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HOMOLOGY AND MATUI'S HK CONJECTURE FOR GROUPOIDS ON ONE-DIMENSIONAL SOLENOIDS

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Abstract

We show that Matui's HK conjecture holds for groupoids of unstable equivalence relations and their corresponding C^*-algebras on one-dimensional solenoids.

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1. Introduction

After Crainic and Moerdijk [2] introduced homology theory for étale groupoids, Matui [7, 8] began to study the homology of étale, minimal, topologically principal groupoids with totally disconnected unit space. He later proposed a conjecture, called the HK conjecture [9], that if G is an étale, minimal, topologically principal groupoid with a compact unit space, then $K_0(C_r^*(G))$ is isomorphic to the direct sum of the even homology groups of G and $K_1(C_r^*(G))$ is isomorphic to the direct sum of the odd homology groups of G.

The HK conjecture was confirmed for many important cases. In [7], Matui proved the conjecture for AF groupoids, groupoids associated with subshifts of finite type (SFT) and transformation groupoids of Cantor minimal systems. He also showed in [9] that the HK conjecture holds for all finite products of groupoids of SFT. Groupoids of k-graphs were shown to satisfy the conjecture for $k = 1$ with the row-finite condition by Hazrat and Li [5] and for $k = 1$ or 2 by Farsi, Kumjian, Pask and Sims [4]. Ortega [10] verified the HK conjecture for Katsura–Exel–Pardo groupoids of certain self-similar graph actions.

In this paper, we study Matui's HK conjecture for groupoids associated to equivalence relations on one-dimensional solenoids. Introduced by Williams [15],

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one-dimensional solenoids are a one-dimensional generalisation of SFTs and the K-groups of the related C^*-algebras are already known [17]. Thus it is natural to expect that the HK conjecture holds for groupoids on one-dimensional solenoids. Since one-dimensional solenoids are Smale spaces, there are six different groupoids for each one-dimensional solenoid. Among them, we use the groupoids G_u and $G_u \rtimes \mathbb{Z}$ defined by an unstable equivalence relation on a one-dimensional solenoid and their groupoid algebras U and R_u, respectively. Instead of the spectral sequence used in [4, 5, 7–9], we use the chain complex of skew-products developed by Ortega [10, Lemma 1.4]. The spectral sequence method is useful when the skew-product of the groupoid of interest is an AF-groupoid. In the one-dimensional solenoid case, the skew-product of $G_u \rtimes \mathbb{Z}$ is similar to G_u, that is, not an AF-groupoid. We show that G_u satisfies the HK conjecture because G_u is equivalent to the transformation groupoid of a Cantor minimal system and that the conjecture also holds for $G_u \rtimes \mathbb{Z}$ using the chain complex of Ortega.

Remark 1.1. We have learned that Scarparo [14] showed that the transformation groupoids associated with certain $(\mathbb{Z} \rtimes \mathbb{Z}_2)$-odometers are counterexamples to the general HK conjecture.

2. One-dimensional solenoids and homology

We review the definition of one-dimensional solenoids of Williams [15–17] and homology theory of groupoids [2, 7].

One-dimensional solenoids. Let X be a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ and let $f\colon X \to X$ be a continuous map. We define additional axioms that may be satisfied by (X, f):

(1) (X, f) is indecomposable;
(2) all points of X are nonwandering under f;
(3) (*Flattening Axiom*) there is $k \geq 1$ such that for every $x \in X$ there is an open neighbourhood U of x such that $f^k(U)$ is homeomorphic to $(-\epsilon, \epsilon)$;
(4) there are a metric d compatible with the topology and positive constants C and λ with $\lambda > 1$ such that for all $n > 0$ and all points x, y on a common edge of X, if f^n maps the interval $[x, y]$ into an edge, then $d(f^n x, f^n y) \geq C\lambda^n d(x, y)$;
(5) $f^n|_{X-\mathcal{V}}$ is locally one-to-one for every positive integer n;
(6) $f(\mathcal{V}) \subseteq \mathcal{V}$.

Let $\overline{X}$ be the inverse limit space

$$\overline{X} = X \xleftarrow{f} X \xleftarrow{f} \cdots = \Big\{(x_0, x_1, x_2, \dots) \in \prod_0^\infty X \mid f(x_{n+1}) = x_n\Big\}$$

and $\overline{f}\colon \overline{X} \to \overline{X}$ the induced homeomorphism defined by

$$(x_0, x_1, x_2, \dots) \mapsto (f(x_0), f(x_1), f(x_2), \dots) = (f(x_0), x_0, x_1, \dots).$$

Suppose that Y is a topological space and $g\colon Y \to Y$ a homeomorphism. We call (Y, g) a *one-dimensional solenoid* and g a solenoid map if there exist a directed graph X and a continuous map $f\colon X \to X$ such that (X, f) satisfies all six axioms and $(\overline{X}, \overline{f})$ is topologically conjugate to (Y, g). Here (X, f) is called a *presentation* of Y. If we can choose the direction of each edge in X so that the connection map $f\colon X \to X$ is orientation preserving, then we call (X, f) an *oriented presentation* and Y an *orientable* solenoid. We call a point $x \in X$ a *nonbranch point* if x has an open neighbourhood that is homeomorphic to an open interval, and a *branch point* otherwise. An *elementary presentation* (X, f) of a one-dimensional solenoid is such that X is a wedge of circles and f leaves the unique branch point of X fixed.

Theorem 2.1 [15]. *Suppose that (X, f) is a presentation of a one-dimensional solenoid.*

(1) *The inverse limit spaces of (X, f) and (X, f^n) are homeomorphic for every positive integer n.*
(2) *There exists an integer m such that $(\overline{X}, \overline{f^m})$ has an elementary presentation.*

Hence there is no loss of generality in replacing (X, f) by (X, f^n) where $n = m \cdot k$ is a positive integer such that $(\overline{X}, \overline{f^m})$ has an elementary presentation (Z, h) and for every $z \in Z$ there is an open set U_z such that $h^k(U_z)$ is an open interval by the Flattening Axiom. Thus we can assume that every point $x \in X$ has a neighbourhood U_x such that $f(U_x)$ is an interval.

Standing Assumption. *In this paper, we always assume that (X, f) is an orientable elementary presentation such that all six axioms are satisfied and that every point $x \in X$ has a neighbourhood U_x such that $f(U_x)$ is an interval.*

Suppose that (X, f) is a presentation of a one-dimensional solenoid with the edge set $\mathcal{E}$. We observe that, for an edge $e \in \mathcal{E}$, $f(e)$ is a path $e_1 \cdots e_n$ in X such that $e_i \in \mathcal{E}$. Then the *adjacency matrix* $M = M_{X,f}$ of the one-dimensional solenoid $(\overline{X}, \overline{f})$ is an $n \times n$ matrix given by

$$M(i, j) = \#\{e_j \text{ appearing in } f(e_i)\}.$$

Dimension groups. The dimension group Δ_M of an $n \times n$ nonnegative integer matrix M is the direct limit group

$$\varinjlim_{M} \mathbb{Z}^n = \{(\mathbf{v}, k) \mid \mathbf{v} \in \mathbb{Z}^n, k \in \mathbb{N}\}/\sim$$

with $(\mathbf{v}, k) \sim (\mathbf{v}', k')$ if there exist $i, j \in \mathbb{N}$ such that $i + k = j + k'$ and $\mathbf{v}M^i = \mathbf{v}'M^j$. The dimension group automorphism δ_M is the restriction of M to Δ_M so that δ_M is defined by $\delta_M[\mathbf{v}, k] = [\mathbf{v}M, k]$. See [3] for more details.

Smale spaces. We will omit a formal definition of Smale spaces given by the [,]-operation and refer to [11–13] for details.

Suppose that (Ω, g) is a Smale space where Ω is a compact metric space with a metric d and that $g\colon \Omega \to \Omega$ is a homeomorphism. Two points x and y in Ω are said to be stably equivalent and unstably equivalent if

$$\lim_{n\to+\infty} d(g^n(x), g^n(y)) = 0 \quad \text{and} \quad \lim_{n\to-\infty} d(g^n(x), g^n(y)) = 0, \quad \text{respectively.}$$

We denote the stable and unstable equivalence classes of x by $\Omega^s(x)$ and $\Omega^u(x)$.

Let $G_u = \{(x, y) \in \Omega\times\Omega\colon d(g^n(x), g^n(y)) \to 0 \text{ as } n \to -\infty\}$ and

$$G_u \rtimes \mathbb{Z} = \{(x, n, y) \in \Omega\times\mathbb{Z}\times\Omega\colon (g^n(x), y) \in G_u\}.$$

It is not difficult to verify that G_u is a principal groupoid of the unstable equivalence relation. Each $(g \times g)^n(G_{u,0})$ is given the relative topology of $\Omega\times\Omega$ and G_u is given the inductive limit topology. Then G_u is a second countable locally compact Hausdorff principal groupoid. The Haar system $\{\mu_u^x \mid x \in \Omega\}$ for G_u is described in [12, 3.c]. The product topology of $G_u \times \mathbb{Z}$ is transferred to $G_u \rtimes \mathbb{Z}$ via the map $((x, y), n) \mapsto (x, n, g^n(y))$. Then $G_u \rtimes \mathbb{Z}$ is a second countable locally compact Hausdorff groupoid with a Haar system. The groupoid C^*-algebras of G_u and $G_u \rtimes \mathbb{Z}$ are denoted $U(\Omega, g)$ and $R_u(\Omega, g)$, respectively. We call $U(\Omega, g)$ and $R_u(\Omega, g)$ the *unstable algebra* and *unstable Ruelle algebra*, respectively, for (Ω, g). We summarise some facts for one-dimensional solenoids.

Remark 2.2. Suppose that $(\overline{X}, \overline{f})$ is a one-dimensional solenoid with an adjacency matrix M.

(1) Every one-dimensional solenoid is a mixing Smale space [17].

(2) For $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ in $\overline{X}$, x is stably equivalent to y if and only if there is a nonnegative integer n such that $f^n(x_0) = f^n(y_0)$.

(3) Let $T = \{\bar{x} = (x_0, x_1, \ldots) \in \overline{X}^s(\bar{v})\colon f(x_0) = v\}$, where $\overline{X}^s(\bar{v})$ is the stable equivalence class of a fixed point $\bar{v} = (v, v, \ldots)$, and define

$$G_u(T) = \{(x, y) \in G_u\colon x, y \in T\}$$
$$G_u(T) \rtimes \mathbb{Z} = \{(x, n, y) \in G_u \rtimes \mathbb{Z}\colon x, y \in T\}.$$

Then G_u and $G_u \rtimes \mathbb{Z}$ are equivalent to $G_u(T)$ and $G_u(T) \rtimes \mathbb{Z}$, respectively, in the sense of Muhly–Renault–Williams [13].

(4) $G_u(T) \rtimes \mathbb{Z}$ is an amenable, étale, effective, minimal, second countable, Hausdorff groupoid whose unit space is T, a Cantor set [13].

(5) From [17],

$$K_0(U(\overline{X}, \overline{f})) \cong \Delta_M,$$
$$K_1(U(\overline{X}, \overline{f})) \cong \mathbb{Z},$$
$$K_0(R_u(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \{\Delta_M/\mathrm{Im}(\mathrm{Id} - \delta_M)\} \cong \mathbb{Z} \oplus \mathrm{Coker}(\mathrm{Id} - \delta_M),$$

and

$$K_1(R_u(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \delta_M).$$

Groupoid homology. We briefly review the homology theory of groupoids. See [2, 4, 7, 10] for more details.

For a local homeomorphism $\pi\colon X \to Y$ between locally compact Hausdorff spaces, we define $\pi_*\colon C_c(X,\mathbb{Z}) \to C_c(Y,\mathbb{Z})$ by $g \mapsto \pi_*(g)$ such that

$$\pi_*(g)(y) = \sum_{\pi(x)=y} g(x).$$

Suppose that G is an étale groupoid. For every $n \in \mathbb{N}$, we let

$$G^{(n)} = \{(g_1,\ldots,g_n) \in G^n\colon\ s(g_i) = r(g_{i+1}) \text{ for } i = 1,\ldots,n-1\}.$$

For $n = 1$, we let $d_0, d_1\colon G^{(1)} \to G^{(0)}$ be the source map and the range map, respectively. When $n \geq 2$ and $i = 0, 1, \ldots, n$, we define $d_i\colon G^{(n)} \to G^{(n-1)}$ by

$$d_i(g_1,\ldots,g_n) = \begin{cases} (g_2,\ldots,g_n) & \text{for } i = 0,\\ (g_1,\ldots,g_ig_{i+1},\ldots,g_n) & \text{for } 1 \leq i \leq n-1,\\ (g_1,\ldots,g_{n-1}) & \text{for } i = n. \end{cases}$$

Then we define the boundary operator $\delta_n\colon C_c(G^{(n)},\mathbb{Z}) \to C_c(G^{(n-1)},\mathbb{Z})$ by

$$\delta_n = \sum_{i=0}^{n} (-1)^n d_{i*}.$$

So we have a chain complex

$$0 \longleftarrow C_c(G^{(0)},\mathbb{Z}) \xleftarrow{\delta_1} C_c(G^{(1)},\mathbb{Z}) \xleftarrow{\delta_2} C_c(G^{(2)},\mathbb{Z}) \xleftarrow{\delta_3} \cdots.$$

Definition 2.3 [2, 7]. Let $H_n(G)$ be the homology group of the above chain complex, that is, $H_n(G) = \operatorname{Ker}\delta_n/\operatorname{Im}\delta_{n+1}$.

Remark 2.4. Suppose $(\overline{X},\overline{f})$ is a one-dimensional solenoid and G_u, $G_u(T)$, $G_u \rtimes \mathbb{Z}$ and $G_u(T) \rtimes \mathbb{Z}$ are as in Remark 2.2. Then [4, Lemma 4.3] implies that $H_n(G_u) \cong H_n(G_u(T))$ and $H_n(G_u \rtimes \mathbb{Z}) \cong H_n(G_u(T) \rtimes \mathbb{Z})$ for every nonnegative integer n.

Matui posed the following conjecture in [9, Conjecture 2.6].

HK Conjecture. *Suppose that G is a minimal topologically principal étale groupoid such that $G^{(0)}$ is homeomorphic to a Cantor set. Then*

$$K_i(C_r^*(G)) \cong \bigoplus_{n=0}^{\infty} H_{2n+i}(G) \quad \text{for } i = 0, 1.$$

3. HK conjecture for one-dimensional solenoids

Suppose that $(\overline{X},\overline{f})$ is a one-dimensional solenoid with its adjacency matrix M, unstable groupoids G_u and $G_u \rtimes \mathbb{Z}$, and corresponding unstable C^*-algebra $U(\overline{X},\overline{f})$ and unstable Ruelle algebra $R_u(\overline{X},\overline{f})$.

Definition 3.1 [1]. A closed subset T of a phase space Y of a flow φ is called a *cross section* if the mapping $\varphi\colon T \times \mathbb{R} \to Y$ defined by $(p, s) \mapsto ps$ is a local homeomorphism onto Y. The *first return map* $r_T : T \to T$ of a cross section K is defined by $x \mapsto y = xs_x$ where $x \in K$ and s_x is the smallest positive number such that $xs_x = y \in K$.

Proposition 3.2. *The HK conjecture holds for G_u, and*

$$H_0(G_u) \cong K_0(U(\overline{X}, \overline{f})) \cong \Delta_M,$$
$$H_1(G_u) \cong K_1(U(\overline{X}, \overline{f})) \cong \mathbb{Z},$$

and

$$H_n(G_u) = 0 \quad \textit{for every } n \geq 2.$$

Proof. By [18, Theorem 3.9], $\overline{X}$ is the phase space of a flow $\varphi\colon \overline{X} \times \mathbb{R} \to \overline{X}$ without a rest point so that G_u is topologically isomorphic to the transformation groupoid $\overline{X} \times_\varphi \mathbb{R}$. Then, by [18, Proposition 3.14],

$$T = \{\bar{x} = (x_0, x_1, \dots) \in \overline{X}^s(\bar{v}) \colon f(x_0) = v\}$$

is a cross section of $\overline{X}$ with the first return map r_T induced from φ such that $G_u(T)$ is topologically isomorphic to the transformation groupoid $T \times \mathbb{Z}$ by the r_T-action on T. So the HK conjecture holds for $G_u(T)$ and $H_n(G_u(T)) = 0$ for every $n \geq 2$ by [4, Theorem 6.7]. The conclusions for G_u follow from Remarks 2.2 and 2.4. □

For $G_u \rtimes \mathbb{Z}$ and the unstable Ruelle algebra of a one-dimensional solenoid, we use the skew-product of groupoids. Suppose that G is an étale groupoid and that $\rho\colon G \to \mathbb{Z}$ is a groupoid homomorphism. Then the *skew-product* $G \times_\rho \mathbb{Z}$ is $G \times \mathbb{Z}$ with the following groupoid structure: (g, n) and (h, m) are composable if and only if g and h are composable and $n + \rho(g) = m$, and

$$(g, n) \cdot (h, n + \rho(g)) = (gh, n) \quad \text{and} \quad (g, n)^{-1} = (g^{-1}, n + \rho(g)).$$

See [7, 10] for more details.

Definition 3.3 [7, Definition 3.4]. Let G and H be étale groupoids.

(1) Two homomorphisms α, β from G to H are said to be similar if there is a continuous map $\theta\colon G^{(0)} \to H$ such that

$$\theta(r(g))\alpha(g) = \beta(g)\theta(s(g))$$

for every $g \in G$.

(2) Two groupoids G and H are said to be homologically similar if there are étale homomorphisms $\alpha\colon G \to H$ and $\beta\colon H \to G$ such that $\beta \circ \alpha$ is similar to id_G and $\alpha \circ \beta$ is similar to id_H.

Lemma 3.4 [7, 10]. *Let G be an étale groupoid with $G^{(0)}$ a Cantor set and $\rho\colon G \to \mathbb{Z}$ a groupoid homomorphism. Then*

(1) *$G \times_\rho \mathbb{Z}$ is homologically similar to* $\operatorname{Ker}\rho$*; and*
(2) *there is a long exact sequence*

$$0 \longleftarrow H_0(G) \longleftarrow H_0(G \times_\rho \mathbb{Z}) \xleftarrow{I-\hat{\rho}^1_*} H_0(G \times_\rho \mathbb{Z}) \longleftarrow H_1(G) \longleftarrow \cdots$$
$$\cdots \longleftarrow H_n(G) \longleftarrow H_n(G \times_\rho \mathbb{Z}) \xleftarrow{I-\hat{\rho}^1_*} H_n(G \times_\rho \mathbb{Z}) \longleftarrow H_{n+1}(G) \longleftarrow \cdots$$

where $\hat{\rho}^1 : G \times_\rho \mathbb{Z} \to G \times_\rho \mathbb{Z}$ is given by $(g, n) \mapsto (g, n+1)$.

We consider $G_u(T) \rtimes \mathbb{Z}$ instead of $G_u \rtimes \mathbb{Z}$ as they are equivalent groupoids by Remark 2.2. Define a groupoid homomorphism

$$\rho\colon G_u(T) \rtimes \mathbb{Z} \to \mathbb{Z} \quad \text{given by } (x, n, y) \mapsto n.$$

Trivially, $\operatorname{Ker}\rho = G_u(T)$ so that $(G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}$ is homologically similar to $G_u(T)$ by Lemma 3.4. Hence $H_n((G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}) \cong H_n(G_u(T))$ for every nonnegative integer n by [7, Proposition 3.5].

Lemma 3.5. *Homological similarity between $(G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}$ and $G_u(T)$ is given by the following maps:*

$$\alpha\colon (G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z} \to G_u(T) \text{ by } ((x, n, y), m) \mapsto (\overline{f}^{-m}(x), \overline{f}^{-m-n}(y)),$$
$$\beta\colon G_u(T) \to (G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z} \text{ by } (u, v) \mapsto ((u, 0, v), 0),$$
$$\theta\colon ((G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z})^{(0)} \to (G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}$$
$$\text{by } ((x, 0, x), m) \mapsto (x, -m, \overline{f}^{-m}(x), m), \text{ and}$$
$$\psi\colon (G_u(T))^{(0)} \to G_u(T) \text{ by } (u, u) \mapsto (u, u).$$

Proof. It is routine to check that α and β are local homeomorphisms because $\overline{f}$ is a homeomorphism, $\beta \circ \alpha$ is similar to $\mathrm{id}_{(G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}}$ by θ and $\alpha \circ \beta$ is similar to $\mathrm{id}_{G_u(T)}$ by ψ. □

We note that $\mathrm{id}_{G_u(T)_*} = (\alpha \circ \beta)_* = \alpha_* \circ \beta_*$ and $\beta_* \circ \alpha_*$ are identity maps on $H_n(G_u(T))$ and $H_n((G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z})$, respectively. So the following lemma is trivial.

Lemma 3.6. *Let α and β be as above. Then the induced homomorphisms*

$$\alpha_*\colon H_n((G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}) \to H_n(G_u(T))$$
$$\beta_*\colon H_n(G_u(T)) \to H_n((G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z})$$

are isomorphisms with ${\alpha_*}^{-1} = \beta_*$ for every nonnegative integer n.

Combining Proposition 3.2 and Lemma 3.4, we derive an exact sequence

$$0 \longleftarrow H_0(G_u(T) \rtimes \mathbb{Z}) \longleftarrow \Delta_M \xleftarrow{\mathrm{Id}-\bar{\rho}^1_*} \Delta_M$$
$$\longleftarrow H_1(G_u(T) \rtimes \mathbb{Z}) \longleftarrow \mathbb{Z} \xleftarrow{\mathrm{Id}-\bar{\rho}^1_*} \mathbb{Z} \longleftarrow H_2(G_u(T) \rtimes \mathbb{Z}) \longleftarrow 0.$$

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Here $\bar{\rho}^1_*$ is the induced map on Δ_M and $\mathbb{Z}$ from $\hat{\rho}^1\colon (G_u \rtimes \mathbb{Z}) \times_\rho \mathbb{Z} \to (G_u \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}$. Since $H_0(G_u(T) \rtimes \mathbb{Z}) \longleftarrow \Delta_M$ and $\mathbb{Z} \longleftarrow H_2(G_u(T) \rtimes \mathbb{Z})$ are epimorphism and monomorphism, respectively, we have

$$H_0(G_u(T) \rtimes \mathbb{Z}) \cong \Delta_M/\mathrm{Im}(\mathrm{Id} - \bar{\rho}^1_*),$$
$$H_2(G_u(T) \rtimes \mathbb{Z}) \cong \mathrm{Ker}(\mathrm{Id} - \bar{\rho}^1_*)$$

and a short exact sequence

$$0 \longleftarrow \mathrm{Ker}(\mathrm{Id} - \bar{\rho}^1_*) \longleftarrow H_1(G_u(T) \rtimes \mathbb{Z}) \longleftarrow \mathbb{Z}/\mathrm{Im}(\mathrm{Id} - \bar{\rho}^1_*) \longleftarrow 0.$$

Thus, to determine the homology groups of $G_u(T) \rtimes \mathbb{Z}$, we need to describe

$$\bar{\rho}^1_*\colon \Delta_M \to \Delta_M \quad \text{and} \quad \bar{\rho}^1_*\colon \mathbb{Z} \to \mathbb{Z}.$$

Let α and β be as in Lemma 3.5 and consider

$$\begin{array}{ccc} G_u(T) & \longrightarrow & G_u(T) \\ {\scriptstyle\beta}\big\downarrow & & \big\uparrow{\scriptstyle\alpha} \\ (G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z} & \xrightarrow[\hat{\rho}^1]{} & (G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}. \end{array}$$

When we define

$$\tilde{\rho}^1\colon G_u(T) \to G_u(T) \quad \text{as } \alpha \circ \hat{\rho}^1 \circ \beta,$$

we can observe that $\tilde{\rho}^1_* = (\alpha \circ \hat{\rho}^1 \circ \beta)_*\colon H_n(G_u(T)) \to H_n(G_u(T))$ is the induced map of $\hat{\rho}^1_*\colon H_n((G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}) \to H_n((G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z})$. Then, for $(u, v) \in G_u(T)$,

$$\tilde{\rho}^1(u, v) = (\overline{f}^{-1}(u), \overline{f}^{-1}(v))$$

where $u = (u_0, u_1, \ldots)$ and $\overline{f}^{-1}(u) = (u_1, u_2, \ldots)$ in $\overline{X}$.

Remark 3.7. It is easy to check that $\hat{\rho}^1\colon (G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z} \to (G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z}$ is a groupoid isomorphism. Then the induced maps $\hat{\rho}^1_*$ on $H_n((G_u(T) \rtimes \mathbb{Z}) \times_\rho \mathbb{Z})$, $\tilde{\rho}^1_*$ on $H_n(G_u(T))$ and $\bar{\rho}^1_*$ on Δ_M and $\mathbb{Z}$ are group isomorphisms.

Proposition 3.8. *For the groupoid $G_u(T) \rtimes \mathbb{Z}$ of a one-dimensional solenoid,*

$$H_0(G_u(T) \rtimes \mathbb{Z}) \cong \Delta_M/\mathrm{Im}(\mathrm{Id} - \bar{\rho}^1_*),$$
$$H_1(G_u(T) \rtimes \mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \bar{\rho}^1_*),$$
$$H_2(G_u(T) \rtimes \mathbb{Z}) \cong \mathbb{Z}$$

and

$$H_k(G_u(T) \rtimes \mathbb{Z}) \cong \{0\} \quad \text{for } k \geq 3.$$

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PROOF. We have explained H_0 and H_k for $k \geq 3$ above. Because $\bar{\rho}^1_* : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism by Remark 3.7, $\bar{\rho}^1_* = \mathrm{Id}$ which implies

$$H_2(G_u(T) \rtimes \mathbb{Z}) \cong \mathrm{Ker}(\mathrm{Id} - \bar{\rho}^1_*) = \mathbb{Z} \quad \text{and} \quad \mathrm{Im}(\mathrm{Id} - \bar{\rho}^1_*) = \{0\}.$$

The above short exact sequence becomes

$$0 \longleftarrow \mathrm{Ker}(\mathrm{Id} - \bar{\rho}^1_*) \longleftarrow H_1(G_u(T) \rtimes \mathbb{Z}) \longleftarrow \mathbb{Z} \longleftarrow 0.$$

Since $\mathbb{Z}$ is a free group, this sequence splits so that

$$H_1(G_u(T) \rtimes \mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \bar{\rho}^1_*). \qquad \square$$

For $\bar{\rho}^1_* : \Delta_M \to \Delta_M$, we recall that $G_u(T)^{(0)}$ is homeomorphic to T so that $C(G_u(T)^{(0)}, \mathbb{Z})$ is identified with $C(T, \mathbb{Z})$.

LEMMA 3.9. *There is a group epimorphism* $\phi : C(G_u(T)^{(0)}, \mathbb{Z}) \to \Delta_M$.

PROOF. We note that $T \cong G_u(T)^{(0)}$ is a Cantor set and that the first return time map r_T is a minimal homeomorphism because the orbit under r_T of every $x \in T$ is dense. So (T, r_T) is a Cantor minimal system and there is a Bratteli–Vershik system (B_T, λ_T) that is topologically conjugate to (T, r_T) [6, Theorem 4.7]. Thus we can identify T with the infinite path space B_T^∞ of the Bratteli diagram B_T. Moreover, the Bratteli diagram B_T is stationary and the incidence matrix of B_T is the adjacency matrix M of the one-dimensional solenoid $(\overline{X}, \overline{f})$ [18, Proposition 3.14]. We recall that the (i, j)-term of M is the number of edges from the vertex j in the kth level of B_T to vertex i in the $(k+1)$th level (see [3, 6] for details).

For every natural number k, let B_k be the collection of allowed paths of length k in B_T and, for each $\gamma \in B_k$, define the cylinder set $Z(\gamma)$ by

$$Z(\gamma) = \{e_1 e_2 \cdots \in B_T^\infty : e_1, \ldots, e_k = \gamma\}.$$

Let

$$P_k = \{Z(\gamma) : \gamma \in B_k\}.$$

Then every P_k is a finite partition of B_T^∞ satisfying $P_1 < P_2 < \cdots$. Here $P_i < P_{i+1}$ means that every element of P_{i+1} lies in some element of P_i.

We consider $\gamma = e_1 \cdots e_k \in B_k$ and the characteristic function $\chi_{Z(\gamma)}$ of $Z(\gamma)$. Let us denote the terminal point of γ as $t(\gamma)$. If $t(\gamma) = t(e_k)$ is the vertex i, we define a map

$$\phi_k : \chi_{Z(\gamma)} \mapsto (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n$$

where 1 appears only at the ith place. On the other hand,

$$Z(\gamma) = \bigcup_{\gamma_l = \gamma e_l} Z(\gamma_l)$$

is a union of elements of P_{k+1}, so that

$$\chi_{Z(\gamma)} = \sum \chi_{Z(\gamma_l)}.$$

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Then ϕ_{k+1} maps each $\chi_{Z(\gamma_l)}$ to $(0, \ldots, 1, \ldots, 0)$ where 1 appears at the terminal vertex of e_l.

Suppose that $t(\gamma)$ is the vertex i. Then the number of paths $\gamma_l = \gamma e_l$ in the Bratteli diagram such that $t(\gamma_l) = j$ is equal to the number of edges from i in the $(k+1)$th level of B_T to vertex j in the $(k+2)$th level, that is, the (j, i)-term of M. So

$$\phi_{k+1}(\chi_{Z(\gamma)}) = \sum \phi_{k+1}(\chi_{Z(\gamma_l)}) = (m_{i,1}, \ldots, m_{i,n}) = \phi_k(\chi_{Z(\gamma)})M.$$

Each cylinder set $Z(\gamma)$ is a clopen subset of T, and the collection of all cylinder sets is a countable basis of T. Thus $C(T, \mathbb{Z})$ is generated by the collection of characteristic functions of cylinder sets and ϕ_k extends to $C(T, \mathbb{Z})$. Then it is routine to show that

$$\phi\colon C(T, \mathbb{Z}) \to \Delta_M \quad \text{given by } h \mapsto [\phi_k(h), k] = [\phi_{k+1}(h), k+1]$$

is a well-defined group epimorphism. □

Remark 3.10. Note that $(x, y) \in G_u$ if and only if $d(\overline{f}^{-n}(x), \overline{f}^{-n}(y)) \to 0$ as $n \to \infty$. So there is a large N such that, for every $k \geq N$, x_k and y_k are contained in the same circle of X and they are located near to each other. When we consider $G_u(B_T^\infty) = G_u(T)$ instead of G_u, where B_T is the Bratteli diagram for (T, r_T), the discreteness of B_T^∞ implies that $(x, y) \in G_u(B_T^\infty)$ if and only if $x_k = y_k$ for every $k \geq N$. Hence the cylinder sets of $G_u(B_T^\infty)$ are $Z(\gamma_1) \times Z(\gamma_2)$ with the restrictions of $\gamma_1, \gamma_2 \in B_k$ and $t(\gamma_1) = t(\gamma_2)$. Here, $t(\gamma)$ denotes the terminal vertex of γ. See [18, Section 3] for more details.

Lemma 3.11. *The induced map $\phi_*\colon H_0(G_u(T)) \to \Delta_M$ is an isomorphism.*

Proof. We show that $\text{Ker}\phi = \text{Im}\delta_1$. Consider $\chi_{Z(\gamma_1)\times Z(\gamma_2)} \in C(G_u(T), \mathbb{Z})$. Then

$$\delta_1(\chi_{Z(\gamma_1)\times Z(\gamma_2)}) = r_*(\chi_{Z(\gamma_1)\times Z(\gamma_2)}) - s_*(\chi_{Z(\gamma_1)\times Z(\gamma_2)}) = \chi_{Z(\gamma_1)} - \chi_{Z(\gamma_2)}$$

with the restrictions $\gamma_1, \gamma_2 \in B_k$ and $t(\gamma_1) = t(\gamma_2)$ implies

$$\phi_k \circ \delta_1(\chi_{Z(\gamma_1)\times Z(\gamma_2)}) = \phi_k(\chi_{Z(\gamma_1)}) - \phi_k(\chi_{Z(\gamma_2)}) = 0.$$

Hence $\text{Ker}\phi \supseteq \text{Im}\delta_1$.

Let $h = \sum a_i \chi_{Z(\gamma_i)} \in \text{Ker}\phi$. Since we have an ascending chain $P_1 < P_2 < \cdots$ of partitions, without loss of generality, we can assume that there is a sufficiently large $k \in \mathbb{N}$ such that every $Z(\gamma_i) \in P_k$ and $\phi_k(h) = 0$. We divide $\{\gamma_i\}$ into the union of n subclasses $\cup_{j=1}^{n} \{\gamma_{j,1}, \ldots, \gamma_{j,l(j)}\}$ such that $t(\gamma_{j,1}) = j$. Then

$$h = \sum a_i \chi_{Z(\gamma_i)} = \sum_{j=1}^{n} \sum_{l=1}^{l(j)} a_{j,l} \chi_{Z(\gamma_{j,l})} \quad \text{and} \quad \phi_k(h) = \Big(\sum_{l=1}^{l(1)} a_{j,l}, \ldots, \sum_{l=1}^{l(n)} a_{j,l} \Big) = (0, \ldots, 0).$$

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Now $\sum_{l=1}^{l(j)} a_{j,l} = 0$ implies

$$\begin{aligned}\sum_{l=1}^{l(j)} a_{j,l}\chi_{Z(\gamma_{j,l})} &= \frac{1}{l(j)}\sum_{l=1}^{l(j)}\sum_{m=1}^{l(j)} a_{j,l}(\chi_{Z(\gamma_{j,l})} - \chi_{Z(\gamma_{j,m})})\\ &= \sum_{l=1}^{l(j)}\sum_{m=1}^{l(j)} \frac{a_{j,l}}{l(j)}(r_*(\chi_{Z(\gamma_{j,l})\times Z(\gamma_{j,m})}) - s_*(\chi_{Z(\gamma_{j,l})\times Z(\gamma_{j,m})}))\\ &= \sum_{l=1}^{l(j)}\sum_{m=1}^{l(j)} \frac{a_{j,l}}{l(j)}\delta_1(\chi_{Z(\gamma_{j,l})\times Z(\gamma_{j,m})}),\end{aligned}$$

and

$$h = \sum_{j=1}^{n}\sum_{l=1}^{l(j)} a_{j,l}\chi_{Z(\gamma_{j,l})} = \sum_{j=1}^{n}\sum_{l=1}^{l(j)}\sum_{m=1}^{l(j)} \frac{a_{j,l}}{l(j)}\delta_1(\chi_{Z(\gamma_{j,l})\times Z(\gamma_{j,m})}) \in \mathrm{Im}\delta_1.$$

Thus $\mathrm{Ker}\phi = \mathrm{Im}\delta_1$, which implies

$$C(G_u(T),\mathbb{Z})/\mathrm{Ker}\phi = C(G_u(T),\mathbb{Z})/\mathrm{Im}\delta_1 = H_0(G_u(T)).$$

Therefore the induced map $\phi_* \colon H_0(G_u(T)) \to \Delta_M$ is an isomorphism. □

Recall that $C(G_u(T)^{(0)},\mathbb{Z})$ is identified with $C(T,\mathbb{Z})$ so that, for any $h \in C(T,\mathbb{Z})$ and its corresponding element $[h] \in H_0(G_u(T))$,

$$\tilde{\rho}^1_*[h] = [h \circ \tilde{\rho}^1] = [h \circ \alpha \circ \hat{\rho}^1 \circ \beta] = [h \circ \overline{f}^{-1}].$$

Lemma 3.12. *The following diagram commutes:*

$$\begin{array}{ccc} H_0(G_u(T)) & \xrightarrow{\tilde{\rho}^1_*} & H_0(G_u(T)) \\ \phi_*\downarrow & & \downarrow\phi_* \\ \Delta_M & \xrightarrow[\delta_M^{-1}]{} & \Delta_M. \end{array}$$

Proof. We consider $(\tilde{\rho}^1_*)^{-1}$ instead of $\tilde{\rho}^1_*$. Since $\overline{f} \colon (x_0, x_1, \dots) \mapsto (f(x_0), x_0, x_1, \dots)$, its corresponding map on B_T^∞, which is also denoted by $\overline{f}$, maps $e_1\epsilon_2\cdots \mapsto e_0e_1e_2\cdots$.

For $\gamma = e_1e_2\cdots e_n \in B_k$, we compare $\chi_{Z(\gamma)}$ with $\chi_{Z(\gamma)} \circ \overline{f}$. Obviously

$$e \in \mathrm{supp}(\chi_{Z(\gamma)} \circ \overline{f}) \iff \overline{f}(e) \in Z(\gamma) \iff e \in Z(e_2\cdots e_n)$$

induces $\chi_{Z(\gamma)} \circ \overline{f} = \chi_{Z(e_2\cdots e_n)}$ and

$$\phi_{k-1}(\chi_{Z(e_2\cdots e_n)}) = \phi_k(\chi_{Z(\gamma)}) \in \mathbb{Z}^n.$$

Then we apply ϕ to $\chi_{Z(\gamma)}$ and $\chi_{Z(\gamma)} \circ \overline{f}$:

$$\begin{aligned}\phi(\chi_{Z(\gamma)} \circ \overline{f}) &= \phi(\chi_{Z(e_2\cdots e_n)}) = [\phi_{k-1}(\chi_{Z(e_2\cdots e_n)}), k-1]\\ &= [\phi_k(\chi_{Z(\gamma)}), k-1] = [\phi_k(\chi_{Z(\gamma)})M, k],\\ \phi(\chi_{Z(\gamma)}) &= [\phi_k(\chi_{Z(\gamma)}), k].\end{aligned}$$

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So

$$\phi_* \circ (\tilde{\rho}_*^1)^{-1}[\chi_{Z(\gamma)}] = \phi_*[\chi_{Z(\gamma)} \circ \overline{f}] = [\phi_k(\chi_{Z(\gamma)})M, k] = \delta_M[\phi_k(\chi_{Z(\gamma)}), k] = \delta_M \circ \phi_*[\chi_{Z(\gamma)})],$$

that is, $\phi_* \circ (\tilde{\rho}_*^1)^{-1} = \delta_M \circ \phi_*$. Therefore the diagram commutes. □

Thus we have the following property of $\bar{\rho}_*^1$ on Δ_M.

Proposition 3.13. *The induced map $\bar{\rho}_*^1 \colon \Delta_M \to \Delta_M$ is given by $[\mathbf{v}, k] \mapsto \delta_M^{-1}[\mathbf{v}, k]$ where δ_M is the dimension group automorphism determined by M.*

Recall that δ_M is an automorphism on Δ_M so that $\delta_M \circ (\mathrm{Id} - \delta_M^{-1}) = \delta_M - \mathrm{Id}$ and $\mathrm{Id} - \delta_M$ have the same kernel and cokernel. Combining Propositions 3.8 and 3.13 leads to the following result.

Theorem 3.14. *The HK conjecture holds for $G_u \rtimes \mathbb{Z}$ and*

$$K_0(R_u(\overline{X}, \overline{f})) \cong H_0(G_u \rtimes \mathbb{Z}) \oplus H_2(G_u \rtimes \mathbb{Z}) \cong \mathbb{Z} \oplus \{\Delta_M/\mathrm{Im}(\mathrm{Id} - \delta_M)\},$$
$$K_1(R_u(\overline{X}, \overline{f})) \cong H_1(G_u \rtimes \mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Ker}(\mathrm{Id} - \delta_M),$$
$$H_n(G_u \rtimes \mathbb{Z}) = 0 \quad \text{for every } n \geq 2.$$

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