1 Mastering Friction

Friction is both welcome and unwelcome in Engineering. Negligible friction between moving parts leads to low losses through heat and sound, giving high transducer efficiencies; high friction can give excellent grip and contact between surfaces when needed, as in clutch plates, road tyres *etc*. Understanding friction is therefore central to a good Engineering performance.

Its treatment usually begins with how several, often prismatic bodies interact and maintain statical equilibrium. When friction is insufficient, there is usually slippage between them or, in the extreme, a loss of contact altogether. The initiation of this otherwise dynamic phase can be viewed as a limiting quasi-static problem without inertial forces.

Friction imparts to the problem a *constitutive* statement in the sense of a relationship between forces and kinematics – in this case, of relative motion between bodies. Thus, we may write in addition to force and moment balances, a limiting *inequality* of the ratio of friction force to normal contact reaction, in order to test for slippage or not.

But consider a different viewpoint, of slippage from the outset. The inequality is always satisfied and the friction forces are uniquely related; or, the resultant of friction and normal forces is of both fixed size and fixed direction. There is now a single force pointing away from the direction of slippage, which, for the purposes of simple statics problems, can admit immediate information about the character of equilibrium without its explicit solution.

For example, consider two cases of equilibrium of a familiar heavy ladder of uniform mass standing on a horizontal floor and leaning against a vertical wall: when the wall is smooth and the floor is rough, and *vice versa* (Fig. 1.1). Let the coefficient of friction be μ, any friction force denoted by F, and normal reactions by N.

From the (planar) free-body diagram of the ladder by itself in Fig. 1.1(a), we may traditionally write two equations of force equilibrium and one of moment about a normal axis through its lower end, along with limiting friction:

$$N_1 = W \quad \text{(a)}, \quad F = N_2 \quad \text{(b)}, \quad N_2 L \sin\theta - W(L/2)\cos\theta = 0 \quad \text{(c)}, \quad F = \mu N_1 \quad \text{(d)}. \tag{1.1}$$

There are four unknowns (N_1, N_2, F, μ) in four equations, thus enabling a complete solution in terms of the layout specified by L and θ, and the self-weight, W. Equations (1.1(a) and (d)) tell us that $F = \mu W$, which substitutes for N_2 in Eq. (1.1)(b) and ultimately in Eq. (1.1)(c) where, after tidying up, we have $2\mu = \cot\theta$.

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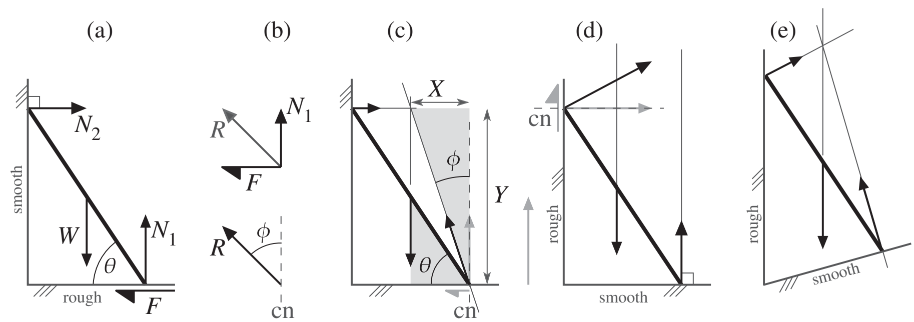

Figure 1.1 (a) External forces acting on a uniform inclined ladder. (b) Combining the normal reaction and friction force into a single frictional resultant, R, inclined at ϕ, the angle of friction at slippage, to the common normal (cn). (c) Reduction of (a) to three concurrent forces, with pertinent geometry. (d) Non-concurrent external forces acting on the same ladder when the floor is smooth and the wall is rough. (e) However, concurrent forces are possible when the floor from (d) is inclined.

This is a statement between the friction coefficient and layout, where the length and self-weight are not present. Of course, they set the absolute force levels, but the initial geometry ultimately dictates the equilibrium requirement. It also fits with our expectations of the ladder's stability: a larger angle causes $\cot\theta$ and hence μ to decrease, with a steeper ladder less prone to slippage.

Writing down all equations gives assurance but is inefficient – led, deliberately, by our imprecise question. If, as we proposed earlier, we combine F and N_1 into a single resultant, which we denote by R, it must be inclined to the common normal, the line of action of N_1, by angle ϕ (say), our *angle of friction* where $\tan\phi = \mu$ from limiting friction, and conveyed in a simple vector diagram, Fig. 1.1(b).

We have now reduced our problem to one with three force resultants by declaring *a priori* a known vector statement; and by reducing the number of unknowns, an easier solution beckons.

In working directly now with R, Eqs. (1.1a–d) no longer apply. We can write another set of equilibrium equations in terms of R, but this is tantamount to substituting the fourth equation into the remaining three. Instead, we pay attention to how these three forces are drawn in Fig. 1.1(c).

The normal contact force at the wall is horizontal, W is vertical, and the intersection of their action lines defines where R should pass through, for moment equilibrium about the same point. The equilibrium relationship now emerges from this geometry, where the height, Y, and half-width, X, between end points yield

$$\tan\phi = \frac{X}{Y};\quad \tan\theta = \frac{Y}{2X} \quad\rightarrow\quad \tan\phi = \mu = \frac{\cot\theta}{2}.$$

This is a bolder, *quicker* solution, which empowers that for the second case. Figure 1.1(d) shows the same three forces where the smooth floor produces a vertical normal reaction parallel to W. No matter the inclination of R at the rough wall, we

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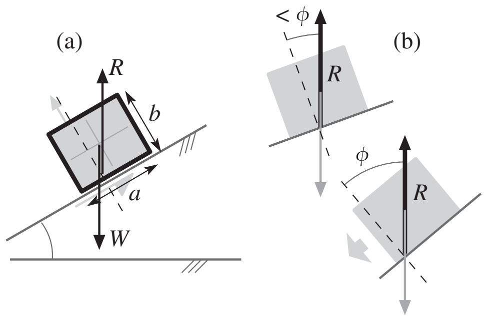

Figure 1.2 (a) External forces acting on a uniform block resting on a rough slope: the dashed line is the common normal. (b) Limiting friction and slippage when R becomes inclined at ϕ, now the slope angle of repose. (c) Limiting equilibrium with R at corner.

can never achieve force concurrency and possible statical equilibrium except for when all forces are parallel: ϕ will be equal to 90°, but this is impossible for it sets μ to be infinity.

Our informal assessment of similar problems is also expedited by a graphical approach, for example, we see in Fig. 1.1(e) that if the smooth floor were itself inclined, equilibrium *becomes* possible because all of the action lines of forces can intersect.

We have proposed R to be inclined at ϕ for limiting friction, but if equilibrium is upheld for a smaller inclination, there is clearly no slippage. For example, a uniform block sits on a rough slope, which is progressively steepened until the block slips, as shown in Fig. 1.2(a).

The two resultant forces, W and R, must be collinear, with R becoming more inclined to the rotating common normal as the slope increases. Slippage occurs when R 'reaches' the required inclination of ϕ, which is also the angle of the slope, more commonly known as the *angle of repose*. This gives us a straightforward experimental check for the coefficient of friction between two surfaces.

1.1 Toppling *vs* Sliding

There is another limiting outcome for the block's repose in Fig. 1.2. As the slope steepens, R migrates towards the lowest point on the block, as shown in Fig. 1.2(c), and arrives there without slippage occurring provided $\tan\phi$ is greater than a/b. It cannot move outside the block, and further steepening leads to W and R separating. Moment equilibrium is now violated and the block will topple first before slipping.

The transition between slippage and toppling is therefore marked by R, already inclined at ϕ, passing through the block corner, which makes for a very precise relationship between the block size and μ in Fig. 1.2. The problem in Fig. 1.3 makes for their more convenient interaction, where the direction of toppling can also change.

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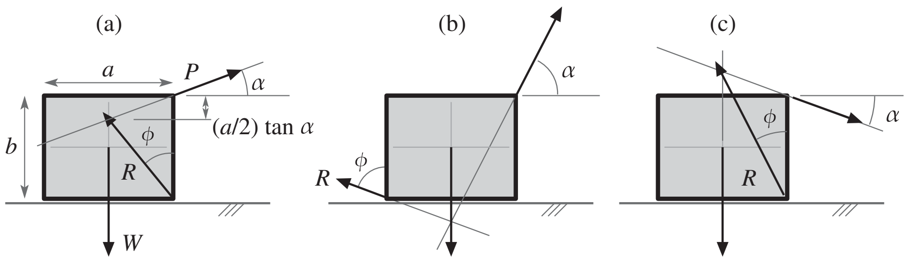

Figure 1.3 (a) Concurrent external forces acting on a uniform block about to topple forward due to force P inclined upwards. (b) Layout of forces just before the block topples backwards. (c) Their layout for toppling forward when P points downwards.

We have a horizontal block being towed to the right by a tensile force, P, applied to the top right corner, as shown in Fig. 1.3(a), and directed at positive angle α above the horizontal. There are three limiting toppling scenarios. First, P is directed upwards ($\alpha > 0$), and its line of action intersects that of W above ground. Since R resists the intended direction of movement, it always points backwards to the left at angle ϕ. Limiting moment equilibrium occurs when R is located at the front bottom corner, with the block tending to rotate forward.

For increasing α, P and W ultimately intersect below ground, but R cannot be inclined beyond $\phi = 90°$. It must be located instead at the rear bottom corner, as shown in Fig. 1.3(b), with the block rotating backwards and lifting off. Third, P points downwards ($\alpha < 0$) with R now again at the front corner, as shown in Fig. 1.3(c), as the block topples forward.

If we define ρ to be the aspect ratio of the block, a/b, the geometry of the concurrent forces in Fig. 1.3(a) reveals the following:

$$\tan\phi = \frac{a/2}{b - (a/2)\tan\alpha} \quad \rightarrow \quad \tan\phi = \frac{\rho}{2 - \rho\tan\alpha}. \tag{1.2}$$

Figure 1.3(c) also expresses this relationship when α takes negative values, so it is valid from $\alpha = -90°$ up to $\alpha = \arctan(2/\rho)$ when the denominator equals zero. At this value of $\alpha = \alpha^*$, we have $\phi = 90°$ with R horizontal. This marks the transition to the second case, where Fig. 1.3(b) can be used to show that $\tan\phi = \rho/(\rho\tan\alpha - 2)$ for $\alpha > \alpha^*$.

A compact *interaction diagram* of ϕ *vs* α expresses these theoretical limits in Fig. 1.4(a) by plotting the above equations around $\alpha = \alpha^*$. Sensibly, we see at either $\alpha = \pm 90°$ that $\phi = 0$, suggesting that, in theory, toppling always occurs first when the surface is smooth; after some rotation, of course, the block may slip as the configuration of forces changes, but that is a different problem.

In practice μ is fixed by the 'available' friction via $\tan\phi = \mu$. To now interpret the diagram, we first plot a horizontal line $\phi = \phi^*$, chosen to be 30°, with $\rho = 1$ for a square block. Where this line lies above the boundary curve, we have toppling; where

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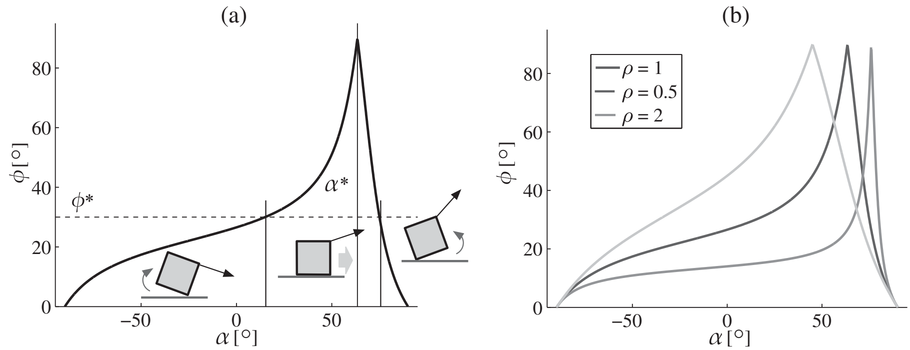

Figure 1.4 Limiting variation between toppling (above curve) and slippage (underneath) of a uniform block being towed along a rough floor. The angle of friction is ϕ, the aspect ratio of the block is ρ, and the towing force is inclined at α to the horizontal. A fixed coefficient of friction sets $\phi = \phi^*$ (dashed line); at α^* the direction of toppling reverses.

it lies below, the value of ϕ cannot meet the value set by the toppling requirement, so the block slips.

Reading from left to right, the block will topple forward, slip, then tip backwards, where the change in behaviour at different values of α will depend on ρ, which gives slightly different curves, as shown in Fig. 1.4(b).

1.2 Different Shapes

The ratio, ρ, tells us about the size of the contacting face relative to the height of the applied force. If we had a different shape of block, such as a triangle or parallelogram, then the form of previous limiting toppling equations remains the same.

On the other hand, a circular cylinder makes contact along a horizontal line, or a point if planar, giving us rolling instead of toppling as a limiting equilibrium scenario. In addition to being inclined at ϕ to the common (radial) normal, R is uniquely located at the contact point.

This further sets the geometry of solution and enables graphical solutions for cylinder problems with four forces, as we shall see. First, the cylinder of radius r and weight W in Fig. 1.5(a) is pulled over a rough step of height Y $(\leq r)$ by a horizontal force, P, without slipping. We wish to find the minimum coefficient of friction and corresponding P.

As the cylinder commences overturning, floor contact is lost and the lines of action of the remaining three forces, R, W and P, intersect at the top of the cylinder, as shown in Fig. 1.5(a). Rather than specify the layout in terms of Y, we draw angle α inclined to the horizontal underneath the common normal, as shown in Fig. 1.5(b), which defines $\sin\alpha$ to be $(r - Y)/r$. We also note the horizontal distance X from $\cos\alpha = X/r$ which defines $X^2 = 2rY - Y^2$ using $\sin^2\alpha + \cos^2\alpha = 1$.

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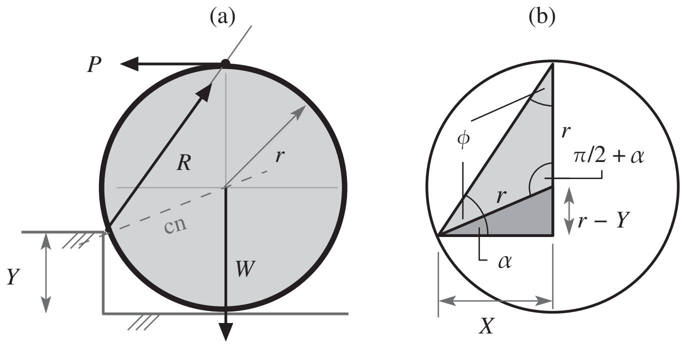

Figure 1.5 (a) Concurrent external forces acting on a uniform cylinder about to be pulled over a vertical step by a horizontal force P. (b) Angle of limiting friction, ϕ, and pertinent geometry.

Two triangles are now highlighted containing ϕ and α. The upper one is isosceles because two of its sides are radii: the obtuse angle must equal $\pi/2+\alpha$ from continuity of the vertical line through the lower triangle, which returns $2\phi+\pi/2+\alpha = \pi$ for the upper one, *i.e.* $\phi = \pi/4 - \alpha/2$. Knowing $\mu = \tan\phi$, the limiting state is expressed as follows:

$$\tan\phi = \tan(\pi/4 - \alpha/2) = \frac{1-\tan(\alpha/2)}{1+\tan(\alpha/2)} = \frac{\cos\alpha}{1+\sin\alpha} \tag{1.3}$$

after using half-angle formulae for $\tan(\alpha/2)$. Furthermore, the right-hand side can be written in terms of the original geometry as $X/(2r-Y)$, and the force P is simply found by taking moments about the contact point, $P(2r-Y) = WX$, whence P.

The cylinder is now tethered horizontally to a rough slope of inclination α by a rigid cable, see Fig. 1.6(a). Equilibrium is maintained by three forces enclosing the highlighted triangle, where limiting friction sets $\phi = \alpha/2$. The cylinder tends to slip down the slope since R acts against it.

To counter this tendency, we apply a vertical upward force, V, at the most easterly point on the cylinder, as shown in Fig. 1.6(b), until R becomes inclined backwards to the common normal at $\phi = \alpha/2$. Such inclination defines where R and the cable tension, P, intersect, about which we take moments to yield limiting V in terms of W directly. If X is the distance from the intersection point to V, as shown in Fig. 1.6(b), then $V = W(X-r)/X$.

The cylinder also tends to slip up-slope when V is applied vertically *downwards* on the other side, as shown in Fig. 1.6(c), and the same four forces suggest a similar solution approach. From moment equilibrium, we immediately see that V is negative if R and P intersect to the left of V. Their intersection point must therefore lie to the right of V, with two possible outcomes. Either α is small enough so that ϕ can be equal to $\alpha/2$ to give limiting R and slippage, or R is less inclined than $\alpha/2$ without slippage.

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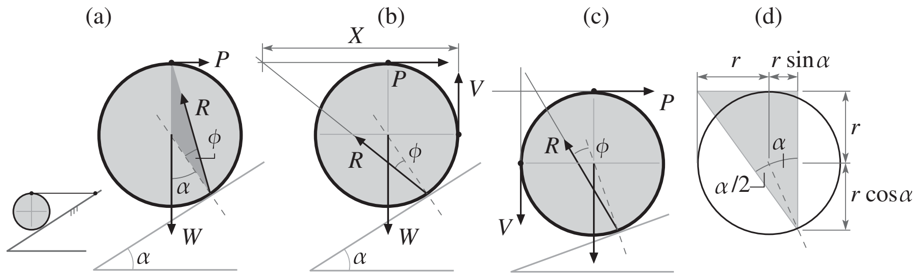

Figure 1.6 Uniform cylinder tethered horizontally to a rough slope. (a) Limiting friction with the cylinder slipping down the slope. (b) Slippage up the slope due to an extra, vertical force, V, applied upwards on the right side. (c) V applied downwards on the left side. (d) Geometry pertaining to (c).

The crossover occurs when the lines of action of R, V and P coincide, with Fig. 1.6(d) showing that $\tan(3\alpha/2) = (1 + \sin\alpha)/(1 + \cos\alpha)$ *i.e.* $\alpha = 0.42$ rad $= 24.3°$.

A larger α allows R, V and P to become very large without slippage, but obviously not W. To find the relationships between them, we could ignore W on grounds that it is much smaller and treat this case as a simpler three-force problem. We should not be surprised (or alarmed) that the friction forces can become infinitely large, notwithstanding local damage at the contact point: V *reinforces* the normal reaction, being generally opposed to it, which increases the friction force tolerance given $F \leq \mu N$; in Fig. 1.6(b) the action of V diminishes N to bolster earlier slippage.

Since no slippage is possible no matter how large the forces involved, the configuration can become 'locked', and usefully so. For example, the wall-mounted 'paper-clip' shown schematically in Fig. 1.7(a) has a cylinder nestled inside a channel with one face inclined inwards to prevent the cylinder falling out under gravity.[1] The cylinder is rough with friction present at its contact lines.

The device works by feeding a paper sheet up through the gap between the vertical channel side and the cylinder, which accommodates by moving into the expanding space; if we now pull the sheet down or let it go, the cylinder rolls into place with contact between all surfaces, as shown, and the sheet is held. The channel is rigid, causing the sheet to tear under a large enough pulling force. In reality though, the channel is flexible and can bend outwards instead, thereby releasing the sheet.

If the weight of the cylinder can be neglected, only two equal and opposite forces, R, are applied to it during locking, as shown in Fig. 1.7(b). Their points of application are normal to the channel sides where contact is made, with one side inclined at angle θ to the vertical. The forces are also collinear, which dictates their orientations and thus inclinations to each common normal. These angles are both $\theta/2$, setting $\mu = \tan(\theta/2)$ as the minimum required coefficient of friction.

[1] F G J Norton, *Advanced Level Applied Mathematics*, Heinemann Educational Books, London, 1974, p. 282, qu. 16.

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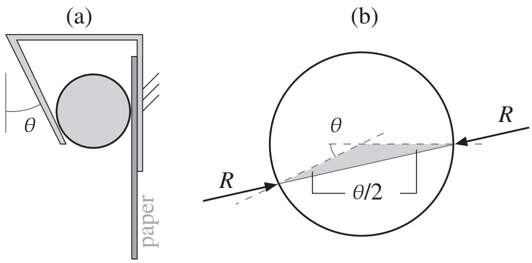

Figure 1.7 (a) Mounted paper clip/cylinder for holding sheets vertically. (b) Equal and opposite resultant forces on the cylinder.

1.3 Distributed Friction

When dealing with prismatic blocks, we have assumed friction forces to be concentrated. In practice, however, there is a normal contact pressure and thus a distributed frictional *intensity*. But because every contacting point experiences the same kinematical tendency, we may consider limiting behaviour in terms of their resultants – applied to/acting through the correct points, as the previous sections attest.

In the following example, slippage occurs in opposite directions *simultaneously* for a single body, which commands a different solution approach. The plan-view in Fig. 1.8(a) shows a narrow rod of uniform weight W and length L sitting on a rough horizontal plane; gravity acts normal to this plane. A force, P, is applied normal to the rod in the same plane at a point αL from one end, causing the rod to slip on the plane.

The rod is shallow in height and does not topple, and its width in plan is negligibly small compared to its length. Importantly, the rod does not translate uniformly (except when $\alpha = 1/2$: see later) but must also rotate initially for force and moment equilibrium. The direction of rotation is assumed to be anti-clockwise as shown, which stipulates a starting value of $\alpha = 1/2$ if P is to be positive for the same sense of rotation (up to a maximum value of $\alpha = 1$).

The point of rotation is generally located a distance βL from the same end as α, as shown in Fig. 1.8(b). The portion of rod before this point therefore slips backwards against P, and the rest forwards.

The out-of-plane contact pressure from gravity on the rectangular base of the rod is uniformly distributed. We can therefore divide the distribution of weight across the rotation point by length alone to give two normal out-of-plane reaction forces, respectively $(W/L) \cdot \beta L$ and $(W/L) \cdot (1 - \beta)L$. These act at the centre of each portion with corresponding friction forces $F_1 = \beta f$ and $F_2 = (1 - \beta) f$ in Fig. 1.8(c) after defining $f = W\mu$.

There are no left–right forces, so we resolve normally to the rod to find

$$P + F_1 - F_2 = 0 \quad \rightarrow \quad P = f(1 - 2\beta). \tag{1.4}$$

Taking moments about, say, the left end and noting the lever arm distances to the line of action of each force, we also find

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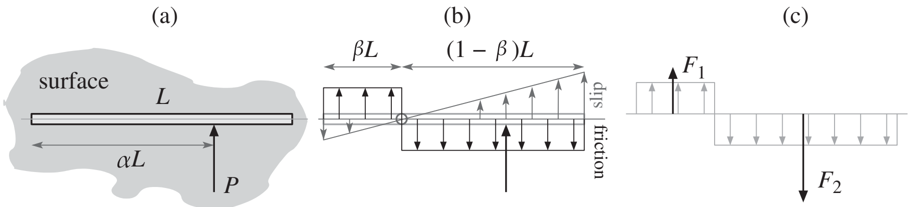

Figure 1.8 (a) Uniform rod on a horizontal plane (grey) acted upon by a normal force, P. (b) Rod slips on the plane by pivoting about a point βL from the left end, generating distributed, opposing frictional loading intensities. (c) Resultant friction forces acting though the centroid of intensities in (b).

$$P\alpha L + F_1\frac{\beta L}{2} - F_2\left[\beta L + \frac{(1-\beta)L}{2}\right] \quad\rightarrow\quad P\alpha = \frac{f}{2}(1-2\beta^2). \tag{1.5}$$

Eliminating P between both equations returns a quadratic in β, which may be solved to give

$$(1-2\beta) = \frac{1}{2\alpha}(1-2\beta^2) \quad\rightarrow\quad \beta = \alpha \pm \sqrt{\alpha^2 - \alpha + 1/2}. \tag{1.6}$$

The larger of these two roots sets $\beta > 1$; or, the rotation point ahead of P, which is impossible in the sense of anti-clockwise rotation: the smaller root only is valid. For example, when $\alpha = 1$, $\beta = 1 - \sqrt{0.5} \approx 0.3$, about 20% of the rod length from the centre (where $P/f \approx 0.41$).

Finally, when P is halfway and $\alpha = 1/2$, then β is found to be zero, giving a point of rotation at the left end of the rod. However, F_1 is also zero, with only F_2 present; the now symmetrical loading precludes any possible rotation of the rod. We can, of course, let P approach halfway in the limiting case, with solutions for β also approaching the left end: these are valid solutions as far as, but not including, the case of $\beta = 0$.

1.4 Static *vs* Kinetic Friction

When a body slips, its ‘dynamic’ coefficient of friction is marginally smaller than the static value before motion takes place: from our experiences of pushing a block on a surface, it is slightly easier to maintain slippage than to initiate it. This difference can also disrupt the symmetry of motion when two or more sliding bodies interact; and the example in Fig. 1.9 elegantly demonstrates this point.

A heavy horizontal rod rests initially on two cylinders asymmetrically positioned about the rod centre. Equal and opposite forces are then applied to each cylinder in order to initiate their approaching movement, as shown in Fig. 1.9(a.i). The cylinder on the right, being farthest away, exerts a smaller normal reaction on the rod compared to that on the left; both apply the same axial force inwards. The resultant frictional force,

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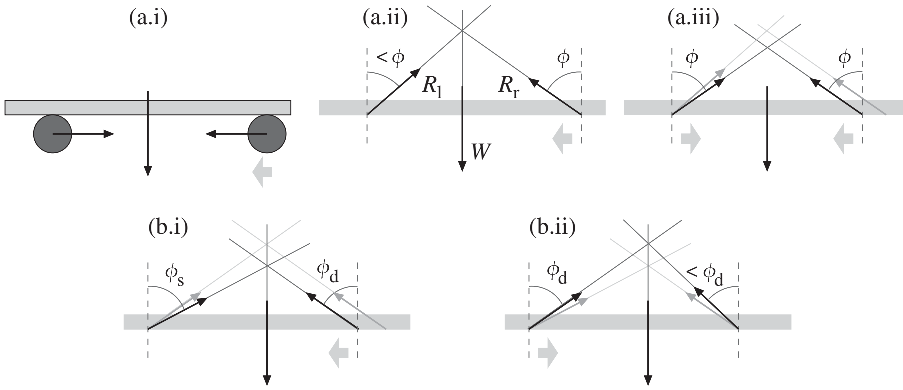

Figure 1.9 (a.i) Uniform rod resting horizontally on a pair of rough asymmetrical cylinders, which are then drawn towards each other; (a.ii) external concurrent forces acting on the rod initially; (a.iii) symmetrical configuration after slippage of the right cylinder. (b.i) Continued slippage of the right cylinder and forces: ϕ_d is the dynamic angle of friction compared to the (larger) static value, ϕ_s; (b.ii) reversal in slippage to the left cylinder.

R_r, is thus more inclined than R_l, and both are concurrent with the third force, W, the weight of the rod, for moment equilibrium.

Assume first that the coefficients of friction are the same and equal to μ ($= \tan\phi$). As the axial forces increase, R_r and R_l lean further away from their common normals, but R_r reaches ϕ first. The right cylinder therefore moves first to the left (quasi-statically), Fig. 1.9(a.ii). Since the inclination of R_r remains fixed, the intersection point of the three forces lowers during its movement, also making R_l more inclined. Eventually, the inclination of R_l reaches ϕ, giving a symmetrical layout of forces, Fig. 1.9(a.iii). The left cylinder can now slip, and both move together symmetrically at the same rate.

Different coefficients of friction do not affect the initial motion provided the right-side cylinder is appreciably off-centre. During slippage, $\mu = \mu_d$ with a corresponding ϕ_d from $\tan\phi_d = \mu_d$, with both parameters being smaller than their respective static values, ϕ_s and μ_s. When the cylinders are symmetrically displaced, R_l is inclined at ϕ_d but needs to be inclined at the larger ϕ_s to reach limiting statical friction; this occurs after some more movement of the right cylinder, Fig. 1.9(b.i).

As soon as the left cylinder can slip, R_l immediately reverts to the smaller inclination, ϕ_d, causing the intersection point to move up slightly. Since the right cylinder is closer to the rod centroid, the inclination of R_r drops below ϕ_d and, thus, below the limiting value altogether. The right cylinder stops moving, Fig. 1.9(b.ii), and we have, in effect, reversed the initial arrangement of slippage between the sides.

This state of motion continues until the left cylinder is sufficiently closer to the middle for R_r to become inclined again at ϕ_s and to start slipping; the left cylinder stops, and so forth until the cylinders meet close to the middle. This example can be easily demonstrated using a long ruler placed on two index fingers.

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1.5 Final Remarks

Friction cannot be ignored even if it features minimally. Its constitutive behaviour is an inequality relationship between the responsible force components, which is satisfied only when slippage occurs. A ‘safe’ solution accords both equilibrium outcomes, of slippage or none, where the inequality has to be evaluated; this is over-wrought.

Pre-supposing slippage specifies the complexion of forces exactly, making geometrical or graphical solutions more amenable, especially when dealing with prismatic sections. The ‘point mass’ viewpoint of equilibrium so readily applied to, say, a block resting on a slope, is only applicable if moment equilibrium does not matter – even then, this requires a robust judgment. The actual geometry – of the body shape, support/contact conditions, and where the forces are applied – must be regarded for accuracy; their interaction equilibrium-wise, however, can become more varied, introducing other possible kinematical outcomes such as toppling and rolling.