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Perforated Ordered K_0-Groups

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Abstract. A simple C^*-algebra is constructed for which the Murray-von Neumann equivalence classes of projections, with the usual addition—induced by addition of orthogonal projections—form the additive semigroup

$$\{0, 2, 3, \dots\}.$$

(This is a particularly simple instance of the phenomenon of perforation of the ordered K_0-group, which has long been known in the commutative case—for instance, in the case of the four-sphere—and was recently observed by the second author in the case of a simple C^*-algebra.)

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The purpose of this note is to describe a modification of the construction of the second author in [12] of a simple C^*-algebra with perforated ordered K_0-group, which leads to examples in which this ordered group has a particularly simple form.

(Recall that an ordered abelian group, or, rather, its positive cone, is said to be perforated—see [4]—if there exists a non-positive element g such that, for some $n = 2, 3, \dots$, $ng \geq 0$.) (It is said to be strongly perforated, at least in the simple case,—opposite: weakly unperforated—if g may be chosen such that $ng > 0$, *i.e.*, not to be a torsion element. For torsion-free groups the two properties coincide.)

Theorem 1 *For any $n = 2, 3, \dots$ there exists a simple, separable, amenable C^*-algebra with (pre-)ordered K_0-group isomorphic to the group $\mathbb{Z}$ with positive cone*

$$\{0, n, n+1, \dots\}.$$

(In particular, while in general the K_0-group of a C^*-algebra, with the positive cone consisting of the classes of projections in the algebra and in matrix algebras over it, is just a pre-ordered abelian group—in other words, the positive cone is only known to be some subsemigroup, containing 0—in the present case one obtains an ordered group—that is, the positive cone has zero intersection with its negative, and generates the whole group.)

(While the abelian group arising as K_0 of a C^*-algebra is presumably unrestricted (this is known in the countable case—see [10]), the pre-order or even order structures arising are only incompletely known. It is known that every weakly unperforated simple countable ordered group arises in this way—see [5]. Here, simple means that every non-zero positive element is an order unit—some multiple of it majorizes any given element.)

The present construction is based on the inductive limit construction of [12]. It consists in applying the generalized mapping torus construction of [5] (*cf.* also [7]) to the building blocks of [12] (instead of to finite-dimensional algebras), in a way compatible with the embeddings of [12]. As in [5], an initially non-simple inductive limit is made simple by a deformation.

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2 The Building Blocks

The desired ordered K_0-group will be obtained already at the level of building blocks. (The K_0-map corresponding to the embedding of one building block into the next will be just the identity map.)

The building blocks that we shall use, like those of [5], are obtained by the following generalized mapping torus construction. Given two C*-algebras, C and D, and a pair of maps, ϕ_0 and ϕ_1, from one to the other (from C to D), consider the C*-algebra

$$\begin{aligned} A &= A(C, D, \phi_0, \phi_1) \\ &:= \{(c, d); c \in C, d \in \mathrm{C}([0,1]; D), d(0) = \phi_0(c), d(1) = \phi_1(c)\}. \end{aligned}$$

In a natural way, A is an extension of $\mathrm{S}\,D$, the suspension of D, by C,

$$0 \to \mathrm{S}\,D \to A \to C \to 0$$

(the map $A \to C$ being just the map $(c, d) \mapsto c$). We shall refer to this copy of $\mathrm{S}\,D$ naturally contained in A—the kernel of the map $A \to C$—as the canonical closed two-sided ideal of A.

Theorem 2 *The index map $b_*\colon \mathrm{K}_*\,C \to \mathrm{K}_{1-*}\,\mathrm{S}\,D = \mathrm{K}_*\,D$ in the six-term periodic exact sequence for the extension*

$$0 \to \mathrm{S}\,D \to A \to C \to 0$$

is the difference

$$K_*\phi_1 - K_*\phi_0\colon K_*C \to K_*D.$$

Thus, the six-term exact sequence may be written as the short exact sequence

$$0 \to \mathrm{Coker}\, b_{1-*} \to \mathrm{K}_*\,A \to \mathrm{Ker}\, b_* \to 0.$$

In particular, if b_{1-i} is surjective, then $\mathrm{K}_i\,A$ is isomorphic to its image, $\mathrm{Ker}\, b_i$, in $\mathrm{K}_i\,C$.

Suppose that cancellation holds for D—i.e., that cancellation holds in the semigroup of Murray-von Neumann equivalence classes of projections in D and in matrix algebras over D (equivalently, in $D \otimes \mathcal{K}$). It follows that if b_1 is surjective, so that $\mathrm{K}_0\,A \subseteq \mathrm{K}_0\,C$, then

$$(\mathrm{K}_0\,A)^+ = (\mathrm{K}_0\,C)^+ \cap \mathrm{K}_0\,A.$$

The preceding conclusion also holds if cancellation is only known to hold for each pair of projections in $D \otimes \mathcal{K}$ obtained as the images under the maps ϕ_0 and ϕ_1 of a single projection in $C \otimes \mathcal{K}$. (In other words, if two such projections in $D \otimes \mathcal{K}$ have the same K_0-class then they should be equivalent.) (Of course, b_1 is still assumed to be surjective.)

Proof Consider first the case that $C = D$, $\phi_0 = 0$ and ϕ_1 is the identity map. In this case, A is just the cone over D, and the index map is just the canonical identification of $\mathrm{K}_*\,D$ with $\mathrm{K}_{1-*}\,\mathrm{S}\,D$—in other words, in the present notation, the identity map, *i.e.*, $\mathrm{K}_*\,\phi_1$, as desired.

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The general case follows by functoriality. More specifically, by naturality of the six-term exact sequence, if $\phi_0 = 0$ then the map from A into the cone over D determined canonically by ϕ_1 gives a map between the six-term exact sequence for A and that for the cone over D, considered above, such that the whole diagram is commutative. In particular, the index map for A (composed with the identity on $\mathrm{K}_{1-*}\,\mathrm{S}\,D = \mathrm{K}_*\,D$) is equal to $K_*\phi_1\colon K_*C \to K_*D$ (composed with the identity—the index map for the cone over D). The case that $\phi_1 = 0$ is similar—with the obvious change in sign—and can also be deduced from the case $\phi_0 = 0$ by functoriality applied to the canonical isomorphism $A(\phi_0, \phi_1) \to A(\phi_1, \phi_0)$. The case that neither ϕ_0 nor ϕ_1 is zero can be deduced from the two cases $\phi_0 = 0$ and $\phi_1 = 0$ by applying naturality to the subalgebra A_0 of A consisting of the elements (c, d) with $d(\frac{1}{2}) = 0$. (The kernel of the canonical map of A_0 onto C is isomorphic to $\mathrm{S}\,D \oplus \mathrm{S}\,D$, and comparing the six-term exact sequence for this extension separately with that for $A(\phi_0, 0)$ and $A(0, \phi_1)$ (by restricting to the left and right halves of the interval $[0, 1]$, respectively), one obtains that the index map $\mathrm{K}_*\,C \to \mathrm{K}_*\,D \oplus \mathrm{K}_*\,D$ is $(-\,\mathrm{K}_*\,\phi_0) \oplus \mathrm{K}_*\,\phi_1$. Comparing the six-term exact sequences for the maps of A_0 and A onto C then yields that the index map for the latter is the sum of the two components of that for the former, *i.e.*, $\mathrm{K}_*\,\phi_1 - \mathrm{K}_*\,\phi_0$, as desired.)

Suppose that cancellation holds for D, and let $g \in \mathrm{K}_0\,A$ be such that the image of g in $\mathrm{K}_0\,C$ is the K_0-class of a projection $p \in C \otimes \mathcal{K}$. Since this class, $\mathrm{K}_0\,p$, belongs to $\operatorname{Ker} b_0$, and $b_0 = \mathrm{K}_0\,\phi_1 - \mathrm{K}_0\,\phi_0$, the images of p by $\phi_0 \otimes 1$ and $\phi_1 \otimes 1$ in $D \otimes \mathcal{K}$ have the same class in $\mathrm{K}_0(D)$. By cancellation, these two projections are Murray-von Neumann equivalent in $D \otimes \mathcal{K}$, and hence homotopic. (If two projections are Murray-von Neumann equivalent in a C^*-algebra B, then they are unitarily equivalent in $M_2 \otimes B$—to show this it is enough to show that two orthogonal equivalent projections are unitarily equivalent, and this follows from the fact that if v is a partial isometry with square zero, then $v + v^* + (1 - v^*v - vv^*)$ is a unitary (multiplier) transforming v^*v onto vv^* —, and hence they are homotopic in $M_2 \otimes M_2 \otimes B$—if u is a unitary element of a C^*-algebra B_1 then $u \oplus u^{-1}$ is an element of the connected component of 1 in the unitary group of $M_2 \otimes B_1$.) In other words, there is a projection in $A \otimes \mathcal{K}$ mapping onto p in the quotient $C \otimes \mathcal{K}$.

In other words, every positive element of the image of $\mathrm{K}_0\,A$ in $\mathrm{K}_0\,C$ is the image of a positive element of $\mathrm{K}_0\,A$. In the case that the map $\mathrm{K}_0\,A \to \mathrm{K}_0\,C$ is injective (*i.e.*, b_1 is surjective), this proves that $(\mathrm{K}_0\,A)^+ = (\mathrm{K}_0\,C)^+ \cap K_0A$, as desired.

The preceding proof also establishes the final assertion of the theorem.

3 Maps Between Building Blocks

As in [5], we shall begin by constructing maps between building block algebras which respect the canonical ideals (*i.e.*, which take the ideal of elements vanishing at infinity in the first algebra into the corresponding ideal of the second algebra).

The inductive limit of a sequence of such maps cannot be simple (except possibly in the degenerate case that the canonical ideals are all mapped into zero in the limit). We shall show in Sections 4 and 5, below, how to deform such maps, in certain circumstances, to obtain a simple inductive limit.

The maps that we shall consider (and construct examples of later) have, schematically, the following eigenvalue pattern:

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More specifically, they are given by maps at the level of fibres—four maps, with certain compatibility relations—as follows.

Theorem 3 *Let A_1 and A_2 be building block algebras, as described in Section 2,*

$$A_i = A(C_i, D_i, \phi_0^i, \phi_1^i), \quad i = 1, 2.$$

Let there be given four maps between fibres,

$$\gamma \colon C_1 \to C_2,$$
$$\delta, \delta' \colon D_1 \to D_2, \quad \textit{and}$$
$$\varepsilon \colon C_1 \to D_2,$$

such that δ, δ', and ε have mutually orthogonal images, and

$$\delta\phi_0^1 + \delta'\phi_1^1 + \varepsilon = \phi_0^2\gamma,$$
$$\delta\phi_1^1 + \delta'\phi_0^1 + \varepsilon = \phi_1^2\gamma.$$

Then there exists a unique map

$$\theta \colon A_1 \to A_2,$$

respecting the canonical ideals, giving rise to the map $\gamma \colon C_1 \to C_2$ between the quotients (the fibres at infinity), and such that for any $0 < s < 1$, if e_s denotes evaluation at s, and e_∞ the evaluation at (the fibre at) infinity,

$$e_s\theta = \delta e_s + \delta' e_{1-s} + \varepsilon e_\infty.$$

More generally, the conclusion holds with ε replaced by a continuous family of maps,

$$\varepsilon_s \colon C_1 \to D_2, \quad 0 \le s \le 1,$$

such that the image of ε_s is orthogonal to the images of δ and δ' for every s, and such that

$$\delta\phi_j^1 + \delta'\phi_{1-j}^1 + \varepsilon_j = \phi_j^2\gamma, \quad j = 0, 1,$$

with the condition above replaced by the modified condition

$$e_s\theta = \delta e_s + \delta' e_{1-s} + \varepsilon_s e_\infty, \quad 0 < s < 1.$$

Proof Let $(c, d) \in A_1$, so that $c \in C_1$, $d \in \mathrm{C}([0,1]) \otimes D_1$, and $\phi_0^1(c) = d(0)$, $\phi_1^1(c) = d(1)$; we must show that $(c', d') \in A_2$, where

$$c' = \gamma(c),$$
$$d'(s) = \delta\big(d(s)\big) + \delta'\big(d(1-s)\big) + \varepsilon_s(c), \quad 0 \leq s \leq 1.$$

In other words, we must show that

$$\phi_0^2\big(\gamma(c)\big) = \delta\big(d(0)\big) + \delta'\big(d(1)\big) + \varepsilon_0(c),$$
$$\phi_1^2\big(\gamma(c)\big) = \delta\big(d(1)\big) + \delta'\big(d(0)\big) + \varepsilon_1(c).$$

These equations follow immediately from the relations between the maps γ, δ, δ', and ε, and the assumption $(c, d) \in A_1$.

4 Deforming the Maps

Let us describe a procedure for deforming a map between building block algebras, of the kind constructed in Section 3, with suitable special properties, in such a way as to destroy the compatibility with the canonical ideals—with a view to constructing a simple inductive limit C*-algebra. (Conditions for the inductive limit of a sequence of such deformed maps to be simple will be given in Section 5.)

Theorem 4 *Let A_1 and A_2 be building block* C**-algebras, as described in Section 2,*

$$A_i = A(C_i, D_i, \phi_0^i, \phi_1^i), \quad i = 1, 2.$$

Let $\theta\colon A_1 \to A_2$ be a homomorphism as constructed in Section 3, from maps $\gamma\colon C_1 \to C_2$, $\delta, \delta'\colon D_1 \to D_2$, and $\varepsilon\colon C_1 \to D_2$, such that δ, δ', and ε have orthogonal ranges, and

$$\delta\phi_0^1 + \delta'\phi_1^1 + \varepsilon = \phi_0^2\gamma,$$
$$\delta\phi_1^1 + \delta'\phi_0^1 + \varepsilon = \phi_1^2\gamma.$$

Let there be given a map $\beta\colon D_1 \to C_2$ such that the composed map $\beta\phi_1^1$ is a direct summand of the map $\gamma\colon C_1 \to C_2$, and such that the composed maps $\phi_0^2\beta$ and $\phi_1^2\beta$ are direct summands of the maps δ' and δ, respectively. Suppose that the decomposition of γ as the orthogonal sum of $\beta\phi_1^1$ and another map is such that the image of the second map is orthogonal to the image of β, not just of $\beta\phi_1^1$. This last requirement is automatically satisfied if C_1, D_1, and the map ϕ_1^1 are unital.

It follows that the given map $\theta\colon A_1 \to A_2$, which has eigenvalue pattern

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(cf. Section 3) is homotopic to a map with eigenvalue pattern

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More explicitly, for any $0 < t < \frac{1}{2}$*, the map* $\theta\colon A_1 \to A_2$ *is homotopic to a map* $\theta_t\colon A_1 \to A_2$ *differing from it only as follows: the map* $e_\infty\theta_t$ *has the direct summand* βe_t *instead of one of the direct summands* $\beta\phi_0^1 e_\infty$ *and* $\beta\phi_1^1 e_\infty$ *of* $e_\infty\theta$*, and for each* $0 < s < 1$ *the map* $e_s\theta_t$ *has either the direct summand* $\phi_0^2\beta e_t$ *instead of the direct summand* $\phi_0^2\beta e_{1-s}$ *of* $e_s\theta$*, or the direct summand* $\phi_1^2\beta e_t$ *instead of the direct summand* $\phi_1^2\beta e_s$ *of* $e_s\theta$*, or both.*

Furthermore, let $\alpha\colon D_1 \to C_2$ *be any map homotopic to* β *within the hereditary sub-*C^**-algebra of* C_2 *generated by the image of* β*. Then the map* θ_t *is homotopic to a map* $\theta_t'\colon A_1 \to B_2$ *(with a similar eigenvalue pattern), differing from* θ_t *only in the direct summands mentioned, and such that* $e_\infty\theta_t'$ *has the direct summand* αe_t *instead of* βe_t*, and for each* $0 < s < 1$*,* $e_s\theta_t'$ *has either* $\phi_0^2\alpha e_t$ *instead of* $\phi_0^2\beta e_t$*, or* $\phi_1^2\alpha e_t$ *instead of* $\phi_1^2\beta e_t$*.*

More generally, the conclusion holds with the map ε *replaced by a continuous family of maps as in Theorem 3.*

Proof Let $0 < t < \frac{1}{2}$ be given. For each t' with $t \le t' < 1$ let us construct a map $\theta_{t'}\colon A_1 \to A_2$, such that $\theta_{t'}$ has the specified properties, and, with $\theta_1 = \theta$, for every $a \in A_1$ the map

$$[t, 1] \ni t' \mapsto \theta_{t'}(a) \in A_2$$

is continuous (*i.e.*, $t' \mapsto \theta_{t'}$ is a homotopy).

Fix t' with $t \le t' < 1$. To specify a map $\theta_{t'}\colon A_1 \to A_2$ we must specify the map $A_1 \to C_2$ obtained by evaluating at the fibre at infinity and, also, for each $0 < s < 1$, the map $A_1 \to D_2$ obtained by evaluating at the fibre at s. In other words, in the notation of Section 3, we must specify the maps $e_\infty\theta_{t'}$ and $e_s\theta_{t'}$, $0 < s < 1$.

The map $A_1 \to C_2$ to be considered, as $e_\infty\theta_{t'}$, is constructed as follows. Consider first the map $A_1 \to C_2$ obtained by evaluating the given map, θ, at infinity. This map, namely, $e_\infty\theta$, consists of first evaluating at the fibre of A_1 at infinity, C_1, and then applying the map $\gamma\colon C_1 \to C_2$. In other words,

$$e_\infty\theta = \gamma e_\infty.$$

By hypothesis, γ is the orthogonal sum of $\beta\phi_1^1$ and another map (which, of course, is uniquely determined), and so $\gamma e_\infty\colon A_1 \to C_2$ is the orthogonal sum of $\beta\phi_1^1 e_\infty$ and another map. By hypothesis, the image of the second map is orthogonal to the whole of the image of β (not just the image of $\beta\phi_1^1 e_\infty$). We may therefore alter the first direct summand, $\beta\phi_1^1 e_\infty$, by replacing it by the map $\beta e_{t'}$ (the composition of $e_{t'}\colon A_1 \to D_1$ and $\beta\colon D_1 \to C_2$), and still have orthogonal maps. The map $A_1 \to C_2$ to be considered is the sum of $\beta e_{t'}$ and the original second direct summand of γe_∞ $(= e_\infty\theta)$.

The map $A_1 \to D_2$ to be considered, as $e_s\theta_{t'}$, is constructed as follows. Consider first the map $A_1 \to D_2$ obtained by evaluating the given map, θ, at s. This map, $e_s\theta$, is the sum of the three maps δe_s, $\delta' e_{1-s}$, and εe_∞, which have orthogonal images:

$$e_s\theta = \delta e_s + \delta' e_{1-s} + \varepsilon e_\infty.$$

By hypothesis, δ and δ' contain $\phi_1^2\beta$ and $\phi_0^2\beta$, respectively, as direct summands, and so δe_s and $\delta' e_{1-s}$ contain $\phi_1^2\beta e_s$ and $\phi_0^2\beta e_{1-s}$ as direct summands. In other words, δe_s is the orthogonal sum of $\phi_1^2\beta e_s$ and another map—in the sense that the images are orthogonal—and $\delta' e_{1-s}$ is the orthogonal sum of $\phi_0^2\beta e_{1-s}$ and another map. In particular, $e_s\theta$ is the orthogonal sum of $\phi_1^2\beta e_s$ and $\phi_0^2\beta e_{1-s}$ and another map. Alter the first two summands—without changing their images, and so preserving orthogonality—as follows: if $t' \le s$, replace $\phi_1^2\beta e_s$ by $\phi_1^2\beta e_{t'}$, and if $s \le 1 - t'$ (note that, if $t' \le \frac{1}{2}$, both of these cases may occur simultaneously), replace $\phi_0^2\beta e_{1-s}$ by $\phi_0^2\beta e_{t'}$. The map $A_1 \to D_2$ to be considered is the sum of these two new maps—although note that, depending on the relation between s and t', either or both of these maps may be not in fact new—together with the original third direct summand of $e_s\theta$ (besides $\phi_1^2\beta e_s$ and $\phi_0^2\beta e_{1-s}$).

Let us now verify that there exists a map $\theta_{t'}\colon A_1 \to A_2$ the evaluations of which at the various fibres of A_2 are the maps specified above. This means simply that for any $a \in A_1$, the images of a in the fibres of A_2 by these maps determine an element of A_2.

Note, incidentally, that continuity of the image of each $a \in A_1$ in each single fibre of A_2, as a function of t'—and indeed in a way uniform over all the fibres—, is immediate from the construction. (In other words, the map $\theta_{t'}$ will depend continuously on t', in the pointwise topology, as soon as it is known to exist.)

Let $a = (c, d)$ be an element of A_1. Then $\theta a = (c', d')$ where $c' = \gamma(c)$ and

$$d'(s) = \delta\big(d(s)\big) + \delta'\big(d(1-s)\big) + \varepsilon(c), \quad 0 \le s \le 1.$$

Moreover, d' is continuous on $[0, 1]$ and the boundary conditions

$$\phi_0^2\big(\gamma(c)\big) = \delta\big(d(0)\big) + \delta'\big(d(1)\big) + \varepsilon(c),$$
$$\phi_1^2\big(\gamma(c)\big) = \delta\big(d(1)\big) + \delta'\big(d(0)\big) + \varepsilon(c)$$

hold. With (c'', d'') defined by altering c' and d' as above, *i.e.*, replacing the direct summand $\beta\phi_1^1(c)$ of $c' = \gamma(c)$ by $\beta\big(d(t')\big)$, and, for each $0 \le s \le 1$, replacing the direct summand $\phi_1^2\beta\big(d(s)\big)$ of d' by $\phi_1^2\beta\big(d(t')\big)$ if $t' \le s$, and the direct summand $\phi_0^2\beta\big(d(1-s)\big)$ of d' by $\phi_0^2\beta\big(d(t')\big)$ if $s \le 1 - t'$, to show that (c'', d'') belongs to A_2 we must check that d'' is continuous—this is clear—and that the boundary conditions

$$\phi_0^2(c'') = d''(0), \quad \phi_1^2(c'') = d''(1)$$

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hold. The boundary conditions hold as they hold for (c', d') and also for the difference, $(c'', d'') - (c', d')$—as is immediate from the computations

$$\begin{aligned} c'' - c' &= \beta\big(d(t')\big) - \beta\phi_1^1(c) = \beta\big(d(t')\big) - \beta\big(d(1)\big), \\ (d'' - d')(0) &= \phi_0^2\beta\big(d(t')\big) - \phi_0^2\beta\big(d(1)\big), \\ (d'' - d')(1) &= \phi_1^2\beta\big(d(t')\big) - \phi_1^2\beta\big(d(1)\big). \end{aligned}$$

(In fact, the boundary conditions hold separately for both terms in the above expression of the difference.)

Given a homotopy from β to a map $\alpha\colon D_1 \to C_2$, within the hereditary sub-C^*-algebra of C_2 generated by the image of β, change $\theta_{t'}$ (for fixed t') as follows: Replace the direct summand βe_t of $e_\infty\theta_t$ by αe_t; replace the direct summand $\phi_0^2\beta e_t$ of $e_s\theta_t$ by $\phi_0^2\alpha e_t$ for each $0 < s \le \frac{1}{2}$, and over the interval $\frac{1}{2} \le s \le \max(t, 1-t)$ replace $\phi_0^2\beta e_t$ by the homotopy from $\phi_0^2\alpha e_t$ to $\phi_0^2\beta e_t$ corresponding to the given homotopy from α to β, scaled down to this interval; and, similarly, replace the direct summand $\phi_1^2\beta e_t$ of $e_s\theta_t$ by $\phi_1^2\alpha e_t$ for each $\frac{1}{2} \le s < 1$, and over the interval $\min(t, 1-t) \le s \le \frac{1}{2}$ replace $\phi_1^2\beta e_t$ by the homotopy from $\phi_1^2\beta e_t$ to $\phi_1^2\alpha e_t$ corresponding to the given homotopy from β to α, scaled down to this interval. In this way one obtains a map $\theta_t'\colon A \to B$ (with evaluations $e_\infty\theta_t'$ and $e_s\theta_t'$ for $0 < s < 1$ as specified above). Carrying out the same construction for each map in the given homotopy from β to α (in place of α), one obtains a homotopy from θ_t to θ_t'.

The proof of the modified statement with (ε_s) in place of ε is similar, with ε replaced either by ε_s or by ε_j ($j = 0, 1$), depending on the circumstances.

5 Simple Inductive Limits

Let us describe conditions under which the maps in a whole sequence of building block algebras, constructed step by step as in Section 3, may be deformed by the procedure described in Section 4, in such a way as to yield a simple inductive limit C^*-algebra.

Theorem 5 *Let*

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

be a sequence of separable building block C^*-algebras,

$$A_i = A(C_i, D_i, \phi_0^i, \phi_1^i), \quad i = 1, 2, \dots$$

(see Section 2), with each map $\theta_i\colon A_i \to A_{i+1}$ obtained by the construction of Section 3 (and in particular respecting the canonical ideals). For each $i = 1, 2, \dots$ let $\beta_i\colon D_i \to C_{i+1}$ be a map verifying the hypotheses of Theorem 4.

Suppose that, for every $i = 1, 2, \dots$, the intersection of the kernels of the boundary maps ϕ_0^i and ϕ_1^i from C_i to D_i is zero.

Suppose that, for each i, the image of each of ϕ_0^{i+1} and ϕ_1^{i+1} generates D_{i+1} as a closed two-sided ideal, and that this is in fact also true for the restriction of ϕ_0^{i+1} and ϕ_1^{i+1} to the smallest direct summand of C_{i+1} containing the image of β_i. Suppose that the closed two-sided ideal of C_{i+1} generated by the image of β_i is equal to this direct summand.

Suppose that, for each i, the maps $\delta_i' - \phi_0^{i+1}\beta_i$ and $\delta_i - \phi_1^{i+1}\beta_i$ from D_i to D_{i+1} are injective.

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Suppose that, for each i, the map $\gamma_i - \beta_i\phi_1^i$ takes each non-zero direct summand of C_i into a subalgebra of C_{i+1} not contained in any proper closed two-sided ideal.

Suppose that, for each i, the map $\beta_i \colon D_i \to C_{i+1}$ can be deformed—inside the hereditary sub-C-algebra generated by its image—to a map $\alpha_i \colon D_i \to C_{i+1}$ with the following property: There is a direct summand of α_i, say $\overline{\alpha}_i$, such that $\overline{\alpha}_i$ is non-zero on an arbitrary given element x_i of D_i, and has image a simple sub-C*-algebra of C_{i+1}, the closed two-sided ideal generated by which contains the image of β_i.*

Choose a dense sequence (t_i) in the open interval $]0, \frac{1}{2}[$, such that $t_{2n} = t_{2n-1}$, $n = 1, 2, \ldots$.

Choose a sequence of elements $x_3 \in D_3$, $x_5 \in D_5$, $x_7 \in D_7, \cdots$ (necessarily non-zero) with the following property: For some countable basis for the topology of the spectrum of each of $D_1, D_2, \ldots$, and for some choice of non-zero element of the closed two-sided ideal associated to each of these (non-empty) open sets, under successive application of the maps $\delta_i - \phi_1^{i+1}\beta_i$ (beginning with the i-th one for an element of D_i), each one of these elements is taken into x_j for all j in some set $S \subseteq 2\mathbb{N} + 3 = \{3, 5, \ldots\}$ such that $\{t_j,\ j \in S\}$ is dense in $]0, \frac{1}{2}[$. Choose α_j as above with respect to the element x_j of D_j—i.e., such that $\overline{\alpha}_j(x_j) \neq 0$ for some direct summand $\overline{\alpha}_j$ of α_j—for each $j \in 2\mathbb{N}+3$, and for $j \in 2\mathbb{N}+3+1$ choose α_j with respect to the non-zero element $(\delta'_{j-1} - \phi_0^j\beta_{j-1})(x_{j-1})$ of D_j. (If $j = 1$ or 2, choose $\alpha_j = \beta_j$.)

It follows that, if θ'_i denotes the deformation of θ_i constructed in Theorem 4, with respect to the point $t_i \in]0, \frac{1}{2}[$ and the maps β_i and α_i (and a fixed homotopy of β_i to α_i), then the inductive limit of the sequence

$$A_1 \xrightarrow{\theta'_1} A_2 \xrightarrow{\theta'_2} \cdots$$

is simple.

Proof Since (by [3], [2]) every closed two-sided ideal of the inductive limit is the inductive limit of a sequence of closed two-sided ideals of the algebras in the sequence—with the same maps, restricted to the ideals—it is sufficient to prove that for every non-zero closed two-sided ideal of one of the algebras in the sequence, the image of this at some later stage in the sequence is not contained in any proper closed two-sided ideal (at that stage). (Note that in the case of a general inductive limit this condition is not necessary for simplicity; it is necessary if the maps are injective and the spectrum of each algebra in the sequence is compact.)

Let I be a non-zero closed two-sided ideal of A_i for some $i = 1, 2, \ldots$, and let us show that the image of I in A_j for some $j > i$ generates A_j as a closed two-sided ideal. The hypothesis $\operatorname{Ker}\phi_0^i \cap \operatorname{Ker}\phi_1^i = 0$ states that the spectrum of the canonical closed two-sided ideal of A_i is dense in the spectrum of A_i. In particular, this spectrum has non-empty intersection with the spectrum of I.

Since the canonical ideal is the C*-algebra of continuous functions from $]0, 1[$ to D_i vanishing at infinity, its spectrum is the Cartesian product of the spectrum of D_i and the interval $]0, 1[$. (This may perhaps most easily be seen in terms of the definition of the spectrum by means of pure states.) It follows that the spectrum of I contains a non-empty set $U \times V$ where U and V are open subsets of the spectra of D_i and $]0, 1[$, respectively.

Let us consider separately the cases $V \cap]0, \frac{1}{2}[\neq \varnothing$ and $V \cap]\frac{1}{2}, 1[\neq \varnothing$. Suppose first that $V \cap]0, \frac{1}{2}[\neq \varnothing$. By choice of the sequence $(x_3, x_5, \ldots)$, there exists an odd number

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$j > i$ such that x_j is the image under the product of the maps $\delta_i - \phi_1^{i+1}\beta_i, \dots, \delta_{j-1} - \phi_1^j\beta_{j-1}$ of a non-zero element x of D_i belonging to the closed two-sided ideal of D_i with spectrum U, and such that, furthermore, $t_j \in V$. There is then an element y of I the image of which in the fibre D_j of A_j at t_j has x_j as a direct summand (for instance, the element xf where f is a continuous real-valued function on $]0, 1[$ supported in V and equal to 1 at t_j).

It follows that the image of y, and hence of I, in A_{j+2} is contained in no proper closed two-sided ideal, as desired. To show this, let us show first that the closed two-sided ideal generated by the image of y, and hence of I, in A_{j+1} contains the canonical ideal of A_{j+1}, and that the image of this ideal in the fibre C_{j+1} of A_{j+1} at infinity contains a non-zero direct summand—namely, that generated by the image of $\overline{\alpha}_j$. (The desired conclusion then follows.)

Let π be an irreducible representation of the canonical ideal of A_{j+1}, *i.e.*, an irreducible representation of A_{j+1} supported on the fibre D_{j+1} of A_{j+1} at some $0 < s < 1$—*i.e.*, factoring through the evaluation e_s. Depending on s, the map $e_s\theta_j'$ contains either $\phi_0^{j+1}\overline{\alpha}_j e_{t_j}$ or $\phi_1^{j+1}\overline{\alpha}_j e_{t_j}$ (or both) as a direct summand. By the hypotheses concerning the boundary maps ϕ_0^{j+1} and ϕ_1^{j+1}, and the map $\overline{\alpha}_j$, the restriction of any irreducible representation of D_{j+1} to the image of each of the maps $\phi_0^{j+1}\overline{\alpha}_j e_{t_j}$ and $\phi_1^{j+1}\overline{\alpha}_j e_{t_j}$ is faithful, and, furthermore, each of the elements $\phi_0^{j+1}\overline{\alpha}_j(x_j)$ and $\phi_1^{j+1}\overline{\alpha}_j(x_j)$ is non-zero. In particular, each of these two elements is non-zero in the representation π. Since at least one of them is a direct summand of the image of y in the fibre D_{j+1} of A_{j+1} at s (as x_j is a direct summand of the image of y in the fibre D_j of A_j at t_j, and either $\phi_0^{j+1}\overline{\alpha}_j e_{t_j}$ or $\phi_1^{j+1}\overline{\alpha}_j e_{t_j}$ is a direct summand of $e_s\theta_j'$), it follows that the image of y—and hence of I—in the representation π is non-zero. This shows that the closed two-sided ideal generated by the image of I in A_{j+1} contains the canonical ideal of A_{j+1}.

That the closed two-sided ideal of C_{j+1} generated by the image of y—and hence of I—contains a non-zero direct summand of C_{j+1} follows from the fact that the map $e_\infty\theta_j'$ contains $\overline{\alpha}_j e_{t_j}$ as a direct summand, together with the facts, used also above, that the image of y in the fibre of A_j at t_j contains x_j as a direct summand—so that the image of y in C_{j+1} contains the non-zero element $\overline{\alpha}_j(x_j)$ as a direct summand—and the image of $\overline{\alpha}_j$ is a simple sub-C*-algebra of C_{j+1} the closed two-sided ideal generated by which is a direct summand.

Now consider the closed two-sided ideal generated by the image of I in A_{j+2}. Denote this by J. Since the pre-image of J in A_{j+1} contains the canonical ideal of A_{j+1}, it follows that J contains the canonical ideal of A_{j+2}. (This obtains as $e_s\theta_j'$ contains either $\phi_0^{j+1}\overline{\alpha}_j e_{t_j}$ or $\phi_1^{j+1}\overline{\alpha}_j e_{t_j}$ (or both) as a direct summand, and by hypothesis the image of each of these maps generates D_{j+1} as a closed two-sided ideal.) Since the pre-image of J in A_{j+1}, modulo the canonical ideal, contains a non-zero direct summand of C_{j+1}, since by hypothesis the image of this by $\gamma_{j+1} - \beta_{j+1}\phi_{j+1}^1$ generates C_{j+2} as a closed two-sided ideal, and since by construction $e_\infty\theta_j'$ contains $(\gamma_{j+1} - \beta_{j+1}\phi_1^{j+1})e_\infty$ as a direct summand, it follows that J is equal to all of A_{j+2}, as asserted.

It remains to consider the case $V \cap]\frac{1}{2}, 1[\neq \varnothing$. This is only slightly different from the first case. There exists an odd number $j > i$ such that x_j is the image under the product of the maps $\delta_i - \phi_1^{i+1}\beta_i, \dots, \delta_{j-1} - \phi_1^j\beta_{j-1}$ of a non-zero element x of D_i belonging to the

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closed two-sided ideal of D_i with spectrum U, and such that, furthermore, $1 - t_j \in V$. There is then an element y of I the image of which in the fibre D_j of A_j at $1 - t_j$ has x_j as a direct summand (for instance, xf where f is supported in $]0,1[$ and equal to 1 at $1 - t_j$). Since the map $e_{t_j}\theta'_j$ contains $(\delta'_j - \phi_0^{j+1}\beta_j)e_{1-t_j}$ as a direct summand, and e_{1-t_j} of the image of y in A_j contains x_j as a direct summand, it follows that the image of y in the fibre D_{j+1} of A_{j+1} at t_j contains the element $(\delta'_j - \phi_0^{j+1}\beta_j)(x_j)$ as a direct summand. Since, by construction, $\overline{\alpha}_{j+1}$ is non-zero on this element of D_{j+1}, we may continue as in the first case and conclude that, first, the closed two-sided ideal of A_{j+2} generated by the image of y, and hence of I, contains the canonical ideal of A_{j+2}, and modulo this ideal also contains a non-zero direct summand of C_{j+2}—and, second, as a consequence, the closed two-sided ideal of A_{j+3} generated by the image of I is equal to all of A_{j+3}.

6 Proof of Theorem 1: Realizing the Semigroup $\{0, n, n+1, \dots\}$

Let us construct a sequence of separable, amenable (in fact type I) building block C*-algebras, with the desired ordered K_0-group at every stage, and with maps as described in Section 3—each an order isomorphism at the level of K_0—this will ensure that the inductive limit has the desired ordered K_0-group—fulfilling the conditions of Theorem 5. Deforming the maps in the sequence as described in Theorem 5 will then make the inductive limit C*-algebra simple, and will not change the ordered K_0-group.

We wish, then, to construct a sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

with $A_i = A(C_i, D_i, \phi_0^i, \phi_1^i)$ as in Section 2 and with θ_i constructed as in Theorem 3 from maps

$$\gamma_i \colon C_i \to C_{i+1}, \quad \delta_i, \delta'_i \colon D_i \to D_{i+1}, \text{ and } \varepsilon_i \colon C_i \to D_{i+1},$$

the latter three with orthogonal images (in fact we shall take $\varepsilon_i = 0$), and in order to carry out the desired deformations we wish to have a map

$$\beta_i \colon D_i \to C_{i+1}$$

for each i with the properties specified in Theorem 5.

Let us take C_i to be the cut-down of $\mathrm{C}(X_i) \otimes \mathcal{K}$ by a certain projection p_i, *i.e.*,

$$C_i = p_i\big(\mathrm{C}(X_i) \otimes \mathcal{K}\big)p_i,$$

with p_1 to be specified and p_i to be $\gamma_{i-1}(p_{i-1})$ for $i \geq 2$, and with the compact metrizable space X_1 to be specified and X_i to be the Cartesian product of n_{i-1} copies of X_{i-1} for $i \geq 2$, with $n_1, n_2, \dots$ to be specified. Here $\mathcal{K}$ denotes the C*-algebra of compact operators on an infinite-dimensional separable Hilbert space.

Let us take $D_i = C_i \otimes \mathrm{M}_{k_i \dim(p_i)}$.

Let us take ϕ_j^i ($j = 0, 1$) to be the map from C_i to D_i obtained as the direct sum of l_j^i copies and $k_i - l_j^i$ copies, respectively, of the maps

$$\mu_i \colon a \mapsto p_i \otimes a(x_i) \quad \text{and}$$

$$\nu_i \colon a \mapsto a \otimes 1$$

from C_i to $C_i \otimes \mathrm{M}_{\dim(p_i)}$, where l_j^i $(j = 0, 1)$, k_i, and $x_i \in X_i$ are to be specified—with $l_0^i \neq l_1^i$ and $k_i \neq 0$. Note that this specifies ϕ_j^i only up to a choice of the order of direct summands, but, clearly, it is only necessary to specify ϕ_j^i up to unitary equivalence (*i.e.*, up to composition with an inner automorphism). In constructing the mappings between building blocks, below, we shall in fact have occasion to modify the maps ϕ_j^i, successively, stage by stage, by inner automorphisms.

Note that as C_i and D_i are unital, as each ϕ_j^i is unital and injective, and as C_i has no non-zero proper direct summand, the maps ϕ_j^i fulfil the hypotheses of Theorem 5 concerning them alone.

By Theorem 2, for each $e \in \mathrm{K}_0(C_i)$,

$$\begin{aligned} b_0(e) &= (l_1^i - l_0^i)\big(\mathrm{K}_0(\mu_i) - \mathrm{K}_0(\nu_i)\big)(e) \\ &= (l_1^i - l_0^i)\big(\dim(e) \cdot \mathrm{K}_0(p_i) - \dim(p_i) \cdot e\big). \end{aligned}$$

Since

$$l_1^i - l_0^i \neq 0,$$

if $\mathrm{K}_0\, C_i$ is a finitely generated free abelian group then $\operatorname{Ker} b_0$ is the largest subgroup of this group containing the class $\mathrm{K}_0(p_i)$ of p_i and isomorphic to $\mathbb{Z}$.

Let us take X_1 to be the Cartesian product of n copies of the two-sphere S^2. Then every X_i is a product of copies of S^2. In particular, $\mathrm{K}_0\, C_i$ is a finitely generated free abelian group and $\mathrm{K}_1\, C_i = 0$. It follows that $\operatorname{Ker} b_0$ is as above, and b_1 is surjective. Hence by Theorem 2, $\mathrm{K}_0\, A_i$ is isomorphic as a group to its image, $\operatorname{Ker} b_0$, in $\mathrm{K}_0\, C_i$—which is isomorphic as a group to $\mathbb{Z}$.

In order for $\mathrm{K}_0\, A_i$ to be isomorphic as an ordered group to its image in $\mathrm{K}_0(C_i)$, with the relative order, by Theorem 2 it is sufficient that for any projection q in $C_i \otimes \mathcal{K}$ such that the images of q under $\phi_0^i \otimes 1$ and $\phi_1^i \otimes 1$ have the same K_0-class, these images be in fact equivalent. For any such q, the image of $\mathrm{K}_0(q)$ under $b_0 = \mathrm{K}_0(\phi_1^i) - \mathrm{K}_0(\phi_0^i)$ is zero—in other words, $\mathrm{K}_0(q)$ belongs to $\operatorname{Ker} b_0$. By construction, $\mathrm{K}_0(q)$ belongs to the largest subgroup of $\mathrm{K}_0\, C_i$ containing $\mathrm{K}_0(p_i)$ and isomorphic to $\mathbb{Z}$. The choice of p_i below will be such that $\mathrm{K}_0(p_i)$ is the positive element of this subgroup of smallest dimension (as an element of $\mathrm{K}_0\, \mathrm{C}(X_i)$—dimension being defined pointwise). The dimension of q is therefore at least the dimension of p_i. If this is at least half the dimension of the space X_i, as will be ensured by the choice of p_i below, then the dimension of q and therefore also the dimension of both $\phi_0^i(q)$ and $\phi_1^i(q)$ are at least half the dimension of X_i. Hence by Theorem 8.1.5 of [8], $\phi_0^i(q)$ and $\phi_1^i(q)$ are equivalent (as they have the same K_0-class).

Let us now choose p_1, and the maps $\gamma_i\colon C_i \to C_{i+1}$. We desire that the projection defined as $\gamma_{i-1} \cdots \gamma_1(p_1)$ have the properties specified above: On the one hand, $\mathrm{K}_0(p_i)$ should have the smallest dimension of any positive element of $\mathrm{K}_0\big(\mathrm{C}(X_i)\big)$ in the largest subgroup containing $\mathrm{K}_0(p_i)$ and isomorphic to $\mathbb{Z}$ (*i.e.*, $\operatorname{Ker} b_0$); on the other hand, this dimension should be at least half the dimension of X_i (which is n_{i-1} times the dimension of X_{i-1}). In addition, in order to ensure the desired order structure, we desire that $\mathrm{K}_0(p_i)$ be n times a generator of the subgroup in question, and that any larger multiple of this generator (*i.e.*, any element of the subgroup of dimension larger than $\mathrm{K}_0(p_i)$) also be positive.

As in [12], let us choose a (complex) line bundle over S^2 the Euler class of which is non-zero—for instance, the tangent bundle (when S^2 is considered as the Riemann sphere).

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Denote this bundle by ζ, and consider the Cartesian product ζ^n of n copies of ζ, a bundle of dimension n over $X_1 = (\mathrm{S}^2)^n$. Denote by g_1 the difference

$$[\zeta^n] - [\theta_{n-1}] \in \mathrm{K}^0(X_1) = \mathrm{K}_0\big(\mathrm{C}(X_1)\big),$$

where θ_{n-1} denotes the trivial bundle of dimension $n-1$. Note that g_1 has dimension one in $\mathrm{K}^0(X_1)$, and so hg_1 has dimension h for any integer $h \geq 0$. Since half the dimension of X_1 is n, by Theorem 8.1.2 of [8], $hg_1 \in (\mathrm{K}^0 X_1)^+$ for any $h \geq n$. Furthermore, by Lemma 4 of [12] (with $m = n-1$), $hg_1 \notin (\mathrm{K}^0 X_1)^+$ for $0 < h < n$. It follows immediately that if we take p_1 to be a projection in $\mathrm{C}(X_1) \otimes \mathcal{K}$ such that

$$\mathrm{K}_0(p_1) = ng_1,$$

then p_1 has the four properties specified in the preceding paragraph.

Let us now specify the map $\gamma_i \colon C_i \to C_{i+1}$, $i = 1, 2, \ldots$. Consider first the map

$$\gamma_i' := (\mathrm{id} \otimes 1 \otimes \cdots \otimes 1) \oplus (1 \otimes \mathrm{id} \otimes 1 \otimes \cdots \otimes 1) \oplus \cdots \oplus (1 \otimes \cdots \otimes 1 \otimes \mathrm{id})$$

from $\mathrm{C}(X_i)$ to $\mathrm{M}_{n_i}\big(\mathrm{C}(X_{i+1})\big) = \mathrm{M}_{n_i}\big(\mathrm{C}(X_i^{n_i})\big) = \mathrm{M}_{n_i}\big(\mathrm{C}(X_i) \otimes \cdots \otimes \mathrm{C}(X_i)\big)$, where 1 denotes the unit of $\mathrm{C}(X_i)$, and id denotes the identity map $\mathrm{C}(X_i) \to \mathrm{C}(X_i)$.

Consider also the map

$$\beta_i' := e_{x_i} \cdot 1$$

from $\mathrm{C}(X_i)$ to $\mathrm{C}(X_{i+1})$ where e_{x_i} denotes evaluation at x_i, and $1 = 1_{X_i}$ denotes the unit of $\mathrm{C}(X_{i+1})$. In other words, $\beta_i'(f)(x) = f(x_i)$, $f \in \mathrm{C}(X_i)$, $x \in X_{i+1}$. Let us, incidentally, specify x_i as just the point in X_i—a product of copies of S^2—with all coordinates equal to a fixed point $x_0 \in \mathrm{S}^2$.

Now, inductively, let us take γ_i to be the map from C_i to $\mathrm{C}(X_{i+1}) \otimes \mathrm{M}_2(\mathcal{K})$ consisting of the direct sum of the following two maps: first, the restriction to $C_i \subseteq \mathrm{C}(X_i) \otimes \mathcal{K}$ of the tensor product of γ_i' with the identity map from $\mathcal{K}$ to $\mathcal{K}$, and second, the map from C_i to $\mathrm{C}(X_{i+1}) \otimes \mathrm{M}_{q_i}(\mathcal{K})$ consisting of the composition of the map ϕ_1^i from C_i to D_i with the direct sum of q_i copies of the tensor product of β_i' with the identity map from $\mathcal{K}$ to $\mathcal{K}$ (restricted to $D_i \subseteq \mathrm{C}(X_i) \otimes \mathcal{K}$), where q_i (as well as n_i) is to be specified. The induction consists in first considering the case $i = 1$ (as p_1 has already been chosen), then setting $\gamma_1(p_1) = p_2$, so that C_2 is specified, as the cut-down of $\mathrm{C}(X_{i+1}) \otimes \mathrm{M}_2(\mathcal{K})$ by p_2, and continuing in this way.

With $\beta_i \colon D_i \to C_{i+1}$ taken to be the restriction to $D_i \subseteq \mathrm{C}(X_i) \otimes \mathcal{K}$ of $\beta_i' \otimes \mathrm{id}$ (as considered above), we have by construction that $\beta_i \phi_1^i$ is a direct summand of γ_i—and, furthermore, the second direct summand and β_i map into orthogonal subalgebras (in fact orthogonal blocks)—as desired.

Let us verify that p_i defined as $\gamma_{i-1} \cdots \gamma_1(p_1)$ has the four properties specified above: These may be summarized as the single property that the set of all rational multiples of $\mathrm{K}_0(p_i)$ in the ordered group $\mathrm{K}_0 C_i = \mathrm{K}^0 X_i$ should be isomorphic as a sub ordered group to $\mathbb{Z}$ with the positive cone $\{0, n, n+1, \ldots\}$, with $\mathrm{K}_0(p_i)$ corresponding to the first non-zero positive element n. Since this has been established in the case $i = 1$, all that has to be checked now is that γ_i determines an order isomorphism between the subgroup in question at the i-th stage and the corresponding subgroup at the $(i+1)$-st stage.

Let us show first that at least γ_i gives a group isomorphism between the subgroups in question. Recalling that g_1 is a generator of this subgroup at the first stage, let us show that its image, $g_2 = \gamma_1(g_1)$, is also a generator at the second stage. In other words, we must show that g_2 is not a positive integral multiple of any other element of $\mathrm{K}_0 C_2 = \mathrm{K}^0 X_2$. We shall show this using (besides the property of g_1 mentioned above) only that g_1 and $\mathrm{K}_0(1_{X_1})$ are independent in $\mathrm{K}^0 X_1$. (We shall also use that $\mathrm{K}^0 X_1$ is torsion free and that $\mathrm{K}^1 X_1 = 0$, so that by the Künneth theorem $\mathrm{K}^0 X_2$, as a group, is the tensor product of n_1 copies of $\mathrm{K}^0 X_1$—but this has nothing to do with g_1.) Note that the map $\mathrm{id} \otimes \dim \otimes \cdots \otimes \dim$, where id denotes the identity map of $\mathrm{K}^0 X_1$, and $\dim\colon \mathrm{K}^0 X_1 \to \mathbb{Z}$ the dimension function, takes $\mathrm{K}^0 X_2 = \mathrm{K}^0 X_1 \otimes \cdots \otimes \mathrm{K}^0 X_1$ onto $\mathrm{K}^0 X_1$ and takes g_2 onto g_1 plus a multiple of $\mathrm{K}_0(1_{X_1})$. If g_2 is a multiple of some other element of $\mathrm{K}^0 X_2$, say $g_2 = kg$, then it follows that g_1 plus a multiple of $\mathrm{K}_0(1_{X_1})$ is k times the image of g. Then, modulo the subgroup of $\mathrm{K}^0 X_1$ generated by $\mathrm{K}_0(1_{X_1})$, g_1 is k times some element (the image of g). But the subgroup of $\mathrm{K}^0 X_1$ generated by g_1 has zero intersection with the subgroup generated by $\mathrm{K}_0(1_{X_1})$, and so its image modulo $\mathrm{K}_0(1_{X_1})$ is still isomorphic to $\mathbb{Z}$, and has (the image of) g_1 as a generator. This shows that $k = \pm 1$, as desired.

Since we have shown that g_2 has the same properties as those of g_1 that were used above (namely, g_2 generates a maximal subgroup of rank one, which has zero intersection with the subgroup generated by $\mathrm{K}_0(1_{X_2})$), we may deduce in the same way that $\gamma_2(g_2)$ generates a maximal subgroup of $\mathrm{K}^0 X_3$ of rank one, *i.e.*, γ_2 gives a group isomorphism between the subgroups under consideration (namely, $\operatorname{Ker} b_0$ at the two stages). Clearly, we may proceed in this way to establish that γ_i gives an isomorphism for every i between $\operatorname{Ker} b_0$ at the i-th and at the $(i+1)$-st stage.

Let us now show that, for each i, if n_i is chosen sufficiently large, then γ_i restricted to $\operatorname{Ker} b_0$ is an order isomorphism between the subgroups $\operatorname{Ker} b_0 = \mathbb{Z}g_i$ and $\operatorname{Ker} b_0 = \mathbb{Z}g_{i+1}$ of $\mathrm{K}^0 X_i$ and $\mathrm{K}^0 X_{i+1}$, with the relative order, where $g_i = \mathrm{K}_0(\gamma_i \cdots \gamma_1)g_1$. Since we have shown that

$$(\mathbb{Z}g_1)^+ = \{0, n, n+1, \dots\}g_1,$$

what we must show is that, for each $i = 1, 2, \dots,$

$$(\mathbb{Z}g_{i+1})^+ = \{0, n, n+1, \dots\}g_{i+1}.$$

Since γ_i is positive, we have $(\mathbb{Z}g_{i+1})^+ \supseteq \{0, n, n+1, \dots\}g_{i+1}$. It remains to prove that $hg_{i+1} \notin (\mathrm{K}^0 X_{i+1})^+$ for $0 < h < n$; the proof of this is, as we shall now see, similar to that in the case of g_1. Let us reformulate Lemma 4 of [12] (in the present case $B = \mathrm{S}^2$ and applied to the line bundle ζ over S^2 chosen above): if (as was ensured by the choice of ζ) the Euler class of ζ is non-zero, and if $q, m, h \in \mathbb{N}$ are such that $0 < h(q-m) < q$, then

$$h([\zeta^q] - [\theta_m]) \notin (\mathrm{K}^0 B^q)^+.$$

To apply this, note that $\mathrm{K}_0(\beta_i \phi^i_j)$ takes $\mathrm{K}^0 X_i$ into $\mathbb{Z}\,\mathrm{K}_0(1_{X_{i+1}})$ for all i, so that

$$g_{i+1} = [\zeta^{nn_1\cdots n_i}] - [\theta_{m_i}]$$

for some $m_i \in \mathbb{N}$. With $q = nn_1 \cdots n_{i-1}n_i$ and $m = m_i$, we wish to have

$$0 < (n-1)(q-m) < q,$$

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as then $0 < h(q-m) < q$ for $0 < h < n$.

Note that

$$q - m = \dim g_{i+1} = (n_i + q_i k_i \dim p_i) \dim g_i.$$

(The map $\mathrm{K}_0(\gamma_i)\colon \mathrm{K}^0 X_i \to \mathrm{K}^0 X_{i+1}$ multiplies dimension by $n_i + q_i k_i \dim p_i$.) Thus, what we wish to have is

$$\dim g_{i+1} < \frac{n}{n-1} n_1 \cdots n_{i-1} n_i.$$

(Note that since $\dim g_1 = 1$, $\dim g_i > 0$ for all i.) Assume inductively that $n_1, n_2, \ldots, n_{i-1}$ have been chosen so that

$$\dim g_i < \frac{n}{n-1} n_1 \cdots n_{i-1}.$$

Choose n_i large enough that also

$$\frac{n_i + q_i k_i \dim p_i}{n_i} \dim g_i < \frac{n}{n-1} n_1 \cdots n_{i-1}.$$

(Recall that k_i and p_i have already been chosen; we may suppose that also q_i has already been chosen, in the way specified below—which does not depend on the choice of n_i.) (Actually, q_i will be chosen to be $3k_i(2 \dim p_i + \dim X_i)$.) Then,

$$\dim g_{i+1} = (n_i + q_i k_i \dim p_i) \dim g_i < \frac{n}{n-1} n_1 \cdots n_{i-1} n_i,$$

as desired.

Note that $\gamma_i - \beta_i \phi_1^i$ is non-zero, and so—as required in the hypotheses of Theorem 5—takes C_i into a subalgebra of C_{i+1} not contained in any proper closed two-sided ideal. (C_i is unital, and any non-zero projection of C_{i+1} generates it as a closed two-sided ideal.)

Next, let us construct maps δ_i and δ_i' from D_i to D_{i+1}, with orthogonal images, such that

$$\delta_i \phi_0^i + \delta_i' \phi_1^i = \phi_0^{i+1} \gamma_i,$$
$$\delta_i \phi_1^i + \delta_i' \phi_0^i = \phi_1^{i+1} \gamma_i,$$

and $\phi_0^{i+1} \beta_i$ and $\phi_1^{i+1} \beta_i$ are direct summands of δ_i' and δ_i, respectively. To do this we shall have to modify ϕ_0^{i+1} and ϕ_1^{i+1} by inner automorphisms; as noted earlier, this is permissible (all that is important is the action on K-groups).

In order to carry out this step, the only property of γ_i we shall use (besides $\gamma_i(p_i) = p_{i+1}$, which is just the definition of p_{i+1}) is that

$$e_{x_{i+1}} \gamma_i = \operatorname{mult}(\gamma_i) e_{x_i},$$

where $\operatorname{mult}(\gamma_i)$ denotes the factor by which γ_i multiplies the dimension (or trace). (We have used this number twice before; it is equal to $n_i + q_i k_i \dim p_i$, but we shall not need to use this now.) In other words, $e_{x_{i+1}} \gamma_i$ (recall that $e_{x_{i+1}}$ denotes evaluation at x_{i+1}) is the

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direct sum of $\mathrm{mult}(\gamma_i)$ copies of e_{x_i} (in canonical orthogonal blocks). From this property, on recalling that $\mu_i = p_i \otimes e_{x_i}$ and $\nu_i = \mathrm{id} \otimes 1_{\dim p_i}$, it follows that

$$\begin{aligned}\mu_{i+1}\gamma_i &= p_{i+1} \otimes e_{x_{i+1}}\gamma_i \\ &= \gamma_i(p_i) \otimes \mathrm{mult}(\gamma_i)e_{x_i} \\ &= \mathrm{mult}(\gamma_i)\gamma_i(p_i \otimes e_{x_i}) \\ &= \mathrm{mult}(\gamma_i)\gamma_i\mu_i,\end{aligned}$$

and

$$\begin{aligned}\nu_{i+1}\gamma_i &= \gamma_i \otimes 1_{\dim p_{i+1}} \\ &= \mathrm{mult}(\gamma_i)\gamma_i \otimes 1_{\dim p_i} \\ &= \mathrm{mult}(\gamma_i)\gamma_i\nu_i.\end{aligned}$$

Let us take δ_i and δ_i' to be the direct sum of r_i and s_i copies of γ_i, where r_i and s_i are to be specified. The condition, for $j = 0, 1$,

$$\delta_i\phi_j^i + \delta_i'\phi_{1-j}^i = \phi_j^{i+1}\gamma_i,$$

understood up to unitary equivalence, then becomes the condition

$$r_i\gamma_i\big(l_j^i\mu_i + (k_i - l_j^i)\nu_i\big) + s_i\gamma_i\big(l_{1-j}^i\mu_i + (k_i - l_{1-j}^i)\nu_i\big) = (k_{i+1} - l_j^{i+1})\nu_{i+1}\gamma_i,$$

also up to unitary equivalence. As $\mathrm{K}_0(\mu_i)$ and $\mathrm{K}_0(\nu_i)$ are independent this is equivalent to the two equations

$$\begin{aligned}r_i l_j^i + s_i l_{i-j}^i &= \mathrm{mult}(\gamma_i) l_j^{i+1}, \\ (r_i + s_i)k_i &= \mathrm{mult}(\gamma_i)k_{i+1}.\end{aligned}$$

Let us choose $r_i = 2\,\mathrm{mult}(\gamma_i)$ and $s_i = \mathrm{mult}(\gamma_i)$, so that

$$k_{i+1} = 3k_i,$$

and

$$l_j^{i+1} = 2l_j^i + l_{1-j}^i.$$

Taking $k_1 = 1$, $l_0^1 = 0$, and $l_1^1 = 1$, we have $k_i = 3^{i-1}$ for all i and $l_1^i - l_0^i = 1$ for all i, and in particular these quantities are non-zero, as required above.

Next, let us show that, up to unitary equivalence preserving the equations $\delta_i\phi_j^i + \delta_i'\phi_{1-j}^i = \phi_j^{i+1}\gamma_i$, $\phi_0^{i+1}\beta_i$ is a direct summand of $\delta_i' = \mathrm{mult}(\gamma_i)\gamma_i$, and $\phi_1^{i+1}\beta_i$ is a direct summand of $\delta_i = 2\,\mathrm{mult}(\gamma_i)\gamma_i$.

Note that $\phi_j^{i+1}\beta_i$ is the direct sum of l_j^{i+1} copies of $p_{i+1} \otimes \beta_i$ and $(k_{i+1} - l_j^{i+1})\dim p_{i+1}$ copies of β_i, whereas δ_i' and δ_i contain, respectively, $q_i\,\mathrm{mult}(\gamma_i)$ and $2q_i\,\mathrm{mult}(\gamma_i)$ copies of β_i. Note also, that by Theorem 8.1.2 of [8], a trivial projection of dimension $\dim p_{i+1} +$

$\dim X_{i+1}$ (or even just dimension at least $\dim p_{i+1} + \frac{1}{2}\dim X_{i+1}$) in $\mathrm{C}(X_{i+1}) \otimes \mathcal{K}$ contains a copy of p_{i+1}. Therefore, $\dim p_{i+1} + \dim X_{i+1}$ copies of β_i contain a copy of $p_{i+1} \otimes \beta_i$. It follows that $k_{i+1}(2\dim p_{i+1} + \dim X_{i+1})$ copies of β_i contain a copy of $\phi_j^{i+1}\beta_i$ when j is equal to either 0 or 1. Here, by a copy of a given map from D_i to D_{i+1} we mean another map obtained from it by conjugating by a partial isometry in D_{i+1} with initial projection the image of the unit.

Note that

$$\begin{aligned} k_{i+1}(2\dim p_{i+1} + \dim X_{i+1}) &= 3k_i\big(2\,\mathrm{mult}(\gamma_i)\dim p_i + n_i \dim X_i\big) \\ &\leq 3k_i(2\dim p_i + \dim X_i)\,\mathrm{mult}(\gamma_i), \end{aligned}$$

and that k_i, $\dim p_i$, and $\dim X_i$ have already been specified—and do not depend on n_i. It follows that, with

$$q_i = 3k_i(2\dim p_i + \dim X_i),$$

$q_i\,\mathrm{mult}(\gamma_i)$ copies of β_i contain a copy of $\phi_j^{i+1}\beta_i$ ($j = 0, 1$). In particular δ_i' and δ_i contain copies, respectively, of $\phi_0^{i+1}\beta_i$ and of $\phi_1^{i+1}\beta_i$.

With this choice of q_i, let us show that for each $j = 0, 1$ there exists a unitary $u_j \in D_{i+1}$, commuting with the image of $\phi_j^{i+1}\gamma_i$, *i.e.*, with

$$(\mathrm{Ad}\,u_j)\phi_j^{i+1}\gamma_i = \phi_j^{i+1}\gamma_i,$$

such that $(\mathrm{Ad}\,u_0)\phi_0^{i+1}\beta_i$ is a direct summand of δ_i' and $(\mathrm{Ad}\,u_1)\phi_1^{i+1}\beta_i$ is a direct summand of δ_i. In other words, for each $j = 0, 1$, we must show that the partial isometry constructed in the preceding paragraph, producing a copy of $\phi_j^{i+1}\beta_i$ inside δ_i' or δ_i, may be chosen in such a way that it extends to a unitary element of D_{i+1}—which in addition commutes with the image of $\phi_j^{i+1}\gamma_i$.

Let us consider the case $j = 0$; the case $j = 1$ is similar. Let us first show that the partial isometry in D_{i+1}, transforming $\phi_0^{i+1}\beta_i$ into a direct summand of δ_i', may be chosen to lie in the commutant of the image of $\phi_0^{i+1}\gamma_i$. Note first that the unit of the image of $\phi_0^{i+1}\beta_i$—the initial projection of the partial isometry—lies in the commutant of the image of $\phi_0^{i+1}\gamma_i$. Indeed, this projection is the image by $\phi_0^{i+1}\beta_i$ of the unit of D_i, which, by construction, is the image by ϕ_1^i of the unit of C_i. The property that $\beta_i\phi_1^i$ is a direct summand of γ_i implies in particular that the image by $\beta_i\phi_1^i$ of the unit of C_i commutes with the image of γ_i. The image by $\phi_0^{i+1}\beta_i\phi_1^i$ of the unit of C_i (*i.e.*, the unit of the image of $\phi_0^{i+1}\beta_i$) therefore commutes with the image of $\phi_0^{i+1}\gamma_i$, as asserted.

Note also that the final projection of the partial isometry also commutes with the image of $\phi_0^{i+1}\gamma_i$. Indeed, it is the unit of the image of a direct summand of δ_i', and since D_i is unital it is the image of the unit of D_i by this direct summand; since C_i is unital and $\phi_1^i \colon C_i \to D_i$ is unital, the projection in question is the image of the unit of C_i by a direct summand of $\delta_i'\phi_1^i$. But $\delta_i'\phi_1^i$ is itself a direct summand of $\phi_0^{i+1}\gamma_i$ (as $\phi_0^{i+1}\gamma_i = \delta_i\phi_0^i + \delta_i'\phi_1^i$), and so the projection in question is the image of the unit of C_i by a direct summand of $\phi_0^{i+1}\gamma_i$, and in particular commutes with the image of $\phi_0^{i+1}\gamma_i$.

Note that both direct summands of $\phi_0^{i+1}\gamma_i$ under consideration ($\phi_0^{i+1}\beta_i\phi_1^i$ and a copy of it) factor through the evaluation of C_i at the point x_i, and so are contained in the largest such direct summand of $\phi_0^{i+1}\gamma_i$; this largest direct summand, π_i, let us say, is seen to exist

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by inspection of the construction of $\phi_0^{i+1}\gamma_i$. Since both projections under consideration (the images of $1 \in C_i$ by the two copies of $\phi_0^{i+1}\beta_i\phi_1^i$) are less than $\pi_i(1)$, to show that they are unitarily equivalent in the commutant of the image of $\phi_0^{i+1}\gamma_i$ (in D_{i+1}) it is sufficient to show that they are unitarily equivalent in the commutant of the image of π_i in $\pi_i(1)D_{i+1}\pi_i(1)$. Note that this image is isomorphic to $\mathrm{M}_{\dim p_i}(\mathbb{C})$. By construction, the two projections in question are Murray-von Neumann equivalent—in D_{i+1} and therefore, *a fortiori*, in $\pi_i(1)D_{i+1}\pi_i(1)$—but all we shall use from this is that they have the same class in $\mathrm{K}^0 X_{i+1}$. Note that the dimension of these projections is $(k_{i+1} \dim p_{i+1})(k_i \dim p_i)$, and that the dimension of $\pi_i(1)$ is $k_{i+1} \dim p_{i+1} + l_0^{i+1}(\dim p_{i+1})^2$. Since the two projections under consideration commute with $\pi_i(C_i)$, and this is isomorphic to $\mathrm{M}_{\dim p_i}(\mathbb{C})$, to prove unitary equivalence in the commutant of $\pi_i(C_i)$ in $\pi_i(1)D_{i+1}\pi_i(1)$ it is sufficient to prove unitary equivalence of the product of these projections with a fixed minimal projection of $\pi_i(C_i)$, say e. Since $\mathrm{K}^0 X_{i+1}$ is torsion free (X_{i+1} is a product of spheres), the products of the two projections under consideration with e still have the same class in $\mathrm{K}^0 X_{i+1}$. To prove that they are unitarily equivalent in $eD_{i+1}e$, it is sufficient (and necessary) to prove that both they and their complements (inside e) are Murray-von Neumann equivalent. Since both the cut-down projections and (hence) their complements (inside e) have the same class in $\mathrm{K}^0 X_{i+1}$, to prove that they (*i.e.*, the two pairs) are equivalent it is sufficient, by Theorem 8.1.5 of [8], to show that all four projections have dimension at least $\frac{1}{2}\dim X_{i+1}$ (note that $\dim X_{i+1}$ is even). Dividing the numbers above by $\dim p_i$ (the order of the matrix algebra), we see that the dimension of the first pair of projections is $k_{i+1}k_i \dim p_{i+1} = k_{i+1}k_i \operatorname{mult}(\gamma_i)\dim p_i$ and the dimension of e is $k_{i+1}\operatorname{mult}(\gamma_i) + l_0^{i+1}\operatorname{mult}(\gamma_i)\dim p_{i+1}$, so that the dimension of the second pair of projections is $\operatorname{mult}(\gamma_i)(k_{i+1} + l_0^{i+1}\dim p_{i+1} - k_{i+1}k_i\dim p_i)$. Since $\dim p_1 = \frac{1}{2}\dim X_1$ and $\dim p_{i+1} = \operatorname{mult}(\gamma_i)\dim p_i$, and $\dim X_{i+1} = n_i \dim X_i$ and $\operatorname{mult}(\gamma_i) \geq n_i$ (for all i), we have $\dim p_{i+1} \geq \frac{1}{2}\dim X_{i+1}$ (for all i). Since $k_{i+1}k_i$ is non-zero (for all i), the first inequality holds. Since l_0^{i+1} is non-zero the second inequality holds if $\operatorname{mult}(\gamma_i)$ is strictly bigger than $k_{i+1}k_i$. (One then has, using $\dim p_{i+1} = \operatorname{mult}(\gamma_i)\dim p_i$ twice, that the dimension of the second pair of projections is at least $\dim p_{i+1}$.) Since $k_{i+1}k_i = 3k_i^2$, and k_i was specified before n_i, we may modify the choice of n_i so that $\operatorname{mult}(\gamma_i)$—which is greater than n_i—is sufficiently large.

This shows that the two projections in D_{i+1} under consideration are unitarily equivalent by a unitary in the commutant of the image of $\phi_0^{i+1}\gamma_i$. Replacing ϕ_0^{i+1} by its composition with the corresponding inner automorphism, we may suppose that the two projections in question are equal. In other words $\phi_0^{i+1}\beta_i$ is unitarily equivalent to the cut-down of δ_i' by the projection $\phi_0^{i+1}\beta_i(1)$.

Now consider the compositions of these two maps with ϕ_1^i, namely, $\phi_0^{i+1}\beta_i\phi_1^i$ and the cut-down of $\delta_i'\phi_1^i$ by the projection $\phi_0^{i+1}\beta_i(1)$. Since both of these maps can be viewed as the cut-down of $\phi_0^{i+1}\gamma_i$ by the same projection (on the one hand, $\beta_i\phi_1^i$ is the cut-down of γ_i by $\beta_i\phi_1^i(1)$, and $\phi_0^{i+1}\beta_i(1) = \phi_0^{i+1}\big(\beta_i\phi_1^i(1)\big)$, and, on the other hand, $\delta_i'\phi_1^i$ is a direct summand of $\phi_0^{i+1}\gamma_i$, and so a cut-down of $\delta_i'\phi_1^i$ by a subprojection of $\delta_i'\phi_1^i(1) = \delta_i'(1)$, in particular, the projection $\phi_0^{i+1}(1)$, is a cut-down of $\phi_0^{i+1}\gamma_i$), they are in fact the same map.

Therefore, any unitary inside the cut-down of D_{i+1} by $\phi_0^{i+1}\beta_i(1)$ taking $\phi_0^{i+1}\beta_i$ into the cut-down of δ_i' by this projection—such a unitary is known to exist—must commute with the image of $\phi_0^{i+1}\beta_i\phi_1^i$, and hence with the image of $\phi_0^{i+1}\gamma_i$—since this commutes with the projection $\phi_0^{i+1}\beta_i(1) = \phi_0^{i+1}\big(\beta_i\phi_1^1(1)\big)$. The extension of such a partial unitary to a unitary u_0 in D_{i+1} equal to one inside the complement of this projection then belongs to the

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commutant of the image of $\phi_0^{i+1}\gamma_i$, and transforms $\phi_0^{i+1}\beta_i$ into the cut-down of δ_i' by this projection, as desired.

As stated above, the proof of the corresponding result for $\phi_1^{i+1}\gamma_i$ is similar.

Inspection of the construction shows that the maps $\delta_i' - \phi_0^i\beta_i$ and $\delta_i - \phi_1^i\beta_i$ are injective, as required in the hypotheses of Theorem 5.

Replacing ϕ_j^{i+1} by $(\mathrm{Ad}\, u_j)\phi_j^{i+1}$, we have completed the (inductive) construction of the desired sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots ,$$

with each A_i as in Section 2, and with each θ_i is in Theorem 3, with the properties specified in the hypotheses of Theorem 5. (The existence of α_i homotopic to β_i, non-zero on a given element of D_i, defined by another point evaluation—and so satisfying the requirement of Theorem 5 with $\overline{\alpha}_i = \alpha_i$—is clear.) (An application of Theorem 5 with $\overline{\alpha}_i$ different from α_i—and with the other hypotheses fulfilled in a less trivial way—will be given in the Appendix, below.)

By Theorem 5, there exists a sequence

$$A_1 \xrightarrow{\theta_1'} A_2 \xrightarrow{\theta_2'} \cdots$$

with θ_i' homotopic to θ_i (and so agreeing with θ_i on K_0), the inductive limit of which is simple. This inductive limit then has the properties specified in the statement of Theorem 1.

7 Appendix: The Weakly Unperforated Case

Theorems 2, 3, 4, and 5 are sufficiently general to include also the construction described in [5]—of a stable simple separable amenable C^*-algebra with arbitrary weakly unperforated invariant. (Recall that the invariant considered in [5] was the pre-ordered K_0-group, paired with the convex cone of positive, densely defined, lower semicontinuous traces, together with the K_1-group—and that there are only two special properties of this structure which need to be reflected in the axioms: the lattice nature of the cone, and the surjectivity of the map from traces to positive functionals on the ordered K_0-group. Weak unperforation was defined in this setting as the property that no non-positive element of K_0 is strictly positive on all non-zero traces.)

To be more explicit, the building blocks constructed in [5] are of the kind considered in Section 2, and the calculation of their K_0- and K_1-groups is a precursor of Theorem 2; the maps initially constructed between building blocks, preserving canonical ideals, are of the kind constructed in Theorem 3; and the deformation of the maps to make the inductive limit simple is (essentially) of the kind described in Theorems 4 and 5.

In order, perhaps, to elucidate the deformation construction of [5], and the proof that the resulting, deformed, sequence has a simple inductive limit, let us show that the hypotheses of Theorem 5 are fulfilled by the initial sequence of maps constructed in [5]—with some minor modifications.

(The deformation procedure described in [5] is actually slightly different from that of the proof of Theorem 5, and is not applicable to the maps constructed in Section 6, above, in which each map ε_i in the notation of Theorem 4 is zero. The construction given in Section 6 could have been modified to introduce non-zero maps ε_i, and the deformation then carried out exactly as in [5], but that seemed more complicated. In fact, the deformation procedure described in Theorem 5 is somewhat simpler than that used in [5].)

Theorem **(5.2.3.2 of [5])** *Let G_0 be a countable simple pre-ordered abelian group and let C be a topological cone with a compact convex base which is a metrizable Choquet simplex. Let $G_0 \times C \to \mathbb{R}$ be a weakly unperforated pairing. Let G_1 be a countable abelian group. It follows that there exists a separable, simple, stable, amenable* C^**-algebra A such that the invariant for A—the triple $(\mathrm{K}_0 A, \mathrm{K}_1 A, \mathrm{T}^+ A)$, including the positive cone $(\mathrm{K}_0 A)^+$ and the pairing $\mathrm{K}_0 A \times \mathrm{T}^+ A \to \mathbb{R}$ of K_0 with (positive, densely defined, lower semicontinuous) traces—is isomorphic to the triple (G_0, G_1, C).*

Proof The case $C = 0$ is included for completeness; the construction, given in [11] and [6], is quite different in this case.

In [5], a construction was described of a sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

of basic building blocks as in Section 2 and maps preserving canonical ideals as in Section 3. Let us recall the essential properties of this construction.

The algebra A_i is the building block $A(C_i, D_i, \phi_0^i, \phi_1^i)$ where D_i is finite-dimensional, C_i is the direct sum of algebras each of which is the tensor product of the C^*-algebra Int_k of continuous M_k-valued functions on the interval [0,1] equal to a scalar multiple of the unit at 0 and 1, for some $k = 1, 3, \ldots$, the C^*-algebra $\mathrm{C}(\mathbb{T})$, and the C^*-algebra M_l for some $l = 1, 2, \ldots$, and $\phi_j^i \colon C_i \to D_i$ $(j = 0, 1)$ factors through the quotient of C_i with spectrum the union of the canonical pairs of points (0,1), (1,1) belonging to the various copies of the cylinder $[0, 1] \times \mathbb{T}$ in the spectrum of C_i (one copy for each minimal direct summand). In addition, for each $j = 0, 1$, the map ϕ_j^i has the property (assumed in Theorem 5) that its image is not contained in any proper two-sided ideal of the finite-dimensional algebra D_i.

The map $\theta_i \colon A_i \to A_{i+1}$ is constructed as in Theorem 3 from maps

$$\begin{aligned} \gamma_i \colon C_i &\to C_{i+1}, \\ \delta_i, \delta_i' \colon D_i &\to D_{i+1}, \quad \text{and} \\ \varepsilon_i \colon C_i &\to D_{i+1}, \end{aligned}$$

such that δ_i, δ_i', and ε_i have orthogonal images, and

$$\delta_i \phi_j^i + \delta_i' \phi_{1-j}^i + \varepsilon_i = \phi_j^{i+1} \gamma_i, \quad j = 0, 1.$$

The inductive limit of the sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

has the desired properties except for simplicity.

In order to apply Theorem 5 directly to this construction, we would need to know that $\operatorname{Ker} \phi_0^i \cap \operatorname{Ker} \phi_1^i = 0$ (before even coming to the question of what the maps β_i should be). Patently, this does not hold. (The spectrum of the ideal $\operatorname{Ker}\phi_0^i \cap \operatorname{Ker}\phi_1^i$ is a union of cylinders each with just two points missing.)

Fortunately, it is possible to view each algebra A_i as a building block in another way—with respect to a slightly different pair of algebras as the fibre at infinity and the generic fibre, and correspondingly different boundary maps,

$$A_i = A\big(C_i', D_i', (\phi_0^i)', (\phi_1^i)'\big),$$

—in such a way that, with a suitable choice of the maps γ_i (namely, to be of the kind described below), the maps θ_i still arise as in Theorem 3, and, furthermore, the kernel hypothesis holds (that is, $\operatorname{Ker}(\phi_0^i)' \cap \operatorname{Ker}(\phi_1^i)' = 0$).

Namely, each direct summand $\mathrm{Int}_k \otimes \mathrm{C}(\mathbb{T}) \otimes \mathrm{M}_l$ of C_i should be replaced by the direct sum of two copies of $\mathrm{C}(\mathbb{T}) \otimes \mathrm{M}_l$, labelled by 0 and 1, and, correspondingly, in each case (*i.e.*, for each summand $\mathrm{Int}_k \otimes \mathrm{C}(\mathbb{T}) \otimes M_l$ of C_i), a single direct summand $\mathrm{C}(\mathbb{T}) \otimes \mathrm{M}_l \otimes \mathrm{M}_k$ should be added to D_i. Each of the maps ϕ_0^i and ϕ_1^i should then be modified to include (as a direct summand) a copy of the canonical map

$$\mathrm{id} \otimes 1\colon\ \mathrm{C}(\mathbb{T}) \otimes \mathrm{M}_l \to \mathrm{C}(\mathbb{T}) \otimes \mathrm{M}_l \otimes \mathrm{M}_k$$

between the direct summand labelled by 0 or 1, respectively, of the new fibre at infinity—let us denote this new fibre by C_i'—corresponding to each direct summand $\mathrm{Int}_k \otimes \mathrm{C}(\mathbb{T}) \otimes \mathrm{M}_l$ of C_i and the corresponding direct summand of D_i'—the new generic fibre. As maps into the direct summand D_i of D_i', the new maps, $(\phi_j^i)'$, should be the same as the old ones—after factoring through the evaluations at the special pairs of points—note that C_i' is just the quotient of C_i with spectrum the union of the pairs of circles forming the boundary of the spectrum of C_i (the spectrum C_i is a union of cylinders $[0,1] \times \mathbb{T}$, each with boundary $\{0,1\} \times \mathbb{T}$).

Note that $(\phi_0^i)'$ is injective on the direct summands of C_i' labelled 0, and $(\phi_1^i)'$ is injective on the direct summands labelled 1. Therefore,

$$\operatorname{Ker}(\phi_0^i)' \cap \operatorname{Ker}(\phi_1^i)' = 0,$$

as is required in order to apply Theorem 5.

On the other hand, the hypothesis that the image of each of $(\phi_0^i)'$ and $(\phi_1^i)'$ should generate D_i' as a closed two-sided ideal simply does not hold. This deficiency will be attended to later.

Let us suppose that each map $\gamma_i\colon C_i \to C_{i+1}$ is chosen in the following way (as is consistent with the choice specified in [5]). Each of the partial maps making up γ_i, from a minimal direct summand $\mathrm{Int}_k \otimes \mathrm{C}(\mathbb{T}) \otimes M_l$ of C_i to a minimal direct summand of $\mathrm{Int}_{k'} \otimes \mathrm{C}(\mathbb{T}) \otimes \mathrm{M}_{l'}$ of C_{i+1}, should be a direct sum, with respect to diagonal matrix blocks in $M_{l'}$, of copies of the following eight maps from $\mathrm{Int}_k \otimes \mathrm{C}(\mathbb{T})$ to $\mathrm{Int}_{k'}\, \mathrm{C}(\mathbb{T})$ (tensored with the identity map $\mathrm{M}_l \to \mathrm{M}_l$)—at least one copy of each:

- (i) The tensor product of the canonical embedding of Int_k in Int_{pk} with the identity map $\mathrm{C}(\mathbb{T}) \to \mathrm{C}(\mathbb{T})$;
- (i)′ The preceding with the map $\mathrm{C}(\mathbb{T}) \to \mathrm{C}(\mathbb{T})$ corresponding to the map $z \mapsto \bar{z}$ from $\mathbb{T}$ to $\mathbb{T}$ in place of the identity map;

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(ii) Evaluation of $\mathrm{Int}_k \otimes \mathrm{C}(\mathbb{T})$ on the subset $\{1\} \times \mathbb{T}$ of the spectrum (the right hand boundary of the cylinder $[0,1] \times \mathbb{T}$) followed by the map from the quotient $\mathrm{C}(\mathbb{T})$ to $\mathrm{Int}_{k'} \otimes \mathrm{C}(\mathbb{T})$ taking the canonical unitary $u\colon z \mapsto z$ of $\mathrm{C}(\mathbb{T})$ to $1 \otimes u \in \mathrm{Int}_{k'} \otimes \mathrm{C}(\mathbb{T})$;

(ii)′ The preceding with $1 \otimes u^*$ in place of $1 \otimes u$;

(ii)′′ The preceding with $v \otimes 1$ in place of $1 \otimes u$, where v denotes the canonical unitary generator of $\mathrm{Int}_{k'}$, *i.e.*, the map $t \mapsto \exp(2\pi i t e_{11})$ where e_{11} denotes the upper left matrix unit of $\mathrm{M}_{k'}$;

(iii) Evaluation of $\mathrm{Int}_k \otimes \mathrm{C}(\mathbb{T})$ on the subset $[0,1] \times \{1\}$ of the spectrum (the canonical generating line of the cylinder $[0,1] \times \mathbb{T}$), followed by the canonical map from the quotient Int_k to $\mathrm{Int}_{k'} \otimes \mathrm{C}(\mathbb{T}) = \mathrm{Int}_{pk} \otimes \mathrm{C}(\mathbb{T})$ (*i.e.*, the map $f \mapsto (f \otimes 1_p) \otimes 1$ where $f \otimes 1_p$ denotes the canonical image of f in Int_k (this is a slight abuse of notation));

(iii)′ The preceding with the canonical map $\mathrm{Int}_k \to \mathrm{Int}_{kp} \otimes \mathrm{C}(\mathbb{T})$ composed with the map from Int_k to Int_k consisting of flipping the interval $(0,1)$ (*i.e.*, $f \mapsto f'$ where $f'(t) = f(1-t)$);

(iv) Evaluation of $\mathrm{Int}_k \otimes C(\mathbb{T})$ at the point (1,1) of the spectrum $[0,1] \times \mathbb{T}$, followed by the canonical map from the quotient $\mathbb{C}$ to the algebra $\mathrm{Int}_{k'} \otimes \mathrm{C}(\mathbb{T})$ (*i.e.*, the map taking 1 to 1).

With γ_i a map such as stipulated above, the map $\theta_i\colon A_i \to A_{i+1}$, when viewed in terms of the new building block structure, as a map

$$A\bigl(C_i', D_i', (\phi_0^i)', (\phi_1^i)'\bigr) \to A\bigl(C_{i+1}', D_{i+1}', (\phi_0^{i+1})', (\phi_1^{i+1})'\bigr),$$

still arises as in Theorem 3 from (unique) maps

$$\overline{\gamma}_i\colon C_i' \to C_{i+1}',$$
$$\overline{\delta}_i, \overline{\delta}_i'\colon D_i \to D_{i+1}, \quad \text{and}$$
$$(\overline{\varepsilon}_i)_s\colon C_i' \to D_{i+1}, \quad 0 \le s \le 1$$

(we use the notation "bar" instead of "prime" here to avoid confusion with the prime of Theorem 3: the map $\theta_i\colon A_i \to A_{i+1}$ arises from the maps δ_i, δ_i', and ε_i). More specifically, $\overline{\gamma}_i$ is the map induced by γ_i between the quotients C_i' and C_{i+1}' of C_i and C_{i+1}; $\overline{\delta}_i\colon D_i' \to D_{i+1}'$ is the direct sum of $\delta_i\colon D_i \to D_{i+1}$ with a map between the remaining direct summands of D_i' and D_{i+1}' (orthogonal to D_i and D_{i+1}, respectively), each partial map between minimal direct summands with spectrum $\mathbb{T}$ being a direct sum of maps, corresponding to certain of the direct summands of the corresponding component of γ_i, namely, one for each direct summand of class (i), (i)′, or (iii)—these are copies, respectively, of the maps taking u to u, u^*, and 1; similarly, $\overline{\delta}_i'$ is the direct sum of $\delta_i'\colon D_i \to D_{i+1}$ with a map between the remaining direct summands of D_i' and D_{i+1}' (orthogonal to D_i and D_{i+1}, respectively), each partial map between minimal direct summands with spectrum $\mathbb{T}$ being again a direct sum of maps, corresponding to certain of the direct summands of the corresponding component of γ_i, namely, one for each direct summand of type (iii)′—these are again copies of the map taking u to 1; finally, $(\overline{\varepsilon}_i)_s$ is the direct sum of the map $C_i' \to D_{i+1}$ corresponding directly to $\varepsilon_i\colon C_i \to D_{i+1}$, factoring through the finite-dimensional quotient of C_i' which is canonically identified with the finite-dimensional quotient of C_i through which ε_i factorizes (this direct summand of $(\overline{\varepsilon}_i)_s$ is independent of s), with a map from C_i' to the remaining

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direct summand of D'_{i+1} (orthogonal to D_i), each partial map between a pair of minimal direct summands labelled 0 and 1 (each with spectrum $\mathbb{T}$) and a direct summand of D'_{i+1} with spectrum $\mathbb{T}$ being a direct sum of maps, corresponding to the remaining direct summands of the corresponding component of γ_i, namely, one for each direct summand of type (ii), (ii)$'$, (ii)$''$, or (iv)—these are copies, respectively, of the map taking the canonical generator u of the direct summand labelled 1 to the unitary u, u^*, $\exp 2\pi is$ (this is the only case with a dependence on s), or 1.

Let us now choose maps $\beta_i \colon D'_i \to C'_{i+1}$ with the properties specified in Theorem 5. First, $\beta_i(\phi^i_1)'$ should be a direct summand of $\overline{\gamma}_i$. Note that $(\phi^i_1)'$ is unital, so that the requirement in Theorem 5 concerning the complementary direct summand of $\overline{\gamma}_i$ is redundant (as pointed out in the statement of Theorem 4).

Following the construction of [5], let us choose β_i as an isomorphism of D'_i with a subalgebra of a single standard direct summand of C_{i+1}, *i.e.*, a direct summand consisting of a canonical pair of copies of a certain matrix algebra over $\mathrm{C}(\mathbb{T})$, labelled by 0 and 1—the quotient of some minimal direct summand of C_{i+1} corresponding to the boundary of its spectrum (the boundary of a cylinder being two circles). If the K_0-multiplicities of $\overline{\gamma}_i$ (separately between direct summands both labelled 0 or both labelled 1—the multiplicities occur in equal pairs, labelled 0 or 1) are sufficiently large—as we may suppose them to be—such a subalgebra (isomorphic to D'_i) exists. Bearing in mind that the partial maps of $(\phi^i_1)'$ and of $\overline{\gamma}_i$ are similar—in each case, direct sums of copies of the identity map $\mathrm{C}(\mathbb{T}) \to \mathrm{C}(\mathbb{T})$, the composition of this with the automorphism of $\mathrm{C}(\mathbb{T})$ determined by the automorphism $z \mapsto \bar{z}$ of the spectrum, $\mathbb{T}$, and evaluation at the point 1—all of these tensored with the identity map on a matrix algebra—and that, passing farther out in the sequence, we may assume that each partial map of $\overline{\gamma}_i$ (between minimal direct summands of C'_i and C'_{i+1} labelled either both 0 or both 1) contains arbitrarily many copies of each of these maps, let us choose the subalgebra of C'_{i+1} and the isomorphism β_i of D'_i with it to have the following properties. The image by the composed map $\beta_i(\phi^i_1)'$ of the unit of C'_i should commute with the image of $\overline{\gamma}_i$, and the cut-down of $\overline{\gamma}_i$ by this projection should be equal to this map. Furthermore, the restriction of β_i to each minimal direct summand should be homotopic, inside the image of the unit of that direct summand, to a point evaluation (*i.e.*, to a homomorphism onto a simple subalgebra).

The choice of β_i as desired may be effected as follows. Express $(\phi^i_1)'$ as the direct sum of copies of the three special maps mentioned above, and for each one of these—defined on a minimal direct summand of C'_i labelled either 0 or 1—choose a map from the same direct summand of C'_i to the correspondingly labelled member of the canonical pair of minimal direct summands of C'_{i+1}, the direct sum of which is to contain the image of β_i, as follows—and of course in such a way that all the maps constructed in this way have orthogonal images. If the map in question is a copy of one of the first two special maps mentioned (either the identity map from $\mathrm{C}(\mathbb{T})$ to $\mathrm{C}(\mathbb{T})$, tensored with the identity map on a matrix algebra, or what might be called the flip of this), choose the map into C'_{i+1} to be the direct sum of one copy of each of these two special maps. If the map in question as a copy of the map $\mathrm{C}(\mathbb{T}) \to \mathbb{C}$ consisting of evaluation at $1 \in \mathbb{T}$, tensored with the identity map on a matrix algebra (this is the only other possibility, for minimal direct summands of $(\phi^i_1)'$), choose the map into C'_{i+1} to be a copy of the point evaluation at 1 considered as a map from $\mathrm{C}(\mathbb{T})$ to $\mathrm{C}(\mathbb{T})$—tensored with the identity map on a matrix algebra. An isomorphism β_i from D'_i to a subalgebra of C'_{i+1}—uniquely determined on the cut-down

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of D_i' by $(\phi_1^i)'(1)$—then exists with the property that $\beta_i(\phi_1^i)'$ is equal to the cut-down of $\overline{\gamma}_i$ by $\beta_i(\phi_1^i)'(1)$. On the image of each of the direct summands of $(\phi_1^i)'$ considered above, β_i is either the direct sum of a copy of each of the two maps from $\mathrm{C}(\mathbb{T})$ to $\mathrm{C}(\mathbb{T})$ (the identity and the flip), tensored with the identity on a matrix algebra—this is the case if the direct summand of D_i' in question has spectrum $\mathbb{T}$—, or a copy of the unital map $\mathbb{C} \to \mathrm{C}(\mathbb{T})$, tensored with the identity on a matrix algebra. We may suppose that β_i has the same form on the whole of each minimal direct summand of D_i'—either the direct sum of copies of the two maps from $\mathrm{C}(\mathbb{T})$ to $\mathrm{C}(\mathbb{T})$, one copy for each of the two direct summands of C_{i+1}' in question (labelled 0 and 1), or the direct sum of two copies of the map $\mathbb{C} \to \mathrm{C}(\mathbb{T})$, tensored with an embedding of matrix algebras in either case. By the proof of Corollary 5 of [9], the map β_i constructed in this way is homotopic to a sum of evaluations at 1, as stipulated.

We must also ensure that $(\phi_0^{i+1})'\beta_i$ is a direct summand of $\overline{\delta}_i'$, and that $(\phi_1^{i+1})'\beta_i$ is a direct summand of $\overline{\delta}_i$. Recall that in the construction of the maps γ_i, δ_i, δ_i', and ε_i in 5.2.1 and 5.2.3.1 of [5], it was specified that C_{i+1} should have more than one minimal direct summand, and that the multiplicities of the map δ_i and δ_i' should be large compared with the multiplicities of the restriction of ϕ_1^{i+1} to a particular one of these minimal direct summands—which we may choose (as in [5]) to be the one involved in the construction of β_i. With the present approach, we wish to make, in addition, a similar specification with respect to the restriction of ϕ_0^{i+1} to this direct summand. We also wish to ensure that the analogous situation holds with respect to $\overline{\delta}_i$, $\overline{\delta}_i'$, and $(\phi_j^{i+1})'$, $j = 0, 1$, in other words, with respect to the remaining direct summand of D_i' (orthogonal to D_i). But for this it is enough to note that in the construction of γ_i in [5], the K_0-multiplicities may be chosen large compared with the dimension drops in C_{i+1}. (This amounts to a slight refinement of the choice of the inductive limit decomposition of the simple graded dimension group with torsion entering into the construction of [5].)

The preceding considerations are sufficient to ensure that, at least up to unitary equivalence, $(\phi_0^{i+1})'\beta_i$ and $(\phi_1^{i+1})'\beta_i$ are direct summands, respectively, of $\overline{\delta}_i'$ and $\overline{\delta}_i$. Let us show that the unitaries in question may be chosen to commute with the images of $(\phi_0^{i+1})'\overline{\gamma}_i$ and $(\phi_1^{i+1})'\overline{\gamma}_i$, respectively, so that they may be absorbed in the choice of the maps $(\phi_0^{i+1})'$ and $(\phi_1^{i+1})'$.

The proof of this is very similar to the corresponding part of the proof of Theorem 1, in Section 6, above. Let us consider only the case of the map $(\phi_0^{i+1})'$; the case of the map $(\phi_1^{i+1})'$ is similar.

By construction, the image under β_i of the projection $(\phi_1^i)'(1)$ commutes with the image of $\overline{\gamma}_i$. Let us show that we may choose β_i in such a way that this is true also for the image under β_i of the unit of D_i'. Since we are aiming at constructing a stable C^*-algebra, we may suppose not only that the map γ_i is non-unital, but, that the projection $\beta_i(1) - \beta_i(\phi_1^i)'(1)$ is equivalent in C_{i+1}' to a projection orthogonal to $\overline{\gamma}_i(1)$. These two projections are then unitarily equivalent by a unitary equal to 1 on $\beta_i(\phi_1^i)'(1)$, and so, replacing β_i by its composition with the corresponding inner automorphism of C_{i+1}', we may suppose that, in fact, $\overline{\gamma}_i(1)$ and $\beta_i(1) - \beta_i(\phi_1^i)'(1)$ are orthogonal. In particular, $\beta_i(1)$ commutes with the image of $\overline{\gamma}_i$, as desired. Furthermore—and this is needed to fulfil the hypotheses of Theorems 4 and 5 concerning β_i—, $\overline{\gamma}_i(1) - \beta_i(\phi_1^i)'(1)$, *i.e.*, the image of the unit of C_i' under the map $\overline{\gamma}_i - \beta_i(\phi_1^i)'$, is orthogonal to $\beta_i(1)$, not just to $\beta_i(\phi_1^i)'(1)$.

It follows that $(\phi_0^{i+1})'\beta_i(1)$, *i.e.*, the unit of the image of $(\phi_0^{i+1})'\beta_i$, commutes with the

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image of $(\beta_0^{i+1})'\overline{\gamma}_i$. Note that this is also true for any direct summand of the map $\overline{\delta}_i'$, in particular, one unitarily equivalent to $(\phi_0^{i+1})'\beta_i$, chosen as above. Indeed, the unit of the image of such a map of course commutes with the image of $\overline{\delta}_i'$, and at the same time is majorized by $\overline{\delta}_i'(1)$; this, on account of the identity

$$\overline{\delta}_i(\phi_0^i)' + \overline{\delta}_i'(\phi_1^i)' + \overline{\varepsilon}_i = (\phi_0^{i+1})'\overline{\gamma}_i,$$

implies immediately that the projection in question commutes with the image of $(\phi_0^{i+1})'\overline{\gamma}_i$ (the right hand side of the identity), as desired.

Since the two projections in question—the units of the images of $(\phi_0^{i+1})'\beta_i$, and of a fixed direct summand of $\overline{\delta}_i'$ unitarily equivalent to this map—are unitarily equivalent in D_{i+1}' and belong to the commutant of the image of $(\phi_0^{i+1})'\overline{\gamma}_i$, which, because of the simple nature of the maps involved is just a finite direct sum of matrix algebras over $\mathbb{C}$ or $\mathrm{C}(\mathbb{T})$, these projections are unitarily equivalent in the commutant of the image of $(\phi_0^{i+1})'\overline{\gamma}_i$. Replacing $(\phi_0^{i+1})'$ then by its product with an inner automorphism of D_{i+1}', determined by a unitary in the commutant of the image of the map $(\phi_0^{i+1})'\overline{\gamma}_i$—so that this map, and the identity above (on which the map θ_i depends), remain unchanged—we may suppose that the two projections are equal.

But the two maps in question, $(\phi_0^{i+1})'\beta_i$ and the cut-down of $\overline{\delta}_i'$ by $(\phi_0^{i+1})'\beta_i(1)$, when composed with $(\phi_1^i)'$, are (by the identity above) both direct summands of $(\phi_0^{i+1})'\overline{\gamma}_i$—and since they are then cut-downs of $(\phi_0^{i+1})'\overline{\gamma}_i$ by the same projection (the common image of the unit of C_i'), these two composed maps, $(\phi_0^{i+1})'\beta_i(\phi_1^i)'$ and the cut-down of $\overline{\delta}_i'(\phi_1^i)'$ by the projection in question, are in fact equal. Therefore (*cf.* Section 6), any unitary inside the cut-down of D_{i+1}' by this projection taking $(\phi_0^{i+1})'\beta_i$ into the cut-down of $\overline{\delta}_i'$ by this projection—such a unitary is known to exist—must commute with the image of $(\phi_0^{i+1})'\beta_i(\phi_1^i)'$, and hence with the image of $(\phi_0^{i+1})'\overline{\gamma}_i$ (as this map contains $\overline{\delta}_i'(\phi_1^i)'$, and so also $(\phi_0^{i+1})'\beta_i(\phi_1^i)'$, as a direct summand). The extension of such a partial unitary to a unitary in D_{i+1}' equal to 1 on the complement of this projection then belongs to the commutant of the image of $(\phi_0^{i+1})'\overline{\gamma}_i$, and transforms $(\phi_0^{i+1})'\beta_i$ into the cut-down of $\overline{\delta}_i'$ by the projection, as desired.

Since, by construction, β_i is homotopic to a direct sum of point evaluations—within the projection $\beta_i(1)$—β_i is also homotopic, within $\beta_i(1)$, to a map $\alpha_i\colon D_i' \to C_{i+1}'$ with the property enunciated in the statement of Theorem 5: There is a direct summand $\overline{\alpha}_i$ of α_i such that $\overline{\alpha}_i$ is non-zero on an arbitrary given non-zero element x_i of D_i', and has image a simple sub-C*-algebra of C_{i+1}', the closed two-sided ideal generated by which contains the image of β_i.

There are certain other properties that β_i should have. First of all, since the map $\beta_i(\phi_1^i)'$ is not assumed to be unital, it must be ensured that (as required in Theorems 4 and 5) the map $\overline{\gamma}_i - \beta_i(\phi_1^i)'$ has image orthogonal to the whole of the image of β_i (not just to the image of $\beta_i(\phi_1^i)'$). This was attended to in the construction of β_i.

Second, the closed two-sided ideal of C_{i+1}' generated by the image of β_i should be a direct summand. This follows from the fact that D_i' is unital, and every closed two-sided ideal of $\mathrm{C}(\mathbb{T})$ generated by a projection is a direct summand.

Third, the image of each of $(\phi_0^{i+1})'$ and $(\phi_1^{i+1})'$ restricted to the direct summand of C_{i+1}' generated by the image of β_i should generate D_{i+1}' as a closed two-sided ideal. As pointed

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out above, this does not hold. This property may be ensured, though, by the following minor modification of $(\phi_0^{i+1})'$ and $(\phi_1^{i+1})'$. From each minimal direct summand of C'_{i+1} with spectrum $\mathbb{T}$, to each minimal direct summand of D'_{i+1}, let us add the map consisting of evaluation at $1 \in \mathbb{T}$, tensored with an embedding of matrix algebras, of multiplicity to be specified. Of course, this means enlarging each minimal direct summand of D'_{i+1}. Let us choose these maps in such a way that their images are orthogonal to the old unit of D'_{i+1}. At the same time, this must also be done for the maps $(\phi_0^i)'$ and $(\phi_1^i)'$—or we may assume inductively that this already has been done—and we must also extend the maps $\overline{\delta}_i$ and $\overline{\delta}_i'$ to the enlarged algebra D'_i, in such a way that the relation

$$\overline{\delta}_i(\phi_j^i)' + \overline{\delta}_i'(\phi_{1-j}^i)' + \overline{\varepsilon}_i = (\phi_j^{i+1})'\overline{\gamma}_i, \quad j = 0, 1,$$

with $\overline{\varepsilon}_i$ possibly modified but with $\overline{\gamma}_i$ as before, is preserved. This will be done below. By Theorem 3 there then still exists a unique map $\theta_i \colon A_i \to A_{i+1}$, inducing the map $\overline{\gamma}_i \colon C'_i \to C'_{i+1}$ between the canonical quotients, such that

$$e_s\theta_i = \overline{\delta}_i e_s + \overline{\delta}_i' e_{1-s} + \overline{\varepsilon}_i e_\infty, \quad 0 < s < 1.$$

Choosing β_i now as before (in this modified setting), we find that the property in question holds.

It remains, in order to ensure this third property of β_i, to extend $\overline{\delta}_i$ and $\overline{\delta}_i'$ to the enlarged algebra D'_i, and to modify $\overline{\varepsilon}_i \colon C'_i \to D'_{i+1}$, in such a way that the identity

$$\overline{\delta}_i(\phi_j^i)' + \overline{\delta}_i'(\phi_{1-j}^i)' + \overline{\varepsilon}_i = (\phi_j^{i+1})'\overline{\gamma}_i, \quad j = 0, 1,$$

with $\overline{\gamma}_i$ as before, is preserved. We wish, of course, to preserve the property that the maps $\overline{\delta}_i$, $\overline{\delta}_i'$, and $\overline{\varepsilon}_i$ have orthogonal images.

Choose extensions of $\overline{\delta}_i$ and $\overline{\delta}_i'$—these are unique up to unitary equivalence—such that the images of the difference between the new unit of D'_i and the old are orthogonal to the old unit of D'_{i+1}, and also orthogonal to each other. (This may require a further enlargement of D'_{i+1}.) Fix $j = 0, 1$. With these choices, each of the (new) maps $\overline{\delta}_i(\phi_j^i)'$ and $\overline{\delta}_i'(\phi_{1-j}^i)'$ is now the sum of two maps, one with image contained in the old unit of D'_{i+1}, and one with image orthogonal to this. This is also true for the (new) map $(\phi_j^{i+1})'\overline{\gamma}_i$. Furthermore, the above identity of course still holds for the new maps after cutting down by the old unit of D'_{i+1}. It therefore remains to compare the cut-downs of the (new) maps $\overline{\delta}_i(\phi_j^i)' + \overline{\delta}_i'(\phi_{1-j}^i)'$ and $(\phi_j^{i+1})'\overline{\gamma}_i$ by the difference between the new unit of D'_{i+1} and the old. For these maps, the only unitary invariant is multiplicity (*i.e.*, K_0), as they have finite-dimensional image, and involve the same irreducible representations of C'_i (evaluation at 1 for each copy of $\mathbb{T}$ in the spectrum). Therefore, choosing the multiplicities of the new summand of $(\phi_j^{i+1})'$ sufficiently large, and noting that $\overline{\gamma}_i$ is not zero on any direct summand of C'_i (as γ_i—as we may suppose—is not zero on any direct summand of C_i), we may suppose that the multiplicities of the cut-down of $\overline{\delta}_i(\phi_j^i)' + \overline{\delta}_i'(\phi_{1-j})'$ in question are smaller than the multiplicities of the cut-down of $(\phi_j^{i+1})'\overline{\gamma}_i$. In order to choose a new direct summand of $\overline{\varepsilon}_i$, with image orthogonal to the old unit of D'_{i+1}, and also to the image of $\overline{\delta}_i$ and $\overline{\delta}_i'$, and factoring through the

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same finite-dimensional quotient of C_i' as referred to above, with multiplicities equal to the differences of the multiplicities just mentioned, we must ensure that the differences of these multiplicities are the same in the two cases $j = 0$ and $j = 1$. This is achieved by choosing the new direct summands of $(\phi_0^{i+1})'$ and $(\phi_1^{i+1})'$ to have the same multiplicities—and, inductively, doing this also at earlier stages. (The multiplicities of both maps in question, the components of $\overline{\delta}_i(\phi_j^i)' + \overline{\delta}'(\phi_{1-j}^i)'$ and of $(\phi_j^{i+1})'\overline{\gamma}_i$ orthogonal to the old unit of D_{i+1}', are then independent of j, and in particular the differences are, as desired.)

Once a new direct summand of $\overline{\varepsilon}_i$ is chosen, with multiplicities as described above (and factoring through the finite-dimensional quotient of C_i' under consideration), and with image orthogonal to the old unit of D_{i+1}', so that the desired identity holds up to unitary equivalence inside the difference between the new and old units of D_{i+1}' (and still holds exactly inside the old unit), the new summand of $(\phi_j^{i+1})'$ may be modified by composition with an inner automorphism (determined by a unitary of D_{i+1}' equal to 1 on the old unit)—separately in the two cases $j = 0$ and $j = 1$—to ensure that the desired identity holds exactly.

Fourth, the maps $\overline{\delta}_i' - (\phi_0^i)'\beta_i$ and $\overline{\delta}_i - (\phi_1^i)'\beta_i$ from D_i' to D_{i+1}' should be injective. Injectivity of the maps $\overline{\delta}_i$ and $\overline{\delta}_i'$ is ensured by a suitable choice of δ_i and δ_i' in [5]. Inspection of the construction of β_i reveals sufficient freedom for the differences $\overline{\delta}_i' - (\phi_0^i)'\beta_i$ and $\overline{\delta}_i - (\phi_1^i)'\beta_i$ to be required to be still injective.

Finally, the map $\overline{\gamma}_i - \beta_i(\phi_1^i)'$ should take each non-zero direct summand of C_i' into a subalgebra of C_{i+1}' not contained in any proper closed two-sided ideal. As above, this property of $\overline{\gamma}_i$ may be ensured by a suitable choice of γ_i in [5], and inspection of the construction of β_i shows that there is sufficient freedom for the difference $\overline{\gamma}_i - \beta_i(\phi_1^i)'$ to be made to retain this property.

As in [5], the inductive limit of the deformed sequence $A_1 \xrightarrow{\theta_1'} A_2 \xrightarrow{\theta_2'} \cdots$ has the same ordered K_0-group and the same K_1-group—this is obvious—and, furthermore, the same tracial cone, in duality in the same way with K_0—this as the deformation is carried out inside a subalgebra at each stage on which all traces attain only a small fraction of their value.

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