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Phase retrieval on circles and lines

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Abstract. Let f and g be analytic functions on the open unit disk $\mathbb{D}$ such that $|f| = |g|$ on a set A. We give an alternative proof of the result of Perez that there exists c in the unit circle $\mathbb{T}$ such that $f = cg$ when A is the union of two lines in $\mathbb{D}$ intersecting at an angle that is an irrational multiple of π, and from this, deduce a sequential generalization of the result. Similarly, the same conclusion is valid when f and g are in the Nevanlinna class and A is the union of the unit circle and an interior circle, tangential or not. We also provide sequential versions of this result and analyze the case $A = r\mathbb{T}$. Finally, we examine the most general situation when there is equality on two distinct circles in the disk, proving a result or counterexample for each possible configuration.

1 Background

A fundamental question of phase retrieval is the following: *For which measurable sets $A \subset \overline{\mathbb{D}}$ is an analytic function $f \in H^1$ determined uniquely to within a unimodular constant by the values of $|f|$ on A*? Here, $\mathbb{D}$ denotes the unit disk and $H^1 = H^1(\mathbb{D})$ the associated Hardy class of analytic functions. Such questions arise in signal processing, crystallography, quantum mechanics, microscopy, and many other applications.

In [8], Pohl, Li, and Boche worked with the class H^1_R of functions in $H^1 = H^1(\mathbb{D})$ such that the inner factor is a Blaschke product (no singular part). They showed (their Theorem 3) that any such function f is determined to within a unimodular constant by the values of $|f|$ on $\mathbb{T}$ and $r\mathbb{T}$ for any $0 < r < 1$. This restriction on the inner factor was removed by Perez [7].

Then in [2], Jaming, Kellay, and Perez solved a phase retrieval problem in $L^2(\mathbb{R}, e^{2c|x|}\,dx)$ for $c > 0$. This is isometric by the Fourier transform to the Hardy space of a strip $\mathcal{S} = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < c\}$ (similar calculations occur in [9]).

By conformal mapping, the phase retrieval problem can then be reformulated on $H^2(\mathbb{D})$ in terms of finding an expression for all pairs $F, G \in H^2$ such that $|F| = |G|$ on $(-1, 1)$. For example, with the notation F^* for the function defined by $F^*(z) = \overline{F(\bar{z})}$, they showed that for $F, G \in H^2$ one has $|F| = |G|$ on $(-1, 1)$ if and only if there exist $u, v \in \mathrm{Hol}(\mathbb{D})$ (the space of all holomorphic functions on $\mathbb{D}$) such that $F = uv$ and $G = uv^*$.

Earlier results along similar lines are due to McDonald [5], who studied the problem of determining an entire function f when $|f(x)|$ is known for every real x.

Finally, a recent paper of Liehr [4] looks at Gabor phase retrieval and Pauli-type uniqueness problems.

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This paper deals with several cases, for which it provides positive and negative results: intersecting curves, disjoint circles, disjoint lines, and a full analysis of the two-circle situation. We also show that many results have sequential counterparts, thus extending and generalizing the work of [7].

2 Results

2.1 Intersecting curves

Perez [7] proved a version of the following result, which generalizes [2, Lemma 4.5]. We give an independent proof, introducing a method that allows us to provide further extensions.

Theorem 2.1 *Let $f, g : \mathbb{D} \to \mathbb{C}$ be analytic functions and L_1, L_2 two line segments in $\mathbb{D}$, intersecting at an angle $\theta \in (0, \pi/2) \setminus \pi\mathbb{Q}$, such that $|f(z)| = |g(z)|$ for all $z \in L_1 \cup L_2$. Then $f = \beta g$ for some constant $\beta \in \mathbb{T}$.*

Proof Clearly by considering the functions on a small disk $\{z \in \mathbb{C} : |z - a| \le r\}$, where a is the intersection point of L_1 and L_2, and by translating and rescaling as necessary, we may suppose that f and g lie in the disk algebra and the two lines intersect at 0.

We may also suppose that the functions are nonvanishing except possibly at 0.

If they do vanish at 0, then the orders of the zeros are equal, since for some N $\lim_{z\to 0} |f(z)|/|z|^N$ exists and is nonzero, and the same for g. So by dividing out any the zeros at 0, and restricting to a smaller disk if necessary, we may suppose that f and g are invertible functions in the disk algebra.

Now $h := f/g$ is also in the disk algebra and it has modulus 1 on both lines L_1 and L_2. Thus, h maps L_1 and L_2 into the unit circle in the complex plane.

Suppose that for some $k \ge 1$, we have

$$h(z) = c_0 + c_k z^k + O(z^{k+1})$$

near 0, with $|c_0| = 1$ and $c_k \ne 0$; then near to 0, it magnifies angles by a factor of k, and so if h is nonconstant, then $k\theta$ must be an integer multiple of π. ∎

Example 2.2 Take $L_1 = [-1, 1]$ and $L_2 = i[-1, 1]$. Then

$$f(z) = \frac{z^2 - 2i}{z^2 + 2i} \qquad \text{and} \qquad g(z) = \frac{z^2 - 3i}{z^2 + 3i}$$

are both unimodular on $L_1 \cup L_2$. Similar examples can be constructed for other rational multiples of π.

Corollary 2.3 *The conclusions of Theorem 2.1 also hold if L_1 and L_2 are two C^1 curves intersecting at an angle $\theta \in (0, \pi/2] \setminus \pi\mathbb{Q}$.*

Proof The conformality argument works with minor changes. ∎

We may prove sequential versions of the above results with the aid of the following easy lemma.

Lemma 2.4 *Let h be a function holomorphic on a neighborhood of 0, $\theta \in [0, 2\pi]$ and (a_n) a real sequence such that $a_n \to 0$ and $h(a_n e^{i\theta})$ is real for all n. Then all the derivatives of the function $z \mapsto h(ze^{-i\theta})$ at 0 are real.*

Proof Clearly, we may assume without loss of generality that $\theta = 0$. Then $h'(0) = \lim_{n\to\infty} h(a_n)/a_n \in \mathbb{R}$. The map $h_1 : z \mapsto h(z)/z - h'(0)$ is also holomorphic around 0, and satisfies $h_1(a_n) \in \mathbb{R}$. Then, in a same way, we get $h_1'(0) = h''(0) \in \mathbb{R}$. Iterating this reasoning, we see that all the derivatives of h at 0 must be real. ∎

Corollary 2.5 *Let $f, g : \mathbb{D} \to \mathbb{C}$ be analytic functions and L_1, L_2 two line segments in $\mathbb{D}$, intersecting at a point p, at an angle $\theta \in (0, \pi/2) \setminus \pi\mathbb{Q}$, such that there are sequences $(a_n) \subset L_1$ and $(b_n) \subset L_2$ tending to p such that $|f(a_n)| = |g(a_n)|$ and $|f(b_n)| = |g(b_n)|$ for all n. Then $f = \beta g$ for some constant $\beta \in \mathbb{T}$.*

Proof We use the argument of the proof of Theorem 2.1 up to the point where we have that $h(a_n)$ and $h(b_n)$ lie in $\mathbb{T}$ for all n. By composing with suitable bilinear transformations, we can construct a function $\tilde{h}$ holomorphic near 0 with sequences $c_n \to 0$ and $d_n \to 0$ such that the c_n are real and the d_n have argument θ, but with $\tilde{h}(c_n)$ and $\tilde{h}(d_n)$ both real for each n. By Lemma 2.4, we conclude that all the derivatives of $z \mapsto \tilde{h}(z)$ and $z \mapsto \tilde{h}(ze^{i\theta})$ are real. By looking at the Taylor series, we see that either $\tilde{h}$ is constant (and thus h is constant) or $e^{in\theta} = 1$ for some $n > 0$. ∎

Clearly, a similar generalization of Corollary 2.3 can be derived.

2.2 Disjoint circles

As described in the introduction to this paper, Perez [7] proved a more general version of the main result from [8], removing the restriction on the singular factor. We give an alternative proof, which leads us to more general results in the same area.

Recall that the Nevanlinna class $\mathcal{N}$ consists of those holomorphic functions in the unit disk $\mathbb{D}$ that are expressible as the ratio of two H^1 functions.

Theorem 2.6 *Let $f, g \in \mathcal{N}$ satisfy $|f| = |g|$ a.e. on $\mathbb{T}$ and $r\mathbb{T}$ for some $0 < r < 1$. Then $f = \lambda g$ for some $\lambda \in \mathbb{T}$.*

Proof By dividing out by the common outer factor u given by

$$u(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})|\,dt\right),$$

we may suppose without loss of generality that f and g are inner functions.

Now, if there are any zeros of f or g on $r\mathbb{T}$, they are at the same points, and finite in number. By dividing out a finite Blaschke product, we may suppose without loss of generality that f and g have no zeros on $r\mathbb{T}$, so indeed f/g is analytic on an annulus $\{z : r - \delta < |z| < r + \delta\}$ for some $\delta > 0$ and unimodular on $r\mathbb{T}$.

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Indeed, f/g has finitely many poles in $\{z : |z| < r + \delta\}$ so, multiplying by a function h such that $h(z/r)$ is a finite Blaschke product $B_1(z)$, fh/g is a function k such that $k(z/r)$ is also a finite Blaschke product $B_2(z)$ since it is analytic in a neighborhood of the disk $r\mathbb{T}$.

Our conclusion is that f/g is a rational function $F(z) = B_2(rz)/B_1(rz)$ in $r\mathbb{D}$; by analytic continuation (and removing any isolated singularities) $f = Fg$ throughout the whole disk. But if F is nonconstant, then it is not unimodular on $\mathbb{T}$ as the zeros and poles are not placed at inverse points. ∎

With the aid of the following lemma, which gives a slight extension of [3, Lemma 1.3], we can prove a stronger result.

Lemma 2.7 *Let F be a function meromorphic on an open set containing the unit circle $\mathbb{T}$ and suppose that $|F(z)| = 1$ for infinitely many $z \in \mathbb{T}$. Then $|F(z)| = 1$ for all $z \in \mathbb{T}$.*

Proof This result uses a modification of an argument introduced in [1, Theorem 1.4]. Namely, we restrict F to a continuous map from $\mathbb{T}$ to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, which we continue to denote by F. Then $X = F^{-1}(\mathbb{T})$ is a closed subset of $\mathbb{T}$.

If X is an infinite proper subset of $\mathbb{T}$, we claim that X contains a point that is a limit of two sequences, one in X and one in $\mathbb{T} \setminus X$. This is clear if $\mathbb{T} \setminus X$ consists of a finite number of intervals. If there are infinitely many intervals, then their endpoints lie in X and accumulate at a point $z_0 \in X$ with the required property.

This point satisfies $|g(z_0)| = 1$, and there exist two sequences $(u_n) \subset X$ and $(v_n) \subset \mathbb{T}\setminus X$ which tend to z_0, such that $|F(u_n)| = 1$ and $|F(v_n)| \neq 1$ (by passing to a subsequence, we may avoid any poles of F).

By composing F with conformal mappings between $\mathbb{T}$ and $\mathbb{R} \cup \{\infty\}$, we obtain a function h holomorphic around 0, and two sequences $(a_n), (b_n) \subset \mathbb{R}$ such that $a_n \to 0$, $b_n \to 0$, $h(a_n) \in \mathbb{R}$ and $h(b_n) \notin \mathbb{R}$. By Lemma 2.4, we have that all the derivatives of h at 0 must be real, so $h(b_n) \in \mathbb{R}$ (since $b_n \in \mathbb{R}$). This contradiction shows that X cannot be an infinite proper subset of $\mathbb{T}$. ∎

Theorem 2.8 *Let $f, g \in H^1(\mathbb{D})$ satisfy $|f| = |g|$ a.e. on $\mathbb{T}$ and on an infinite subset $X \subset r\mathbb{T}$ for some $0 < r < 1$. Then $f = \lambda g$ for some $\lambda \in \mathbb{T}$.*

Proof Let f and g have inner–outer factorizations $f = u_1 v_1$, $g = u_2 v_2$, where u_1 and u_2 are inner and v_1 are v_2 outer with $v_1(0) > 0$ and $v_2(0) > 0$. Since $|f| = |g|$ on $\mathbb{T}$, we have $v_1 = v_2$. We may therefore suppose without loss of generality that f and g are inner.

Note that the function $F := f/g$ is meromorphic in the disk and $|F(z)| = 1$ for an infinite subset of $\mathbb{T}$.

By Lemma 2.7, $|F| = 1$ on $\mathbb{T}$ and so $|f| = |g|$ on $r\mathbb{T}$. The result now follows from Theorem 2.6. ∎

Corollary 2.9 *Let $f, g \in H^1(\mathbb{D})$ (or more generally in the Nevanlinna class) satisfy $|f| = |g|$ a.e. on $\mathbb{T}$ and on an infinite subset $X \subset \Psi_{\omega,\alpha}(r\mathbb{T})$ for some $0 < r < 1$, where $\omega \in \mathbb{T}$, $\alpha \in \mathbb{D}$, and $\Psi_{\omega,\alpha}(z) := \omega\frac{\alpha - z}{1 - \overline{\alpha} z}$. Then $f = \lambda g$ for some $\lambda \in \mathbb{T}$.*

Proof We apply Theorem 2.8 to $f \circ \Psi_{\omega,\alpha}$ and $g \circ \Psi_{\omega,\alpha}$, which implies that $f \circ \Psi_{\omega,\alpha} = g \circ \Psi_{\omega,\alpha}$, and thus $f = g$ since $\Psi_{\omega,\alpha}$ is an automorphism of the open unit disk. ■

Note that every circle contained in $\mathbb{D}$ can be written as $\Psi_{\omega,\alpha}(r\mathbb{T})$ for a suitable choice of $\omega \in \mathbb{T}$, $\alpha \in \mathbb{D}$, and $0 < r < 1$.

Remark 2.10 It is clearly not enough to suppose that $|f| = |g|$ a.e. on $\mathbb{T}$ and on some finite subset $X \subset r\mathbb{T}$. For example, we can let f and g be two different inner functions constructed in the following way:

$$f = \Psi_\alpha \circ (Bu) \text{ and } g = \Psi_\alpha \circ (Bv),$$

where B is the finite Blaschke product associated with the finite set X, u and v are inner functions with $u \neq cv$ for any $c \in \mathbb{T}$ and $\Psi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}$ for an arbitrary $\alpha \in \mathbb{D}$. Since f and g are inner, their radial limits are of modulus one almost everywhere on $\mathbb{T}$, $f(z) = \alpha = g(z)$ for all $z \in X$. The choice of u and v implies that there is no $c \in \mathbb{T}$ such that $f = cg$.

If the inner parts of f and g are finite Blaschke products B_1 and B_2, then, depending on their degrees, $|f| = |g|$ a.e. on $\mathbb{T}$ and at only finitely many points on $r\mathbb{T}$ is a sufficient condition for Theorem 2.8 to hold.

Theorem 2.11 *Let B_1 and B_2 be Blaschke products of degrees M and N, respectively, and take $0 < r < 1$. If $|B_1(z)| = |B_2(z)|$ for more than $2N + 2M - 1$ distinct points on $r\mathbb{T}$, then B_2 is a constant multiple of B_1.*

Proof Note that on $r\mathbb{T}$, we have $\bar{z} = r^2/z$; so for an elementary Blaschke factor with $a \in \mathbb{D}$, we have

$$\frac{z - a}{1 - \bar{a}z} \frac{\bar{z} - \bar{a}}{1 - a\bar{z}} = \frac{(z - a)(r^2 - \bar{a}z)}{(1 - \bar{a}z)(z - r^2 a)},$$

and thus the equation

$$B_1(z)\overline{B_1(z)} = B_2(z)\overline{B_2(z)}$$

can be rewritten as $R_1(z) = R_2(z)$, where R_1 and R_2 are rational functions of degrees $2M$ and $2N$, respectively. If $\alpha_1, \dots, \alpha_M$ are the zeros of B_1 and $\beta_1, \dots, \beta_N$ are the zeros of B_2, we get

$$R_1(z) = \left(\frac{z - \alpha_1}{1 - \overline{\alpha_1} z}\right) \cdots \left(\frac{z - \alpha_M}{1 - \overline{\alpha_M} z}\right) \left(\frac{r^2 - \overline{\alpha_1} z}{z - \alpha_1 r^2}\right) \cdots \left(\frac{r^2 - \overline{\alpha_M} z}{z - \alpha_M r^2}\right)$$

and

$$R_2(z) = \left(\frac{z - \beta_1}{1 - \overline{\beta_1} z}\right) \cdots \left(\frac{z - \beta_N}{1 - \overline{\beta_N} z}\right) \left(\frac{r^2 - \overline{\beta_1} z}{z - \beta_1 r^2}\right) \cdots \left(\frac{r^2 - \overline{\beta_N} z}{z - \beta_N r^2}\right).$$

Denote by $P_1(z)$ (resp. $P_2(z)$) the numerator of $R_1(z)$ (resp. $R_2(z)$) and $Q_1(z)$ (resp. $Q_2(z)$) the denominator of $R_1(z)$ (resp. $R_2(z)$). This reduces to a polynomial equation

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of degree at most $2N + 2M - 1$, noting that the coefficient of z^{2N+2M} of the polynomial P_1Q_2 is $(-1)^{N+M}\prod_{i=1}^{M}\overline{\alpha_i}\prod_{j=1}^{N}\overline{\beta_j}$, which coincide with the coefficient of z^{2N+2M} of the polynomial P_2Q_1. Therefore, if $P_1Q_2 - P_2Q_1$it has more than $2N + 2M - 1$ roots, then it is identically zero. That is, $|B_1| = |B_2|$ on the whole of $r\mathbb{T}$ and thus the result follows from the previous discussions. ■

In general, we have the following parameterization of functions of equal modulus on $r\mathbb{T}$.

Theorem 2.12 *Suppose that $f, g \in \mathrm{Hol}(\mathbb{D})$ and $|f| = |g|$ on $r\mathbb{T}$ for some $0 < r < 1$. Then there exist finite Blaschke products B_1, B_2 such that*

$$B_1(z/r)f(z) = B_2(z/r)g(z).$$

Proof We have that f/g is meromorphic on a neighborhood of $r\overline{\mathbb{D}}$ and $|f/g| = 1$ on $r\mathbb{T}$. By choosing a suitable finite Blaschke product B_1 and writing $\tilde{B}_1(z) = B_1(z/r)$, we have that $\tilde{B}_1 f/g$ is still unimodular on $r\mathbb{T}$ and holomorphic on a neighborhood of $r\overline{\mathbb{D}}$. Thus, it has the form $\tilde{B}_2$, where $\tilde{B}_2(z) = B_2(z/r)$ for some finite Blaschke product B_2. ■

2.3 Disjoint lines

In [8], it is claimed that a similar result to Theorem 2.6 can be proved for the corresponding space $H^1_R(\mathbb{C}^+)$ of analytic functions in the upper half-plane $\mathbb{C}^+$, considering the values on $\mathbb{R}$ and $i + \mathbb{R}$, although no proof is given. The result cannot be deduced directly from the disk result as this strip $\mathbb{S} = \{s \in \mathbb{C} : 0 < \mathrm{Re}\, s < 1\}$ is not conformally equivalent to the annulus.

We note also that there are nonconstant meromorphic functions F defined on $\mathbb{C}^+$ such that $|F| = 1$ on both $\mathbb{R}$ and $i + \mathbb{R}$. One example is

$$F(s) = \frac{i - \exp(\pi s)}{i + \exp(\pi s)},$$

which provides a conformal map of the strip $\mathbb{S}$ onto the unit disk. However, the function does not extend to a quotient of H^p functions, since its zeros $\{i(\frac{1}{2} + 2n) : n \geq 0\}$ do not form a Blaschke sequence.

By means of the Weierstrass factorization theorem, F can be modified to give distinct analytic functions G and H on $\mathbb{C}^+$ such that $|G| = |H|$ on $\mathbb{R}$ and $1 + i\mathbb{R}$.

In fact, the result we require does hold for the half-plane. It will be convenient to work with the right half-plane $\mathbb{C}_+$.

Theorem 2.13 *Suppose that $f, g \in H^1(\mathbb{C}_+)$ and $|f| = |g|$ a.e. on $i\mathbb{R}$, while $|f| = |g|$ on $1 + i\mathbb{R}$. Then $f = cg$ for some unimodular constant c.*

Proof We may assume without loss of generality that f, g are inner.

Now, the functions F and G defined by $F(s) = f(1 + s)$ and $G(s) = g(1 + s)$ satisfy $|F| = |G|$ on $i\mathbb{R}$; and these functions are also in $H^1(\mathbb{C}_+)$, and holomorphic in a neighborhood of $i\mathbb{R}$. Let us consider the inner–outer factorization $F = uh$, say.

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Clearly, the inner factor u has no zeros accumulating at a point on $i\mathbb{R}$. Moreover, the outer factor h is continuous on the closed half-plane $\mathbb{C}_+$ (see, for example, the arguments in [6, Section 4.3.8]), which means that the inner factor u is also continuous except possibly at the discrete set consisting of the zeros of uh on $i\mathbb{R}$. On factoring out a zero at iy_0, by considering uh/b, where $b(s) = \dfrac{s - iy_0}{1+s}$ (an outer function) we see that u is also continuous at the zeros of uh. Thus, u has no singular part except possibly an exponential factor $e^{-\alpha s}$ for some $\alpha \geq 0$. We conclude that $u(s) = B_1(s)e^{-\alpha s}$, where B_1 is a Blaschke product whose zeros do not accumulate at any point in $\overline{\mathbb{C}_+}$. The function B_1, which is a product of factors of the form $\dfrac{s - a_n}{s + \overline{a_n}}$, therefore has a meromorphic extension to $\mathbb{C}$.

So suppose that B_1 has a zero at w. Then the Blaschke factor of f, say, b_1, has a zero at $w+1$, and hence a pole at $-\overline{w}-1$. Thus B_1 has a pole at $-\overline{w}-2$ and hence a zero at $w+2$.

Repeating this argument, we find that B_1 has zeros at $w_n := w + 2n$, $n \in \mathbb{N}$. But these do not satisfy the Blaschke condition $\sum_{n=0}^{\infty} \frac{\operatorname{Re} w_n}{1+|w_n|^2} < \infty$, and that is a contradiction.

So B_1 is a unimodular constant and we may apply a similar argument to the inner function v. We conclude that $f(s)/g(s) = c\exp(\lambda(s-1))$, for a unimodular constant c, which, since f and g are inner, means that $|\exp(\lambda(iy-1))| = 1$ a.e., and $\lambda = 0$. ■

The translation of the previous theorem into the open unit disk via a standard conformal map between the unit disk and the right half-plane is the following.

Corollary 2.14 *Suppose that $f, g \in H^1(\mathbb{D})$ and $|f| = |g|$ a.e. on $\mathbb{T}$, while $|f| = |g|$ on $\{z \in \mathbb{C} : |z - r| = 1 - r\}$ for some $r \in (0,1)$. Then $f = cg$ for some unimodular constant c.*

Remark 2.15 Corollary 2.14 also holds when f and g are in thee Nevalinna class $\mathcal{N}$ and when $|f| = |g|$ on $\mathbb{T}$ and on a sequence included in $\{z \in \mathbb{C} : |z - r| = 1 - r\}$.

2.4 Two circles contained in the unit disk

Suppose now that f and g are holomorphic on $\mathbb{D}$ and $|f| = |g|$ on two distinct circles C_1, C_2 contained in $\mathbb{D}$, which bound open disks D_1 and D_2, respectively. There are five cases to consider (we can of course swap the roles of C_1 and C_2 if we wish):

(1) C_1 and C_2 are "internally" disjoint with $C_2 \subset D_1$;
(2) C_1 and C_2 are "externally" disjoint with $C_2 \subset \mathbb{D} \setminus \overline{D_1}$;
(3) C_1 and C_2 are "internally" tangential with $C_2 \subset \overline{D_1}$;
(4) C_1 and C_2 are "externally" tangential with $C_2 \subset \mathbb{D} \setminus D_1$;
(5) C_1 and C_2 intersect in two points, at which they make an angle $\theta \in (0, \pi/2]$.

Theorem 2.16 *Suppose that C_1 and C_2 satisfy one of the five conditions listed above and that $|f| = |g|$ on $C_1 \cup C_2$. Then in cases (1)–(4) $f = cg$ for some c with $|c| = 1$. The same holds in case (5) if θ is an irrational multiple of π, but need not hold if θ is a rational multiple of π.*

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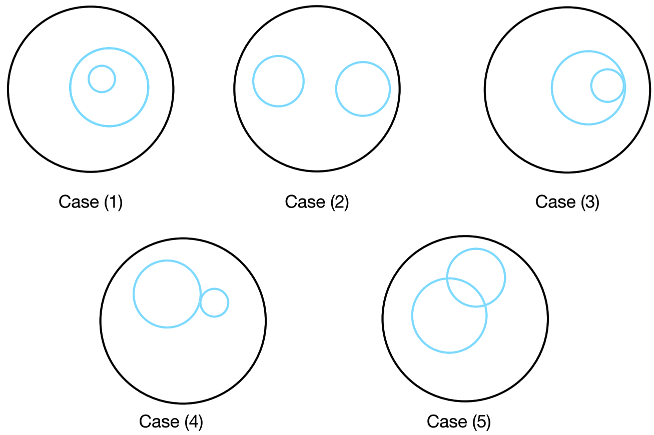

The five cases for two circles contained in $\mathbb{D}$.

Proof For parts (1)–(4), we may suppose without loss of generality, by composing with an automorphism, that C_1 is the circle $r\mathbb{T}$ centered at 0 with radius r for some $0 < r < 1$. For (1), we now consider f_r and g_r defined by $f_r(z) = f(z/r)$ and $g_r(z) = g(z/r)$. Then, (1) follows easily from Corollary 2.9 and the comment following it.

For (2) with $C_1 = r\mathbb{T}$, we note from Theorem 2.12 that $f(z)/g(z)$ is a rational function of the form $B_2(z/r)/B_1(z/r)$ and hence meromorphic on $\mathbb{C} \cup \{\infty\}$. With the change of variable $w = z/r$, we have a meromorphic function $B_2(w)/B_1(w)$ that is unimodular on a smaller circle contained in $\mathbb{D}$. By composition with an automorphism, we may supposed that the smaller circle is centered at 0. But now the fact that f/g is a constant follows an argument similar to that used in the proof of Theorem 2.6 (specifically, that poles and zeros occur in pairs of inverse points with respect to one circle, which means that they cannot occur at inverse points with respect to the other circle).

(3) is easily derived from Corollary 2.14.

For (4), we may again suppose that $C_1 = r\mathbb{T}$ and that $f(z)/g(z) = B_2(z/r)/B_1(z/r)$. Now we take $w = z/r$ again, reducing to the case when a function of the form $B_2(w)/B_1(w)$ is of modulus 1 on a circle internally tangential to $\mathbb{T}$. By means of a conformal mapping, we can transform this to the half-plane, producing a rational function that has absolute value 1 on the lines $i\mathbb{R}$ and $1 + i\mathbb{R}$. Finally, an argument based on the fact that zeros and poles must occur at pairs of inverse points a and $-\overline{a}$ with respect to $i\mathbb{R}$ as well as inverse points a and $-\overline{a} + 2$ with respect to $1 + i\mathbb{R}$ (cf. the proof of Theorem 2.13) leads to a contradiction unless f/g is constant.

For (5), suppose that the two intersection points of C_1 and C_2 are a and b. By changing the variable to $w = 1/(z - a)$, we transform the circles into straight lines, which meet at $1/(b - a)$, still at an angle θ. The result for irrational θ/π now follows from Theorem 2.1, applied to a small disk centered at $1/(b - a)$.

We may similarly construct counterexamples for rational θ/π. Let C_1 and C_2 be circles of radius $1/3$ centered at $\pm 1/(3\sqrt{2})$. Then they meet at $\pm a$, where $a = i/(3\sqrt{2})$.

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The angle between them is a right angle. The transformation $w = \dfrac{z+a}{z-a}$ sends the circles to two perpendicular lines, meeting at 0, from which examples similar to those in Example 2.2 can be constructed easily. ∎

Example 2.17 There are limitations to the "inverse points" argument above if two circles are not concentric. For example, let $C_1 = \{z \in \mathbb{C} : |z - 3/5| = 1/5\}$ and $C_2 = \{z \in \mathbb{C} : |z + 3/5| = 1/5\}$. Then the points $z_\pm := \pm\sqrt{8}/5$ are inverse with respect to both circles, and one can construct rational functions of the form $c\dfrac{z-z_+}{z-z_-}$ that have constant absolute values on both circles. However, they do not have the same absolute values on C_1 and C_2, so do not provide a counterexample to the above result.

Finally, we note that Theorem 2.16 has sequential counterparts, which may be proved using the methods of Section 2.1. We omit the details.

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