Therefore the equations of this axis are

$$x(F+fa') = y(G+ga') = z(H+ha').$$

Similar equations hold for the other axes, with b' and c' instead of a'.

LAWRENCE CRAWFORD.

Proofs of some Inequalities and Limits.

In his article in No. 20, Professor Gibson gives proofs of the inequalities $1-na<(1-a)^n<\dfrac{1}{1+na}$ with certain restrictions as to the values of n and a. The mode of proof, avoiding as it does the use of the Binomial Theorem, is comparatively short and simple. The following mode of proof is, I think, still simpler, as it does not involve the use of Mathematical Induction.

If n is a positive integer and a positive, we have

$$\frac{(1+a)^n-1}{(1+a)-1} = (1+a)^{n-1}+(1+a)^{n-2}+(1+a)^{n-3}+\ldots+(1+a)+1,$$
$$> n,$$
$$\therefore\ (1+a)^n-1>na,$$
$$\therefore\ (1+a)^n>1+na. \qquad \text{(1)}$$

Again, n being a positive integer and a a positive proper fraction, we have

$$\frac{1-(1-a)^n}{1-(1-a)} = 1+(1-a)+(1-a)^2+\ldots+(1-a)^{n-1},$$
$$< n,$$
$$\therefore\ 1-(1-a)^n<na,$$
$$\therefore\ (1-a)^n>1-na. \qquad \text{(2)}$$

Then, since $(1-a)(1+a)=1-a^2$

$$<1,$$
$$\therefore\ 1-a<\frac{1}{1+a},$$
$$\therefore\ (1-a)^n<\frac{1}{(1+a)^n},$$
$$\therefore\ \text{by (1)},\ <\frac{1}{1+na}. \qquad \text{(3)}$$

With the further restriction that na should be a proper fraction, and therefore $1 - na$ positive, we have

$$(1+a)^n < \frac{1}{(1-a)^n},$$

$$\therefore \text{ by (2) } < \frac{1}{1-na}. \qquad\qquad (4)$$

Professor Gibson applies these inequalities to prove

$$\operatorname*{Lim.}_{n \to \infty} \left(\cos\frac{x}{n}\right)^n = 1 \text{ and } \operatorname*{Lim.}_{n \to \infty} \left\{\left(\sin\frac{x}{n}\right)\Big/\frac{x}{n}\right\}^n = 1.$$

It is perhaps worth noting that (2), or something equivalent to it, is required to establish the formula for the "sum to infinity" of a geometrical progression of which the "common ratio" is a proper fraction.

It is, of course, easy to prove that

$$c + cr + cr^2 + \ldots + cr^{n-1} = \frac{c}{1-r} - \frac{cr^n}{1-r},$$

but to deduce the result $\operatorname*{Lim.}_{n \to \infty} (c + cr + cr^2 + \ldots + cr^{n-1}) = \frac{c}{1-r}$

we must first prove that when r is a proper fraction $\operatorname*{Lim.}_{n \to \infty} r^n = 0$. In many text-books of algebra this is left unproved without any acknowledgement of the assumption made. It can, of course, be proved easily by the aid of (1) by noting that if r is a proper fraction, we can write

$$r = \frac{1}{1+a}, \quad \therefore \; r^n = \frac{1}{(1+a)^n},$$

$$< \frac{1}{1+na},$$

which can obviously be made as small as we please by making n large enough. In fact, if ϵ is an arbitrarily small positive quantity, we have only to take $n > \frac{1}{\epsilon a}$, and we have $na > \frac{1}{\epsilon}$,

$$\therefore \; 1 + na > \frac{1}{\epsilon}, \quad \therefore \; \frac{1}{1+na} < \epsilon.$$

It may be useful to collect here certain other fundamental "limit theorems" which are required for a school course of Algebra, with simple proofs annexed.

(i) If $|x| > 1$, $\lim\limits_{n \to \infty} x^n = \infty$.

Putting $|x| = 1 + a$, the proof follows immediately from (1), for the case of n a positive integer.

(ii) If $|x| < 1$, $\lim\limits_{n \to \infty} x^n = 0$.

The proof has been given above.

(iii) If n be a positive integer, $\lim\limits_{n \to \infty} \left(\frac{x^n}{n!}\right) = 0$.

First take the case when n is even.

We have

$$\frac{x^n}{n!} = \left(\frac{x}{1} \cdot \frac{x}{n}\right)\left(\frac{x}{2} \cdot \frac{x}{n-1}\right) \cdots \left(\frac{x}{r} \cdot \frac{x}{n-r+1}\right) \cdots \left(\frac{x}{\frac{n}{2}} \cdot \frac{x}{\frac{n}{2}+1}\right).$$

Now $r(n-r+1) = (r-1)(n-r) + n,$

$$\not< n,$$

the symbol $\not<$ reducing to $>$ except when $r = 1$.

$$\therefore \quad \frac{x^n}{n!} < \left(\frac{x^2}{n}\right)^{\frac{n}{2}}.$$

Now if ϵ is an arbitrarily small positive proper fraction, and if n is any integer greater than $\frac{x^2}{\epsilon}$, we have $\frac{x^2}{n} < \epsilon$, $\therefore \left(\frac{x^2}{n}\right)^{\frac{n}{2}} < \epsilon$.

Thus, when n is even, $\lim\limits_{n \to \infty} \frac{x^n}{n!} = 0,$

When n is odd, let $n = m + 1$, so that m is even.

Then proceeding as before, we get

$$\frac{x^n}{n!} = \frac{x^m}{m!} \cdot \frac{x}{m+1},$$

$$< \left(\frac{x^2}{m}\right)^{\frac{m}{2}} \cdot \frac{x}{m+1} < \left(\frac{x^2}{m}\right)^{\frac{m}{2}} \cdot \frac{x}{\sqrt{m}},$$

$$< \left(\frac{x^2}{m}\right)^{\frac{m+1}{2}},$$

and this, as before, has zero as its limit.

Corollary. $n! > n^{\frac{n}{2}}$,

(iv) $\underset{h \to 0}{\text{Lim.}} \; (a^h - 1) = 0$, or $\underset{h \to 0}{\text{Lim.}} \; a^h = 1$.

Put $h = \frac{1}{n}$, when n is a positive integer, so that $n \to \infty$ when $h \to 0$.

First suppose a positive and greater than 1.

Then $a^{\frac{1}{n}} - 1 = \dfrac{a-1}{a^{\frac{n-1}{n}} + a^{\frac{n-2}{n}} + \ldots + a^{\frac{1}{n}} + 1}$,

$$< \frac{a-1}{n}.$$

Hence $\underset{n \to \infty}{\text{Lim.}} \; (a^{\frac{1}{n}} - 1) = 0$.

Next suppose a to be a positive proper fraction.

Then

$$1 - a^{\frac{1}{n}} = \frac{1-a}{1 + a^{\frac{1}{n}} + a^{\frac{2}{n}} + \ldots + a^{\frac{n-1}{n}}} = \frac{a^{-1} - 1}{a^{-1} + a^{-\frac{n-1}{n}} + a^{-\frac{n-2}{n}} + \ldots + a^{-\frac{1}{n}}}$$

$$< \frac{a^{-1} - 1}{n}.$$

Hence $\underset{n \to \infty}{\text{Lim.}} \; (1 - a^{\frac{1}{n}}) = 0$.

The restriction that n should tend to infinity by integral values is, as pointed out in Professor Gibson's last paragraph, easily removed.

Thus, $\underset{h \to 0}{\text{Lim.}} \; a^h = 1$.

Corollary. a^x is a continuous function of x.

For $a^{x+h} - a^x = a^x (a^h - 1)$,

$\therefore \; \underset{h \to 0}{\text{Lim.}} \; (a^{x+h} - a^x) = 0$, since a^x is finite,

$\therefore \; \underset{h \to 0}{\text{Lim.}} \; a^{x+h} = a^x$,

$\therefore$ a^x is a continuous function of x.

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