Part II

ERGODIC AND STOCHASTIC MOTION

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ERGODITY OF THE MOTIONS IN THE DYNAMICAL SYSTEMS WITH TWO DEGREES OF FREEDOM

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ABSTRACT

We study a process of stochastization of the motions in an example of the Henon-Heiles model. We propose a new method to study this process - a method of the field of directions of motion in the meridional plane. A numerical integration of the equation for the derivative of this field $\partial f/\partial n$ to the normal to a trajectory has been made. We denote the points in which $\partial f/\partial n \to \pm\infty$, and derive the contours of orbit and folds of directions. The growth of ergodity is connected with the increase of a number and an area of the folds. May be, a successive doubling of a number of folds takes place that results in a chaos. In the transition region we found a complex periodical orbit. The interpretation of this fact may be made as an example of cantori. A transition region has very small sizes about 10^{-4}.

INTRODUCTION

In some fields of science (statistical physics,mechanics, biology, biochemistry etc.), we see an increasing interest to the problem of ergodity (or stochasticity) of the motions in the deterministic dynamical systems with a few degrees of freedom (see, e.g., Lichtenberg & Liberman 1982, Zaslavsky & Sagdeev 1988, Nicolis & Prigogine 1989).

It is well-known that a region of initial conditions is separated into a set of complex stochastic or ergodic motions and a set of regular or ordered motions. However, it is not so clear a reason of such behaviour of the solutions. The structure of transition region between regular and ergodic motions is also interesting.

We consider a simple dynamical system with two degrees of freedom-a model by Henon & Heiles(1964). It has a force function

$$U(R,z) = -\frac{1}{2}(R^2 + z^2) - Rz^2 + R^3/3 \qquad (1)$$

Henon and Heiles (1964) have shown that this model may result in the complex trajectories which were named as stochastic versus the ordered trajectories (regular ones). Some differences between regular and stochastic trajectories are revealed in the Poincare map (R,R): the smooth invariant curves correspond to the ordered trajectories, and broken curves - to ergodic ones. A share of the ergodic trajectories (an ergodic sea) increases with a growth of the energy integral.

Benettin et al. (1976), and Contopoulos and Barbanis (1989) have studied a behaviour of the Lyapunov characteristic numbers $\sigma(t)$ for the different trajectories in the Henon-Heiles and Contopoulos models. They have shown nearly exponential divergence of the nearby trajectories in a case of ergodic motions, ($\sigma(t) \approx$ const.) and $\sigma(t) \to 0$, $t \to \infty$ for the ordered motions.

We assume that an appearance of ergodity is connected with the first folds in the direction field, and a growth of stochasticity with an increase of a number of folds, their sizes, and deformation of contours of folds.

METHOD OF STUDY

In order to verify this hypothesis we have considered a few trajectories in the Henon-Heiles model: as ordered and stochastic. For every trajectory under study we integrate numerically an equation for the curvature of the trajectory.

$$\frac{\partial f}{\partial \ell} = \frac{1}{2h}\frac{\partial U}{\partial n}, \qquad (2)$$

where f is an angle between the tangent to trajectory and the axis R,U is the force function (1), h = U+I (I is the energy integral), ℓ and n are the directions of tangent and normal to the trajectory.

Besides, we integrate simultaneously an equation for $\partial f/\partial n$:

$$\frac{\partial}{\partial \ell}\frac{\partial f}{\partial n} + \left(\frac{\partial f}{\partial n}\right)^2 + \frac{1}{2h}\frac{\partial u}{\partial \ell}\frac{\partial f}{\partial n} + \frac{3}{4h^2}\left(\frac{\partial u}{\partial n}\right)^2 + \frac{1}{2h}\gamma = 0, \qquad (3)$$

where

$$\gamma = -\frac{\partial^2 u}{\partial R^2}\sin^2 f + \frac{\partial^2 u}{\partial R \partial z}\sin^2 f \; \frac{\partial^2 u}{\partial z^2}\cos^2 f \qquad (4)$$

(Agekian 1974).

We must find the points in which $\partial f/\partial n = \pm \infty$, at the contours of an orbit and the folds of directions, where the multiplicity of the field varies. In order to avoid a singularity at $\partial f/\partial n \to \pm \infty$, we change the variable quantity $\partial f/\partial n$ to $q = (\partial f/\partial n)^{-1}$, when $|\partial f/\partial n| \geq 1$, and integrate the equation for q. The points $q = 0$ correspond to the contours of orbit and folds.

We consider the trajectories starting from the axis R orthogonally to it. We have two parameters in the problem: a quantity R_o - an abscissa of the starting point, and a quantity I - an integral of energy. We have scanned I at the fixed R_o (some analog of the third integral) or R_o at the fixed I.

RESULTS

We are interested in transition from the order to the chaos. Therefore we try to find some transition regions in the space (R_o, I).

The first example of a complication of the picture is presented in Figures 1-6. We consider $R_o = 0.2$ and $I = 1/12$, 1/8 and 1/6.

We see an increase in the area of folds with the growth of I. The invariant curves are smooth and elongated along the axis R. Then we observe a doubling of invariant curve, a formation of the new folds. In a region of crossing the trajectory and axis R, a multiplicity of direction field is equal to 4. However, the total ergodity does not achieve in this case.

Other examples are $R_o = 0$, and the values of I change from 1/12 to 1/6. The results are shown in Figures 7-12. For $I = 1/12$ we can see three large folds having the smooth contours, the invariant curves are also smooth. Such a behaviour takes near about $I = 1/6.5$. For this, I, we have a small erosion of the invariant curve. A corresponding picture of the direction field shows increase in the size of folds, a formation of some supplementary folds in their angles, some distorsion of the contours. A complication of direction field results in the erosion of invariant curves.

A stronger erosion takes place in Figures 11,12 $(I = 1/6)$. Here a trajectory tends to the whole ergodity. A number of folds increases and the contours are loosed shape.

We have studied a transition between ordered and stochastic motions on the small variation of R_o or I. It appears that in the transition field we have a complex periodic orbit (see Figure 13). Thus, a periodic orbit is a bound between the regular and ergodic behaviour in a space of integrals or quasi-integrals of motion - in this case a space (R_O, I).

Also we have determined the Lyapunov characteristic numbers $\sigma(t)$ as the functions of time by a method of Benettin et al. (1976). In Figure 14, we show two examples of such dependences for the variations of I and the same quantity $R_O = 0$. The Figure shows a qualitatively different behaviour. In the ordered case $\sigma(t)$ tends to zero, and for the stochastic trajectory $\sigma(t) \simeq$ const.

CONCLUSION

Thus a disturbance of the smooth invariant curves and a growth of ergodity of the motions correlate with the increase in the numbers and sizes and/or the twisting of the bounds of folds. An initial stochasticity is connected with the appearence of the first small folds. The transition between the ordered and stochastic regions takes place across a periodic orbit where we observe a sharp decreasing of the Lyapunov exponents.

REFERENCES

[1] Agekian, T.A., 1974, Vestnik Leningr. Univ., 7,140.
[2] Benettin, G., Galgani, L., Strelcyn J.-M., 1976, Phys. Rev. A., 14,2338.
[3] Contopoulos, G., Barbanis, B., 1989, Astron. Astrophys., 222,329.
[4] Henon, M., Heiles, C., 1964, Astron. J., 69,73.
[5] Lichtenberg, A.J., Lieberman, M.A., 1983, Regular and Stochastic Motion, Springer Verlag, New York.
[6] Zaslavsky, G.M., Sagdeev, R.Z., 1988, Introduction into Non-linear Physics: from Pendulum to Turbulence and Chaos, Nauka, Moscow.
[7] Nicolis, G., Prigogine, I., 1989, Exploring Complexity. An introduction, W.H. Freeman and Company, New York.

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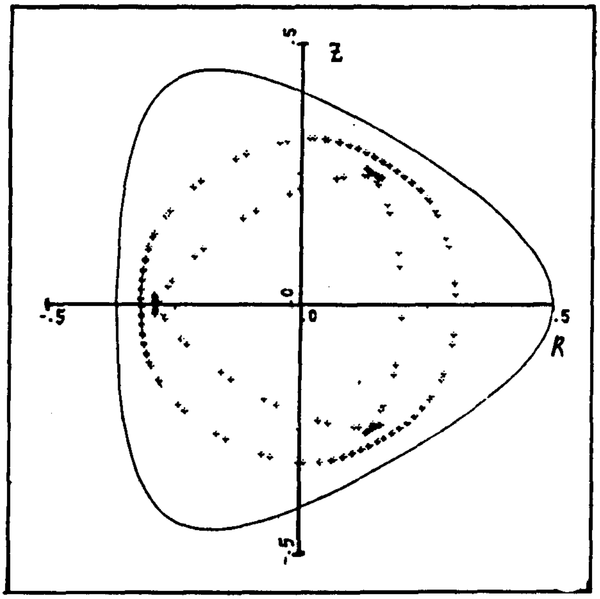

Figure 1: Contours of orbit and folds for the trajectory with $R_o = 0.2$ on $I = 1/12$.

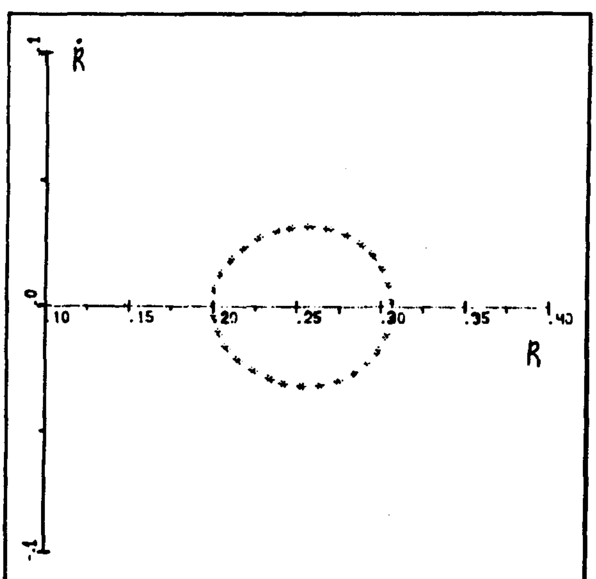

Figure 2: Invariant curve for the trajectory with $R_o = 0.2$ on $I = 1/12$.

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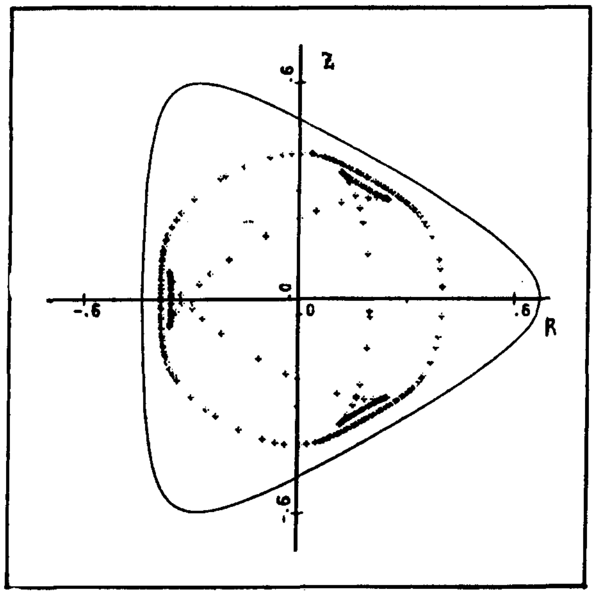

Figure 3: Contours of orbit and folds for R_o = 0.2 and I = 1/8.

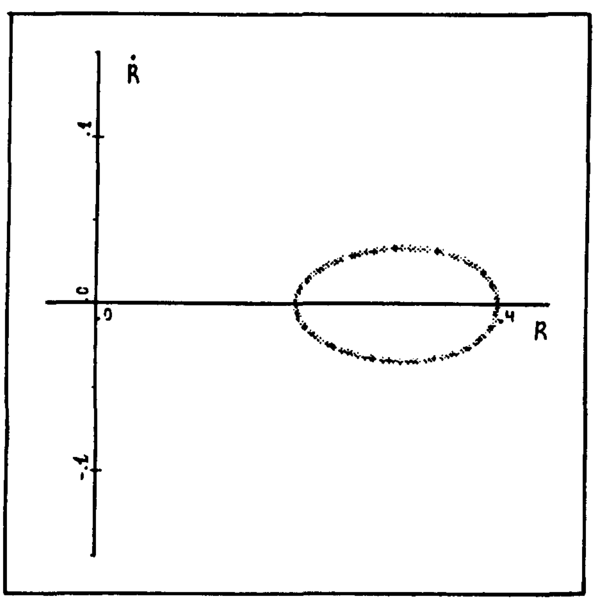

Figure 4: Invariant curve for R_o = 0.2 and I = 1/8.

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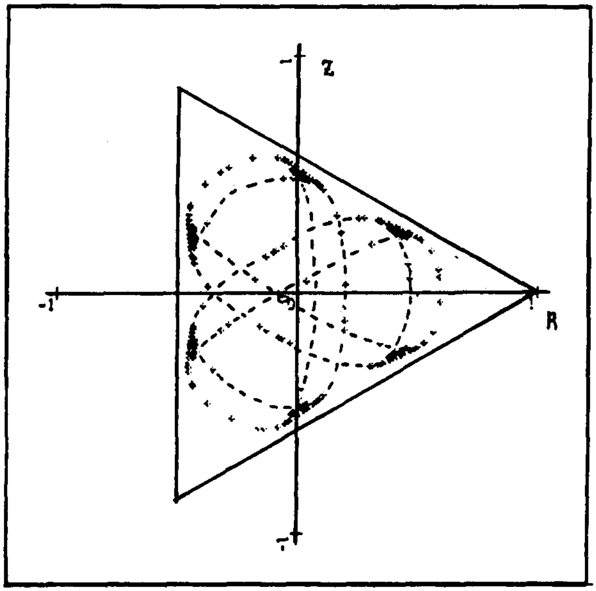

Figure 5: Contours of orbit and folds for R_o = 0.2 and I = 1/6.

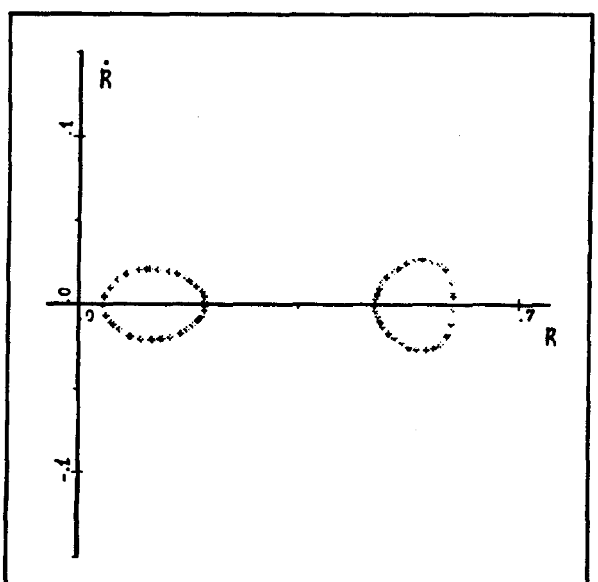

Figure 6: Invariant curves for R_o = 0.2 and I = 1/6.

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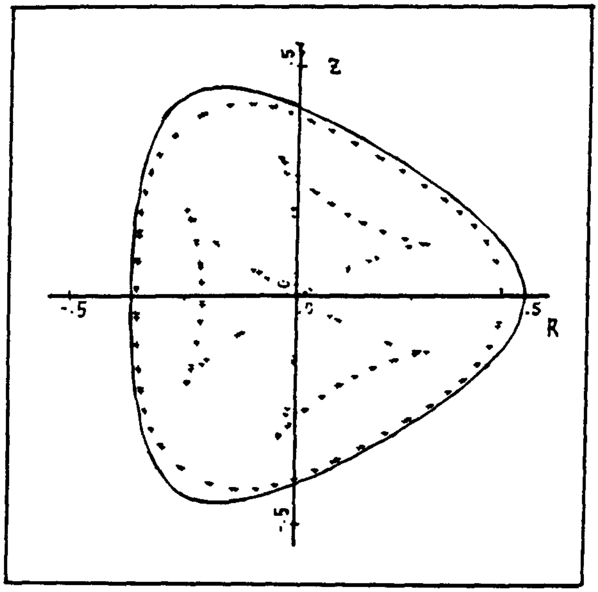

Figure 7: Contours of orbit and folds for R_o = 0 and I =1/12.

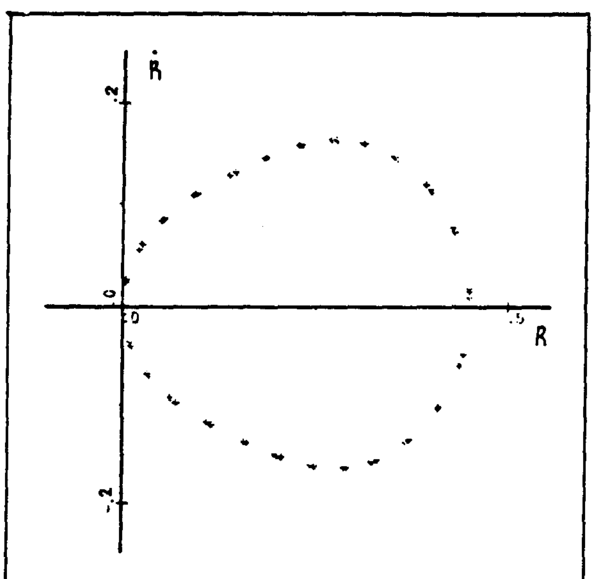

Figure 8: Invariant curve for R_o = 0 and I = 1/12.

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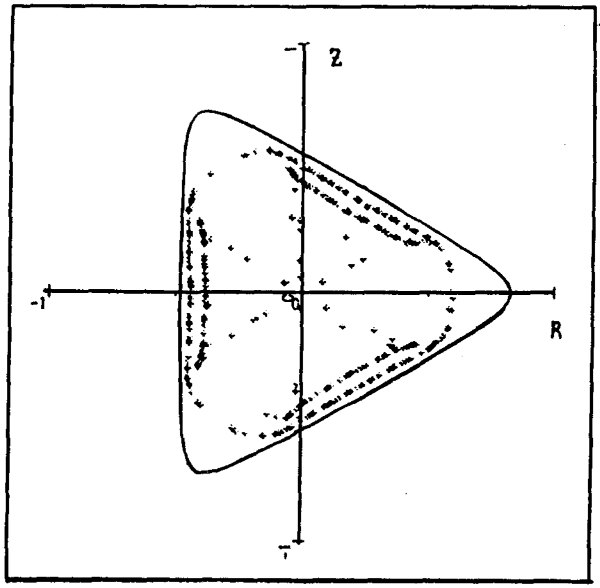

Figure 9: Contours of orbit and folds for R_o = 0 and I = 1/6.5.

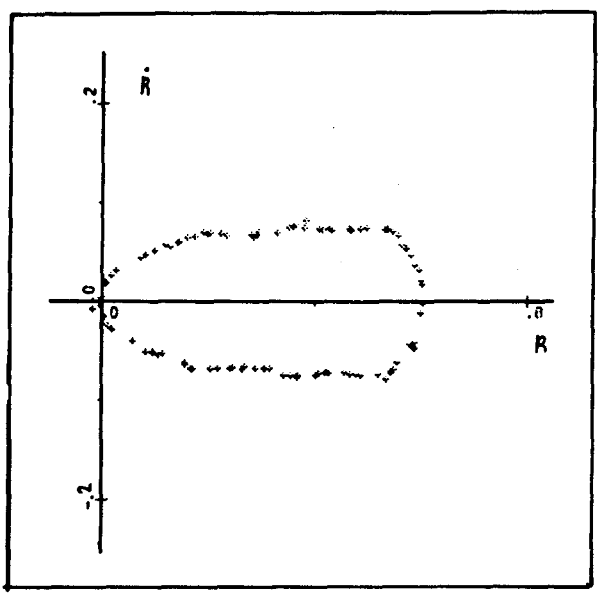

Figure 10: Invariant curve for R_o = 0 and I = 1/6.5.

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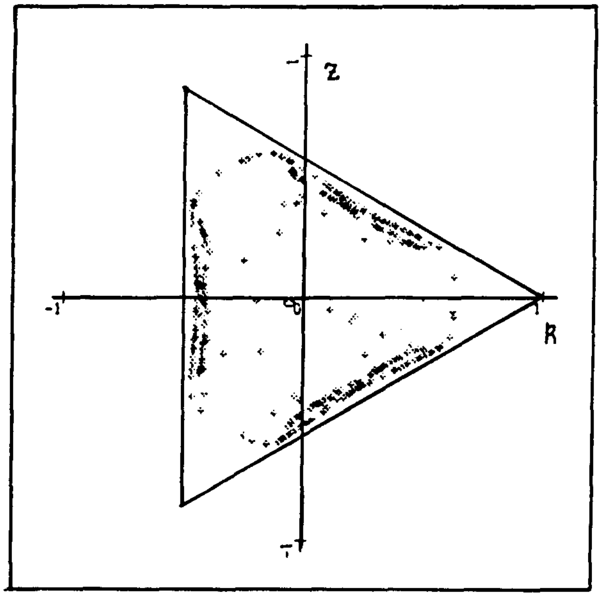

Figure 11: Contours of orbit and folds for R_o = 0 and I = 1/6.

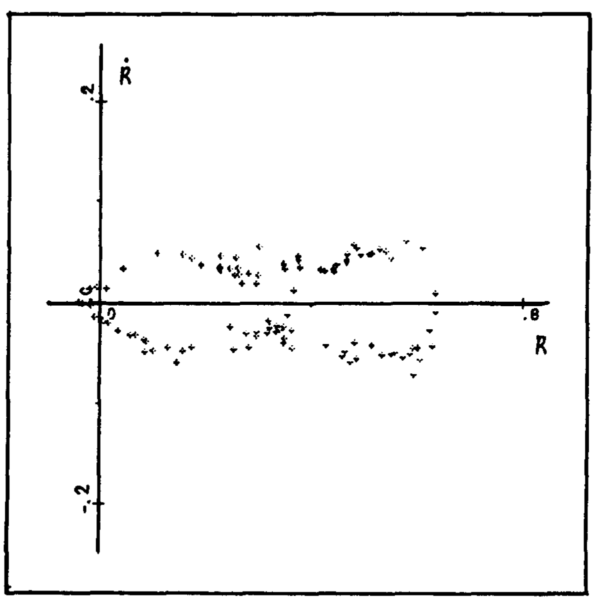

Figure 12: Invariant curve for R_o = 0 and I = 1/6.

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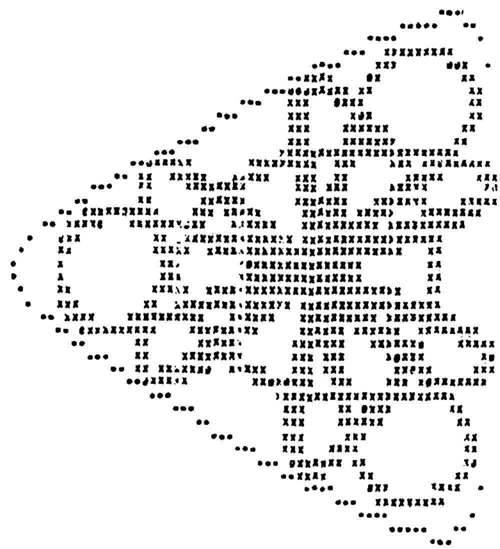

Figure 13: A transient periodic orbit for R_o = - 0.2994 and I = 1/7.

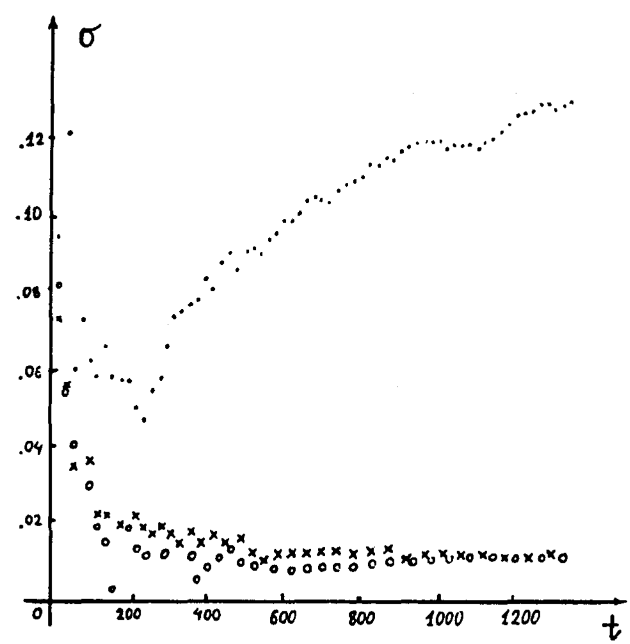

Figure 14: The dependences σ(t) for I = 1/7 (crosses), I = 1/6.5 (circles), and I = 1/6 (dots) for R_o = 0 and $\Delta R_o = 10^{-10}$.

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