(B_w(x,n,2\varepsilon))} \\
&\leq \sum_{n \geq N} \gamma_{2\varepsilon}^{-1}(\theta^n(w)) K_{2\varepsilon}(w) e^{-(\beta n - \log \lambda_w^n)} \sum_{x \in F_n} \nu_w(B_w(x, n, 2\varepsilon)) \\
&\leq d K_{2\varepsilon}(w) \sum_{n \geq N} \gamma_{2\varepsilon}^{-1}(\theta^n(w)) e^{-(\beta - (1/n) \sum_{i=0}^{n-1} \log \lambda_{\theta^i(w)}) n}.
\end{aligned}$$

As from Remark 4.1 the variable $\gamma_{2\varepsilon}^{-1}(\theta^n(w))$ is uniformly bounded and by ergodicity we have $\lim_{n\to\infty}(1/n) \sum_{i=0}^{n-1} \log \lambda_{\theta^i(w)} = \int \log \lambda_w \, d\mathbb{P}(w)$ for almost every $w \in X$, we obtain as N goes to infinity that

$$m_\beta(w, \phi, F|_\theta, 2\varepsilon) = \lim_{N \to +\infty} m_\beta(w, \phi, F|_\theta, 2\varepsilon, N) = 0,$$

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for any $\beta > \int \log \lambda_w \, d\mathbb{P}(w)$ and $\varepsilon > 0$ small. Thus, we necessarily have

$$P_{F|_\theta}(w, \phi) \leq \int_X \log \lambda_w \, d\mathbb{P}(w).$$

Since this is true for $\mathbb{P}$-almost every $w \in X$, we prove the proposition. $\square$

In the next subsection we will prove that $P_{F|_\theta}(\phi) = \int \log \lambda_w \, d\mathbb{P}(w)$.

6.1. *Existence.* Consider the family $\{\nu_w\}_{w\in X}$ of reference measures and $\{h_w\}_{w\in X}$ as in Theorem A. For each $w \in X$ let μ_w be the probability measure defined by $\mu_w = h_w \nu_w$. Recalling that the Jacobian of ν_w is $J_{\nu_w} f_w = \lambda_w e^{-\phi_w}$, it is easy to verify that the Jacobian of μ_w relative to f_w is given by

$$J_{\mu_w} f_w = \frac{\lambda_w e^{-\phi_w} h_{\theta(w)} \circ f_w}{h_w}. \tag{14}$$

Consider the probability measure $\mu_{F,\phi}$ whose disintegration is $\{\mu_w\}_w$, that is,

$$\mu_{F,\phi}(g) = \int_X \int_M g_w \cdot h_w \, d\nu_w \, d\mathbb{P}(w),$$

for every continuous function $g : X \times M \to \mathbb{R}$. As we saw in §5.2, $\mu_{F,\phi}$ is F-invariant. In the next proposition we show its ergodicity.

PROPOSITION 6.2. *The probability measure $\mu_{F,\phi}$ is ergodic.*

Proof. Given a F-invariant set $A \subset X \times M$, for each $w \in X$ denote by A_w the set $A_w = \{z \in M; (w, z) \in A\}$. The F-invariance of A implies that $f_w^{-1}(A_{\theta(w)}) = A_w$. Consider $X_0 = \{w \in X; \mu_w(A_w) > 0\}$. It is straightforward to check that X_0 is a θ-invariant subset of X. Since θ is ergodic with respect to $\mathbb{P}$, we will obtain the ergodicity of $\mu_{F,\phi}$ by showing that for almost every $w \in X_0$ we have $\mu_w(A_w) = 1$, if $\mathbb{P}(X_0) > 0$.

Let φ_w be the characteristic function of A_w, that is, $\varphi_w = \mathbf{1}_{A_w}$. Notice that $\varphi_{\theta^n(w)} \circ f_w^n = \varphi_w$ holds $\mathbb{P}$-almost everywhere. Given $\psi_w \in L^1(\mu_w)$ such that $\int \psi_w \, d\mu_w = 0$, it follows from the decay correlation property of μ_w (Theorem B) that

$$\mu_w((\varphi_{\theta^n(w)} \circ f_w^n) \cdot \psi_w) \to 0 \quad \text{when } n \to +\infty.$$

And thus $\int_{A_w} \psi_w \, d\mu_w = 0$ for any $\psi_w \in L^1(\mu_w)$ satisfying $\int \psi_w \, d\mu_w = 0$. This proves that $\mu_w(A_w) = 1$ for $\mathbb{P}$-almost every $w \in X_0$ which finishes the proof. $\square$

In §5.2 we observe that the measure ν defined by

$$\nu(g) = \int_X \int_M g_w \, d\nu_w \, d\mathbb{P}(w)$$

is non-uniformly expanding. Since $\mu_{F,\phi}$ is absolutely continuous with respect to it, we have that $\mu_{F,\phi}$ is also a non-uniformly expanding measure. In particular, by Lemma 3.2, it admits a generating partition. Thus, we can use the random Rokhlin formula to express the entropy of $\mu_{F,\phi}$ in terms of its Jacobian.

THEOREM 6.1. (Random Rokhlin formula) *Let $\mu \in \mathcal{M}_{\mathbb{P}}(F)$ be an ergodic measure which admits a μ-generating partition. Then*

$$h_\mu(F|\theta) = \int \log J_\mu(F)\, d\mu = \int_X \bigg(\int_M \log J_{\mu_w} f_w(y)\, d\mu_w(y) \bigg)\, d\mathbb{P}(w),$$

where $J_{\mu_w} f_w$ denotes the Jacobian of f_w relative to μ_w.

The reader can consult [23, Theorem 1.9.7] for a proof of the last result. Now we are ready to prove that $\mu_{F,\phi}$ is an equilibrium state for $(F|\theta, \phi)$. We have

$$\begin{aligned} h_{\mu_{F,\phi}}(F|\theta) &= \int_X \int_M \log\ J_{\mu_w} f_w(y)\, d\mu_w(y)\, d\mathbb{P}(w) \\ &= \int_X \int_M \log \bigg(\frac{\lambda_w e^{-\phi_w} h_{\theta(w)} \circ f_w}{h_w} \bigg)(y)\, d\mu_w(y)\, d\mathbb{P}(w) \\ &= \int_X \int_M \log \lambda_w\, d\mu_w(y)\, d\mathbb{P}(w) - \int_X \int_M \phi_w\, d\mu_w(y)\, d\mathbb{P}(w) \\ &\quad + \int_X \int_M \bigg(\log h_{\theta(w)} \circ f_w(y) - \log h_w(y) \bigg)\, d\mu_w(y)\, d\mathbb{P}(w). \end{aligned}$$

From the F-invariance of $\mu_{F,\phi}$ we derive that

$$\int_X \int_M \bigg(\log h_{\theta(w)} \circ f_w(y) - \log h_w(y) \bigg)\, d\mu_w(y)\, d\mathbb{P}(w) = 0.$$

Thus, we can write

$$\begin{aligned} h_{\mu_{F,\phi}}(F|\theta) &= \int_X \int_M \log \lambda_w\, d\mu_w(y)\, d\mathbb{P}(w) - \int_X \int_M \phi_w\, d\mu_w(y)\, d\mathbb{P}(w) \\ &= \int_X \log \lambda_w\, d\mathbb{P}(w) - \int \phi\, d\mu_{F,\phi}. \end{aligned}$$

Applying the variational principle (1) and Proposition 6.1, it follows that

$$\int_X \log \lambda_w\, d\mathbb{P}(w) = h_{\mu_{F,\phi}}(F|\theta) + \int \phi\, d\mu_{F,\phi} \le P_{F|\theta}(\phi) \le \int_X \log \lambda_w\, d\mathbb{P}(w),$$

which implies, in particular, that $P_{F|\theta}(\phi) = \int_X \log \lambda_w\, d\mathbb{P}(w)$ and so $\mu_{F,\phi}$ is an equilibrium state.

6.2. *Uniqueness.* So far we have proved the existence of an equilibrium state for $(F|_\theta, \phi)$. Here we prove uniqueness in the set of non-uniformly expanding measures.

Let η be an ergodic non-uniformly expanding equilibrium state for $(F|_\theta, \phi)$. We prove that the disintegration of η is absolutely continuous to the reference measure. For this we use the following remark from the basic calculus.

Remark 6.1. (Jensen's inequality) Given positive numbers $p_i > 0$ and $q_i > 0$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n p_i = 1$, we have that $\sum_{i=1}^n p_i \log q_i \le \log(\sum_{i=1}^n p_i q_i)$, with equality holding if and only if the q_i are equal.

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PROPOSITION 6.3. *Consider any ergodic non-uniformly expanding equilibrium state $\eta \in \mathcal{M}_{\mathbb{P}}(F)$ of $(F|_\theta, \phi)$ and let $(\eta_w)_w$ be its disintegration. Then, for almost $w \in X$, η_w is absolutely continuous with respect to ν_w.*

Proof. We begin by proving that for almost every $w \in X$ the Jacobian of η_w is given by

$$J_{\eta_w} f_w = \frac{\lambda_w e^{-\phi_w} \cdot h_{\theta(w)} \circ f_w}{h_w}.$$

Indeed, since η is an ergodic non-uniformly expanding equilibrium state we can apply the Rokhlin formula to obtain that

$$h_\eta(F|\theta) = \int_X \bigg(\int_M \log J_{\eta_w} f_w(y)\, d\eta_w(y) \bigg)\, d\mathbb{P}(w) = \int_{X\times M} \log J_{\eta_w} f_w(y)\, d\eta(w, y).$$

Recalling that h_w is a bounded function and that the Jacobian of μ_w is given by $J_{\mu_w} f_w = \lambda_w e^{-\phi_w} \cdot h_{\theta(w)} \circ f_w / h_w$, we have

$$\begin{aligned}
&\int \log \frac{J_{\eta_w} f_w}{J_{\mu_w} f_w}\, d\eta(w, y) \\
&= \int \log J_{\eta_w} f_w \, d\eta - \int \bigg(\log \lambda_w - \phi_w + \log \frac{h_{\theta(w)} \circ f_w}{h_w} \bigg)\, d\eta \\
&= h_\eta(F|\theta) - P_{F|\theta}(\phi) + \int \phi_w + \log h_w - \log h_{\theta(w)} \circ f_w \, d\eta \geq 0.
\end{aligned}$$

From the definition of Jacobian we can write

$$\int \sum_{z=f_w^{-1}(y)} J_{\eta_w} f_w^{-1}(z) \log \frac{J_{\eta_w} f_w}{J_{\mu_w} f_w}(z)\, d\eta(w, y) = \int \log \frac{J_{\eta_w} f_w}{J_{\mu_w} f_w}(y)\, d\eta(w, y) \geq 0. \tag{15}$$

Take $p_i = J_{\eta_w} f_w^{-1}(z_i)$ and $q_i = J_{\eta_w} f_w(z_i)/J_{\mu_w} f_w(z_i)$ where the z_i are the preimages of y. Since $(f_w)_* \eta_w = \eta_{\theta(w)}$ we have $\sum_{i=1}^{\deg(f_w)} p_i = \sum_{z=f_w^{-1}(y)} J_{\eta_w} f_w^{-1}(z) = 1$ for $\eta_{\theta(w)}$-almost every $y \in M$. Therefore, we can apply Remark 6.1 to conclude that

$$\begin{aligned}
\sum_{z=f_w^{-1}(y)} J_{\eta_w} f_w^{-1}(z) \log \frac{J_{\eta_w} f_w}{J_{\mu_w} f_w}(z) &\leq \log \bigg(\sum_{z=f_w^{-1}(y)} \frac{J_{\eta_w} f_w^{-1} \cdot J_{\eta_w} f_w}{J_{\mu_w} f_w} \bigg)(z) \\
&= \log \bigg(\frac{\sum_{z=f_w^{-1}(y)} e^{\phi_w(z)} h_w(z)}{\lambda_w h_{\theta(w)} \circ f_w(z)} \bigg) \\
&= \log \bigg(\frac{\lambda_w h_{\theta(w)}(y)}{\lambda_w h_{\theta(w)}(y)} \bigg) = 0
\end{aligned}$$

for $\eta_{\theta(w)}$-almost every $y \in M$. Recalling the inequality (15), we obtain that

$$0 \leq \int \log \frac{J_{\eta_w} f_w}{J_{\mu_w} f_w}(y)\, d\eta(w, y) = \int \sum_{z=f_w^{-1}(y)} J_{\eta_w} f_w^{-1}(z) \log \frac{J_{\eta_w} f_w}{J_{\mu_w} f_w}(z)\, d\eta(w, y) = 0.$$

Thus, from the second part of Remark 6.1, the values $q_i = J_{\eta_w} f_w(z_i)/J_{\mu_w} f_w(z_i)$ must be the same for all $z_i \in f_w^{-1}(y)$ in a full $\eta_{\theta(w)}$-measure set. In other words, for every $y \in M$

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on the preimage of a full $\eta_{\theta(w)}$-measure set, we have

$$J_{\eta_w} f_w(y) = J_{\mu_w} f_w(y) = \frac{\lambda_w e^{-\phi_w} \cdot h_{\theta(w)} \circ f_w}{h_w}(y).$$

To finish the proof of the proposition we observe that $1/h_w \cdot \eta_w$ is a reference measure associated to λ_w for the dual transfer operator:

$$\begin{aligned}
\mathcal{L}_w^*\bigg(\frac{1}{h_{\theta(w)}} \cdot \eta_{\theta(w)}\bigg)(\psi) &= \int \mathcal{L}_w(\psi)(x)\, d\bigg(\frac{1}{h_{\theta(w)}} \cdot \eta_{\theta(w)}\bigg) \\
&= \int \sum_{y=f_w^{-1}(x)} e^{\phi_w(y)}(y)\psi(y) \cdot \frac{1}{h_{\theta(w)}}(x)\, d\eta_{\theta(w)} \\
&= \int \sum_{y=f_w^{-1}(x)} \lambda_w \frac{\psi(y)}{h_w(y)} \bigg(\frac{h_w(y)}{\lambda_w e^{-\phi_w(y)} \cdot h_{\theta(w)} \circ f_w(y)}\bigg)\, d\eta_{\theta(w)} \\
&= \int \sum_{y=f_w^{-1}(x)} \lambda_w \frac{\psi(y)}{h_w(y)} J_{\eta_w} f_w^{-1}(y)\, d\eta_{\theta(w)} \\
&= \lambda_w \int \psi\, d\bigg(\frac{1}{h_w} \cdot \eta_w\bigg).
\end{aligned}$$

From the uniqueness given by Theorem A we conclude that $1/h_w \cdot \eta_w$ is equivalent to ν_w and thus η_w is absolutely continuous to the latter. $\square$

Finally, we prove the uniqueness of the equilibrium state associated to $(F|_\theta, \phi)$. Suppose that there exist two ergodic equilibrium states μ and η. Let $(\mu_w)_{w\in X}$ and $(\eta_w)_{w\in X}$ be the disintegration of μ and η, respectively.

By the proposition above we have that μ_w and η_w are equivalent measures. From the Radon–Nikodym theorem we know that there exists a mensurable function $q_w : M \to \mathbb{R}$ such that $\mu_w = q_w \eta_w$ for every $w \in X$. Consider $q : X \times M \to \mathbb{R}$, defined by $q(w, x) = q_w(x)$. Given a measurable set $E \subset X \times M$, consider $E_w \subset M$ the intersection $E_w = E \cap M$. Then we have that

$$\mu(E) = \int_X \mu_w(E_w)\, d\mathbb{P}(w) = \int_X \int_{E_w} q_w\, d\eta_w\, d\mathbb{P}(w) = \int_E q\, d\eta.$$

Moreover, it follows from the F-invariance of μ and η that

$$\mu(E) = F_*\mu(E) = (q \circ F)F_*\eta(E) = (q \circ F)\eta(E).$$

Since the Radon–Nikodym derivative is essentially unique, we conclude that $q = q \circ F$ at η-almost every point. By ergodicity we have that q is constant everywhere and thus $\mu = \eta$.

6.3. *Positive Lyapunov exponents.* The main tool in the proof of Proposition 6.3 is the existence of a generating partition for the equilibrium state. In the context of random dynamical systems generated by non-uniformly expanding maps, the existence of generating partitions for ergodic measures with Lyapunov exponents bounded away from

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zero was proved by Bilbao and Oliveira in [9]. Therefore, in this setting we can also apply our Proposition 6.3 to obtain uniqueness of equilibrium states. This is our goal now.

Consider a compact and connected Riemann manifold M^d of dimension d. Let $F : X \times M \to X \times M$ be the skew-product $(\theta(w), f_w(x))$ generated by C^1 local diffeomorphisms $f_w : M \to M$ satisfying conditions (I)–(III). For $1 \leq k \leq d-1$, define

$$C_k(w, x) = \limsup_{n\to+\infty} \frac{1}{n} \log \|\Lambda^k Df_w^n(x)\| \quad \text{and} \quad C_k(w, F) = \max_{x\in M} C_k(w, x),$$

where Λ^k is the kth exterior product. We suppose that, for some $\varepsilon > 0$,

$$\beta(F) := (1-\varepsilon) \int \log \deg(f_w) d\mathbb{P}(w) - \max_{1\leqslant k\leqslant d-1} \int_X C_k(w, F)\, d\mathbb{P}(w) > 0.$$

For potentials $\phi \in \mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M))$ satisfying conditions (IV) and (V) such that for almost every $w \in X$ the inequality $\sup \phi_w - \inf \phi_w < \varepsilon \int \log \deg(f_w)\, d\mathbb{P}(w)$ holds, we obtain the following result.

COROLLARY 6.1. *There exists only one equilibrium state associated to* $(F|_\theta, \phi)$.

Proof. Let $\eta \in \mathcal{M}_{\mathbb{P}}(F|\theta)$ be an ergodic equilibrium state for $(F|_\theta, \phi)$. Denote by $\lambda_1(w, x) \leq \cdots \leq \lambda_d(w, x)$ the Lyapunov exponents of η at (w, x). We claim that they are bigger than $\beta(F) > 0$. If not, by applying the random version of the Marquis–Ruelle inequality [18, Theorem 2.4] we have

$$\begin{aligned} h_\eta(F|\theta) &\leq \int \sum_{i=1}^{d} \lambda_i^+(w, x)\, d\eta(w, x) \\ &= \int \lambda_1^+(w, x)\, d\eta(w, x) + \int \sum_{i\in\{2\ldots d\}} \lambda_i^+(w, x)\, d\eta(w, x) \\ &\leq \beta(F) + \int C_{d-1}(w, x)\, d\eta(w, x) \leq \beta(F) + \max_{1\leq k\leq d-1} \int C_k(w, F)\, d\mathbb{P}(w) \\ &\leq (1-\varepsilon) \int \log \deg(f_w)\, d\mathbb{P}(w). \end{aligned}$$

Thus, for potentials such that $\sup \phi_w - \inf \phi_w < \varepsilon \int \log \deg(f_w)\, d\mathbb{P}(w)$ it follows that

$$\begin{aligned} h_\eta(F|\theta) + \int \phi\, d\eta &\leq (1-\varepsilon) \int \log \deg(f_w)\, d\mathbb{P}(w) + \int \sup \phi_w\, d\eta \\ &< \int \log \deg(f_w)\, d\mathbb{P}(w) + \int \inf \phi_w\, d\eta \leq P_{F|\theta}(\phi). \end{aligned}$$

which is a contradiction. Therefore, the Lyapunov exponents of η are bigger than $\beta(F)$, and so η admits generating partitions with small diameter. Applying the proof of Proposition 6.3, we have that η is absolutely continuous with respect to ν, and thus the uniqueness is proved. □

7. *Equilibrium stability*

Consider a sequence (F_k, ϕ_k) in $\mathcal{H}$ converging to (F, ϕ). For each $k \in \mathbb{N}$, let μ_k be the non-uniformly expanding equilibrium state of (F_k, ϕ_k). We will prove that any accumulation point μ of the sequence (μ_k) is the non-uniformly expanding equilibrium state of (F, ϕ).

For each $k \in \mathbb{N}$, consider the disintegration $\{\mu_{k,w}\}_{w\in X}$ of μ_k. From Theorem A, we know that $\mu_{k,w} = h_{k,w}\nu_{k,w}$ where $h_{k,w}$ and $\nu_{k,w}$ satisfy

$$\mathcal{L}^*_{k,w}\nu_{k,\theta(w)} = \lambda_{k,w}\nu_{k,w} \quad \mathcal{L}_{k,w}h_{k,w} = \lambda_{k,w}h_{k,\theta(w)} \quad \text{with } \lambda_{k,w} = \nu_{k,\theta(w)}(\mathcal{L}_{k,w}(1)).$$

We point out that for any $\psi \in C^\alpha(M)$ we have $\mathcal{L}_{k,w}(\psi)$ converging to $\mathcal{L}_w(\psi)$ in C^0-norm, a proof of which is given in [3].

Let λ_w, ν_w and h_w be as in Theorem A applied to (F, ϕ). The main step in the proof of the equilibrium stability is the following proposition.

PROPOSITION 7.1. *For almost all $w \in X$ we have the convergence*

$$\lambda_{k,w} \to \lambda_w \quad \nu_{k,w} \xrightarrow{w*} \nu_w \quad \textit{and} \quad h_{k,w} \to h_w \quad \textit{as } k \textit{ goes to infinity.}$$

Proof. Recalling that for each $k \in \mathbb{N}$ we have

$$\deg(f_{k,w})e^{\inf \phi_{k,w}} \le \lambda_{k,w} \le \deg(f_{k,w})e^{\sup \phi_{k,w}},$$

the sequence $(\lambda_{k,w})$ admits some accumulation point $\bar{\lambda}_w$. Moreover, taking subsequences, if necessary, there exist probability measures $\bar{\nu}_w$ and $\bar{\nu}_{\theta(w)}$ such that $\nu_{k,w} \xrightarrow{w*} \bar{\nu}_w$ and $\nu_{k,\theta(w)} \xrightarrow{w*} \bar{\nu}_{\theta(w)}$. We will prove that $\mathcal{L}^*_w\bar{\nu}_{\theta(w)} = \bar{\lambda}_w\bar{\nu}_w$.

For any $\psi \in C^\alpha(M)$ we can write

$$\mathcal{L}^*_w\bar{\nu}_{\theta(w)}(\psi) = \bar{\nu}_{\theta(w)}(\mathcal{L}_w(\psi)) = \bar{\nu}_{\theta(w)}(\lim_{k\to\infty} \mathcal{L}_{k,w}(\psi)) = \lim_{k\to\infty} \bar{\nu}_{\theta(w)}(\mathcal{L}_{k,w}(\psi)).$$

From the convergence of $\nu_{k,\theta(w)}$ to $\bar{\nu}_{\theta(w)}$ we have

$$\lim_{k\to\infty} \bar{\nu}_{\theta(w)}(\mathcal{L}_{k,w}(\psi)) = \lim_{k\to\infty} \nu_{k,\theta(w)}(\mathcal{L}_{k,w}(\psi)) = \lim_{k\to\infty} \mathcal{L}^*_{k,w}(\nu_{k,\theta(w)})(\psi).$$

Since $\nu_{k,\theta(w)}$ is a reference measure, the last equality can be rewritten as

$$\lim_{k\to\infty} \mathcal{L}^*_{k,w}(\nu_{k,\theta(w)})(\psi) = \lim_{k\to\infty} \lambda_{k,w}\nu_{k,w}(\psi) = \bar{\lambda}_w\bar{\nu}_w(\psi).$$

Thus, $\mathcal{L}^*_w\bar{\nu}_{\theta(w)}(\psi) = \bar{\lambda}_w\bar{\nu}_w(\psi)$ for any $\psi \in C^\alpha(M)$. Because $C^\alpha(M)$ is dense in $C^0(M)$ we conclude that $\mathcal{L}^*_w\bar{\nu}_{\theta(w)} = \bar{\lambda}_w\bar{\nu}_w$.

Now we will verify that $\bar{\lambda}_w = \lambda_w$. Therefore, from the uniqueness given by Theorem A it follows that $\bar{\nu}_w = \nu_w$.

Given $\varepsilon > 0$ small and $n \in \mathbb{N}$, consider a (w, n, ε)-separated set F_n. Let $\mathcal{U}$ be the open cover of M defined by $\mathcal{U} := \{\bigcap_{j=0}^{n-1} f_w^{-j}(B(f_w^j(x), \varepsilon)); x \in F_n\}$. Because $(\mathcal{L}^n_w)^*\bar{\nu}_{\theta^n(w)} = \bar{\lambda}^n_w\bar{\nu}_w$ it follows that

$$\begin{aligned} 1 = \bar{\nu}_w(M) &= \int (\bar{\lambda}^n_w)^{-1}\mathcal{L}^n_w(1)\, d\bar{\nu}_{\theta^n(w)} \\ &\le (\bar{\lambda}^n_w)^{-1} \sum_{U\subset\mathcal{U}} \int_U e^{S_n\phi_w(z)}\, d\bar{\nu}_{\theta^n(w)} \end{aligned}$$

$$\leq (\bar{\lambda}_w^n)^{-1} \sum_{x \in F_n} e^{S_n\phi_w(x)} \int_U e^{(S_n\phi_w(z) - S_n\phi_w(x))}\, d\bar{\nu}_{\theta^n(w)}$$
$$\leq (\bar{\lambda}_w^n)^{-1} \sum_{x \in F_n} e^{S_n\phi_w(x)} e^{\sum_{j=0}^{n-1} |\phi_{\theta^j(w)}|_\alpha \varepsilon}.$$

Thus, $\sum_{j=0}^{n-1}(\log \bar{\lambda}_{\theta^j(w)} - |\phi_{\theta^j(w)}|_\alpha \varepsilon) \leq \log P_{F|_\theta}(w, n, \varepsilon)$. As $\mathbb{P}$ is ergodic we obtain

$$\int \log \bar{\lambda}_w \, d\mathbb{P}(w) - \varepsilon \int |\phi_w|_\alpha \, d\mathbb{P}(w) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \bar{\lambda}_{\theta^j(w)} - \varepsilon \frac{1}{n} \sum_{j=0}^{n-1} |\phi_{\theta^j(w)}|_\alpha$$
$$\leq \limsup_{n\to\infty} \frac{1}{n} \int \log P_{F|_\theta}(w, n, \varepsilon)\, d\mathbb{P}(w)$$

for every $\varepsilon > 0$ small. Hence, $\int \log \bar{\lambda}_w \, d\mathbb{P}(w) \leq P_F(\phi)$. On the other hand, since $\bar{\nu}_w$ is a reference measure, it satisfies a Gibbs property on hyperbolic times (Proposition 4.1). Thus, we can apply the proof of Proposition 6.1 to obtain $P_F(\phi) \leq \int \log \bar{\lambda}_w \, d\mathbb{P}(w)$. Recalling that $P_F(\phi) = \int \log \lambda_w \, d\mathbb{P}(w)$, we have

$$\int \log \bar{\lambda}_w \, d\mathbb{P}(w) \leq P_F(\phi) = \int \log \lambda_w \, d\mathbb{P}(w) \leq \int \log \bar{\lambda}_w \, d\mathbb{P}(w).$$

Since this is constant for $\mathbb{P}$-almost every $w \in X$, we have proved that $\bar{\lambda}_w = \lambda_w$.

To finish the proof of the proposition it remains to prove the convergence $h_{k,w} \to h_w$. Since $(F_k, \phi_k) \in \mathcal{H}$ we can assume that the transfer operator $\mathcal{L}_{k,w}$ preserves the same cone $C_\delta^{\hat{k}}$ for $\hat{k}$ large enough. Then, recalling the proof of Proposition 5.3, we have each $h_{k,w} \in C_\delta^{\hat{k}}$ with $\int h_{k,w}\, d\nu_{k,w} = 1$. Moreover, it satisfies

$$|h_{k,w}(x) - h_{k,w}(y)| \leq Cm\, d(x, y)^\alpha \quad \text{and} \quad |h_{k,w}(y)| \leq \sup h_{k,w} \leq R.$$

Therefore, $(h_{k,w})$ is a discontinuous and uniformly bounded sequence. From the Arzelà–Ascots theorem there exists some accumulation point $\bar{h}_w$. Notice that $\bar{h}_w$ is Hölder continuous and $\int \bar{h}_w\, d\nu_w = 1$ because $\nu_{k,w} \xrightarrow{w*} \nu_w$. Moreover, $\bar{h}_w$ satisfies

$$\mathcal{L}_w(\bar{h}_w) = \mathcal{L}_w(\lim_{k\to\infty} h_{k,w}) = \lim_{k\to\infty} \mathcal{L}_w(h_{k,w}) = \lim_{k\to\infty} \mathcal{L}_{k,w}(h_{k,w})$$
$$= \lim_{k\to\infty} \lambda_{k,w} h_{k,\theta(w)} = \lambda_w \bar{h}_{\theta(w)}.$$

By the uniqueness of Theorem A we obtain $\bar{h}_w = h_w$ almost everywhere. □

From the previous result we obtain for almost $w \in X$ that the sequence $(\mu_{k,w})$ converges to μ_w defined by $\mu_w = h_w \nu_w$. Therefore, the sequence (μ_k) converges to the probability measure μ whose disintegration is $\{\mu_w\}_{w\in X}$. As in §6, we have that μ is the non-uniformly expanding equilibrium state of $(F|_\theta, \phi)$. Moreover, since in the family $\mathcal{H}$ we have that $P_{F|_\theta}(\phi) = \int \log \lambda_w \, d\mathbb{P}(w)$, we obtain

$$P_{F|_\theta}(\phi) = \int \log \lambda_w \, d\mathbb{P}(w) = \lim_{k\to\infty} \int \log \lambda_{k,w}\, d\mathbb{P}(w) = \lim_{k\to\infty} P_{F_k}(\phi_k),$$

which proves that the random topological pressure varies continuously in the family. This finishes the proof of Theorem D.

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8. *Applications*

In this section we present some classes of systems which satisfy our results. We start by describing a robust class of local diffeomorphisms which contains an open set of non-uniformly expanding maps that are not uniformly expanding. This class was studied in the deterministic case by several authors [1, 2, 12, 29]. The first example is a one-dimensional version of this class.

Example 8.1. Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1-local diffeomorphism defined on the unit circle. Fix $\varepsilon > 0$ small, $\sigma < 1$ and consider a covering $\mathcal{Q}$ of $\mathbb{S}^1$ by injectivity domains of g and a region $A \subset \mathbb{S}^1$ covered by q elements of $\mathcal{Q}$ with $q < \deg(g)$ such that

(H1) $\|Dg^{-1}(x)\| \leq 1 + \varepsilon$, for every $x \in A$;

(H2) $\|Dg^{-1}(x)\| \leq \sigma$, for every $x \in M \setminus A$.

Let us consider a suitable C^1 perturbation of g to produce an open set $\mathcal{F}$ of C^1-local diffeomorphisms satisfying conditions (I)–(II). The perturbation can be chosen small enough to guarantee the uniform openness property in the family $\mathcal{F}$. We also assume that every $g \in \mathcal{F}$ is *topologically exact* and its degree $\deg g$ is constant. Notice that $\mathcal{F}$ may contain expanding maps, perturbations of expanding maps and intermittent maps.

Let $\theta : \mathbb{S}^1 \to \mathbb{S}^1$ be any invertible function preserving an ergodic measure $\mathbb{P}$ on $\mathbb{S}^1$. Thus, any random dynamical system $g = (g_w)_w$ generated by maps $g_w \in \mathcal{F}$ satisfies the hypotheses of our theorems. For potentials $\phi \in \mathbb{L}^1_{\mathbb{P}}(\mathbb{S}^1, C^\alpha(\mathbb{S}^1))$ satisfying (IV) and (V) we can apply our results to obtain the thermodynamic formalism in this class and the existence of only one equilibrium state on the set of non-uniformly expanding measures.

Moreover, if the potential also satisfies the condition $\sup \phi < P_\phi(f)$ then the equilibrium state is unique in the class of ergodic measures. Indeed, using the random versions of Oseledets's theorem and Ruelle's inequality (see [18]), for the equilibrium state μ the Lyapunov exponent $\lambda(\mu)$ satisfies

$$\begin{aligned}\lambda(\mu) \geq h_\mu(g) = P_\phi(g) - \int \phi \, d\mu \geq h_{\text{top}}(g) + \inf \phi - \sup \phi \\ \geq h_{\text{top}}(g) - (\sup \phi - \inf \phi) \geq \log q > 0.\end{aligned}$$

Therefore, $\lambda(\mu)$ is positive and bounded away from zero. In dimension one this implies that the equilibrium state is non-uniformly expanding.

The second example is a generalization of the previous one in higher dimension. The existence of equilibrium state for random transformations given by maps in this setting was considered by Arbieto, Matheus and Oliveira [5].

Example 8.2. Let M^l be a compact l-dimensional Riemannian manifold and $\mathcal{D}$ the space of C^2 local diffeomorphisms on M. Let $(\Omega, T, \mathbb{P})$ be a measure-preserving system where $\mathbb{P}$ is ergodic. Define the skew-product by

$$\begin{array}{rccc} F : & \Omega \times M & \longrightarrow & \Omega \times M \\ & (w, x) & \longmapsto & (T(w), f(w)x) \end{array}$$

where the map $f(w) \in \mathcal{D}$ varies continuously on $w \in \Omega$. Fix positive constants $\delta_0, \delta_1, \delta_2$ small and $p, q \in \mathbb{N}$, satisfying for every $f(w) \in \mathcal{D}$ the following properties.

(H0) There exist $\delta_w > \delta_0 > 0$ such that for every $x \in M$ we can find a neighborhood U_x where $f(w) : U_x \to B_{T(w)}(f(w)(x), \delta_w)$ is invertible.

(H1) There exists a covering $B_1 \ldots B_p \ldots B_{p+q}$ of M by injectivity domains such that:
- $\|Df(x)^{-1}\| \leq (1+\delta_2)^{-1}$ for every $x \in B_1 \cup \cdots \cup B_p$;
- $\|Df(x)^{-1}\| \leq (1+\delta_1)$ for every $x \in M$.

(H2) f is everywhere volume expanding: $|\det Df(x)| \geq \sigma_1$ with $\sigma_1 > q$.

(H3) There exists A_0 such that $|\log \|f\|_{C^2}| \leq A_0$ for any $f \in \mathcal{F} \subset \mathcal{D}$.

Adding other technical hypotheses, the authors in [5] showed the existence of equilibrium states for potentials with small variation. They also proved that these measures are non-uniformly expanding. Now, for potentials satisfying conditions (IV) and (V), we can apply our results to obtain the thermodynamic formalism and the uniqueness of equilibrium state for this class. Moreover, by considering the set $\mathcal{S}$ of skew-products generated by maps of $\mathcal{D}$ where $T : \Omega \to \Omega$ is fixed,

$$F : X \times M \to X \times M;\ F(w, x) = (T(w), f_w(x)),$$

we define the family $\mathcal{H} = \{(F, \phi) \in \mathcal{S} \times \mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M));\ (F, \phi) \text{ satisfying (I)–(VI)}\}$ and observe that $\mathcal{H}$ satisfies the hypothesis of Theorem D. Thus, the equilibrium state and the random topological pressure vary continuously within this family.

Next we present an application of our Corollary 6.1. This example appears in [9] in the context of maximizing entropy measures. Here we prove uniqueness of equilibrium states for potentials with small variation.

Example 8.3. Let $f_0, f_1 : M \to M$ be C^1 local diffeomorphisms of a compact and connected manifold M satisfying our conditions (I)–(III). For $1 \leq k < \dim M = d$ suppose that $\log\|\Lambda^k Df_1\| < \log \deg f_1$ and consider

$$C_k(w, x) = \limsup_{n\to+\infty} \frac{1}{n} \log \|\Lambda^k Df_w^n(x)\| \quad \text{and} \quad C_k(w) = \max_{x\in M} C_k(w, x).$$

Let $\mathbb{P}_\alpha$ be the Bernoulli measure on the sequence space $X = \{0, 1\}^{\mathbb{Z}}$ such that $\mathbb{P}_\alpha([1]) = \alpha$. In [9] Birkhoff proved the existence of $\alpha \in (0, 1)$ close to 1 such that

$$\begin{aligned}\int \lim_{n\to\infty} \frac{1}{n} \log\|\Lambda^k Df_w^n(x)\|\, d\mathbb{P}_\alpha(w) &< \alpha \log \deg(f_1) + (1-\alpha) \log \deg(f_0)\\ &= \int \log \deg(f_w)\, d\mathbb{P}_\alpha(w)\end{aligned}$$

for every $x \in M$. Therefore, for some $\varepsilon > 0$ we have

$$(1-\varepsilon) \int \log \deg(f_w)\, d\mathbb{P}_\alpha(w) - \max_{1\leqslant k\leqslant d-1} \int_X C_k(w)\, d\mathbb{P}_\alpha(w) > 0$$

which means that the hypothesis of Corollary 6.1 was verified.

Thus, for potentials $\phi \in \mathbb{L}^1_{\mathbb{P}}(X, C^\alpha(M))$ satisfying (IV) and (V) with variation $\sup \phi_w - \inf \phi_w < \varepsilon \int \log \deg(f_w)\, d\mathbb{P}_\alpha$ we conclude uniqueness of equilibrium states.

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