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TOPOLOGICAL FULL GROUPS OF C^*-ALGEBRAS ARISING FROM β-EXPANSIONS

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Abstract

We introduce a family of infinite nonamenable discrete groups as an interpolation of the Higman–Thompson groups by using the topological full groups of the groupoids defined by β-expansions of real numbers. They are regarded as full groups of certain interpolated Cuntz algebras, and realized as groups of piecewise-linear functions on the unit interval in the real line if the β-expansion of 1 is finite or ultimately periodic. We also classify them by a number-theoretical property of β.

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1. Introduction

The class of finitely presented infinite groups is one of the most interesting and important classes of infinite groups from the viewpoints of not only group theory but also geometry and topology. The study of finitely presented simple infinite groups has begun with Richard J. Thompson in the 1960s. He [32] discovered the first two such groups. They are now known as the groups V_2 and T_2. Higman [12] and Brown [3] generalized Thompson's examples to an infinite family of finitely presented infinite groups. One of such family is the groups written $V_n, 1 < n \in \mathbb{N}$, which are called the Higman–Thompson groups. They are all finitely presented and their commutator subgroups are all simple. Their abelianizations are trivial if n is even, and $\mathbb{Z}_2$ if n is odd. The Higman–Thompson group V_n is represented as the group of right-continuous piecewise-linear (PL) functions $f : [0, 1) \longrightarrow [0, 1)$ having finitely many singularities such that all singularities of f are in $\mathbb{Z}[1/n]$, the derivative of f at any nonsingular point is n^k for some $k \in \mathbb{Z}$ and $f(\mathbb{Z}[1/n] \cap [0, 1)) = \mathbb{Z}[1/n] \cap [0, 1)$ [32] (see [4, 5, 26]). Nekrashevych [25] showed that the Higman–Thompson group V_n appears as a certain

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subgroup of the unitary group of the Cuntz algebra $\mathcal{O}_n$ of order n. The subgroup of the unitary group of $\mathcal{O}_n$ is the continuous full group Γ_n of $\mathcal{O}_n$, which is also called the topological full group of the associated groupoid (see also [24, Remark 6.3]). Recently, the authors have independently studied full groups of the Cuntz–Krieger algebras and full groups of the groupoids coming from shifts of finite type. The first-named author has studied the normalizer groups of the Cuntz–Krieger algebras [9], which are called the continuous full groups from the viewpoints of orbit equivalence of topological Markov shifts and classification of C^*-algebras (see [16–18] etc.). He [19] proved that the continuous full groups are complete invariants for the continuous orbit equivalence classes of the underlying topological Markov shifts. The second-named author has studied the continuous full groups of more general étale groupoids (see [21–24] etc.). He has called them the topological full groups of étale groupoids. He [24] proved that if an étale groupoid is minimal, the topological full group of the groupoid is a complete invariant for the isomorphism class of the groupoid. He also showed that if a groupoid comes from a shift of finite type, the topological full group is of type F_∞ and in particular finitely presented. He furthermore obtained that the topological full group for a shift of finite type is simple if and only if the homology group $H_0(G)$ of the groupoid G is 2-divisible. Hence, we know an infinite family of finitely presented infinite simple groups coming from symbolic dynamics. Nekrashevych's paper [25] says that the Higman–Thompson groups appear as the topological full groups of the groupoids of the full shifts and as the continuous full groups of the Cuntz algebras. In [13], a family of C^*-algebras $\mathcal{O}_\beta$, $1 < \beta \in \mathbb{R}$ has been introduced. It arises from a family of certain subshifts called the β-shifts, which are the symbolic dynamics defined by the β-transformations on the unit interval $[0, 1]$. The family of the β-shifts is an interpolation of the full shifts. Hence, the C^*-algebras $\mathcal{O}_\beta$, $1 < \beta \in \mathbb{R}$ are considered as an interpolation of the Cuntz algebras $\mathcal{O}_N$, $1 < N \in \mathbb{N}$.

In the present paper, we introduce a family Γ_β, $1 < \beta \in \mathbb{R}$ of infinite discrete groups as an interpolation of the Higman–Thompson groups V_n, $1 < n \in \mathbb{N}$ such that $\Gamma_n = V_n$, $1 < n \in \mathbb{N}$. The groups Γ_β, $1 < \beta \in \mathbb{R}$ are defined as the continuous full groups of the C^*-algebras $\mathcal{O}_\beta$, $1 < \beta \in \mathbb{R}$. They are also considered as the topological full groups of the étale groupoids G_β for the β-shifts. We will first study the groupoid G_β and show that the groupoid G_β for each $1 < \beta \in \mathbb{R}$ is an essentially principal, purely infinite, minimal étale groupoid. The homology groups $H_i(G_\beta)$ are computed as

$$H_i(G_\beta) \cong \begin{cases} K_i(\mathcal{O}_\beta) & \text{if } i = 0, 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

We will show the following theorem.

Theorem 1.1 (Theorem 3.7). *Let $1 < \beta \in \mathbb{R}$ be a real number. Then the group Γ_β is a countably infinite discrete nonamenable group such that its commutator subgroup $D(\Gamma_\beta)$ is simple.*

For a real number $\beta > 1$, let us denote by $d(1,\beta) = \xi_1\xi_2\xi_3\ldots$ the β-adic expansion of 1, which means $\xi_i \in \mathbb{Z}$, $0 \le \xi_i \le [\beta]$ and

$$1 = \frac{\xi_1}{\beta} + \frac{\xi_2}{\beta^2} + \frac{\xi_3}{\beta^3} + \cdots .$$

The expansion $d(1,\beta)$ is said to be finite if there exists $k \in \mathbb{N}$ such that $\xi_m = 0$ for all $m > k$. If there exists $l \le k$ such that

$$d(1,\beta) = \xi_1 \ldots \xi_l \xi_{l+1} \ldots \xi_{k+1} \xi_{l+1} \ldots \xi_{k+1} \xi_{l+1} \ldots \xi_{k+1} \ldots ,$$

the expansion $d(1,\beta)$ is said to be ultimately periodic and written $d(1,\beta) = \xi_1 \cdots \xi_l \dot{\xi}_{l+1} \cdots \dot{\xi}_{k+1}$. It is well known that the Higman–Thompson group $V_n, n \in \mathbb{N}$ is represented as the group of right-continuous PL functions $f : [0,1) \longrightarrow [0,1)$ having finitely many singularities such that all singularities of f are in $\mathbb{Z}[1/n]$, the derivative of f at any nonsingular point is n^k for some $k \in \mathbb{Z}$ and $f(\mathbb{Z}[1/n] \cap [0,1)) = \mathbb{Z}[1/n] \cap [0,1)$. We introduce a notion of β-adic PL functions on the interval $[0,1]$ and show the following theorem.

Theorem 1.2 (Theorems 5.10 and 6.13). *Let $1 < \beta \in \mathbb{R}$ be a real number such that the β-expansion $d(1,\beta)$ of 1 is finite or ultimately periodic. Then the group Γ_β is realized as the group of β-adic PL functions on the interval* $[0,1]$.

It is well known that $d(1,\beta)$ is finite if and only if the β-shift (X_β,σ) is a shift of finite type, and $d(1,\beta)$ is ultimately periodic if and only if the β-shift (X_β,σ) is a sofic shift (see [2, 11]). If $\beta = (1+\sqrt{5})/2$, the number is the positive solution of the quadratic equation $\beta^2 = \beta + 1$, so that the β-expansion is finite: $d(1,\beta) = 110000\cdots$. We will classify the interpolated Higman–Thompson groups Γ_β, $1 < \beta \in \mathbb{R}$ by the number-theoretical property of β in the following way.

Theorem 1.3 (Theorems 7.2, 7.10 and 7.11). *Let $1 < \beta \in \mathbb{R}$ be a real number and $d(1,\beta) = \xi_1\xi_2\xi_3\ldots$ be the β-expansion of* 1.

(i) *If $d(1,\beta)$ is finite, that is, $d(1,\beta) = \xi_1\xi_2\ldots\xi_k 00\ldots$, then the group Γ_β is isomorphic to the Higman–Thompson group $V_{\xi_1+\cdots+\xi_k+1}$ of order $\xi_1 + \cdots + \xi_k + 1$.*

(ii) *If $d(1,\beta)$ is ultimately periodic, that is, $d(1,\beta) = \xi_1 \cdots \xi_l \dot{\xi}_{l+1} \cdots \dot{\xi}_{k+1}$, then the group Γ_β is isomorphic to the Higman–Thompson group $V_{\xi_{l+1}+\cdots+\xi_{k+1}}$ of order $\xi_{l+1} + \cdots + \xi_{k+1}$.*

(iii) *If $1 < \beta \in \mathbb{R}$ is not ultimately periodic, then the group Γ_β is not isomorphic to any of the Higman–Thompson groups V_n, $1 < n \in \mathbb{N}$.*

2. Preliminaries of the C^*-algebra $\mathcal{O}_\beta$

Throughout the paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_+$ the set of nonnegative integers, respectively. We fix an arbitrary real number $\beta > 1$ unless we specify otherwise. Take a natural number N with $N - 1 < \beta \le N$. Put

$\Sigma = \{0, 1, \ldots, N-1\}$. For a nonnegative real number t, we denote by $[t]$ the integer part of t. Let $f_\beta : [0,1] \to [0,1]$ be the function defined by

$$f_\beta(x) = \beta x - [\beta x], \quad x \in [0,1].$$

The β-expansion of $x \in [0,1]$ is a sequence $\{d_n(x,\beta)\}_{n\in\mathbb{N}}$ of integers of Σ determined by (see [27, 30])

$$d_n(x,\beta) = [\beta f_\beta^{n-1}(x)], \quad n \in \mathbb{N}.$$

The numbers $d_n(x,\beta)$ will be denoted by $d_n(x)$ for simplicity. We then obtain the β-expansion of x:

$$x = \sum_{n=1}^{\infty} \frac{d_n(x)}{\beta^n}.$$

We endow the infinite product $\Sigma^{\mathbb{N}}$ with the product topology and the lexicographical order. The lexicographical order in $\Sigma^{\mathbb{N}}$ means that for $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$, the inequality $x < y$ holds if

$$x_1 = y_1, \ldots, x_k = y_k \quad \text{and} \quad x_{k+1} < y_{k+1} \text{ for some } k.$$

We denote by σ the shift on $\Sigma^{\mathbb{N}}$ defined by $\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$. Let $\xi_\beta = (\xi_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ be the supremum element of $\{(d_n(x))_{n\in\mathbb{N}} \mid x \in [0,1)\}$ with respect to the lexicographical order in $\Sigma^{\mathbb{N}}$, which is defined by

$$\xi_\beta = \sup_{x\in[0,1)} (d_n(x))_{n\in\mathbb{N}}.$$

Define the σ-invariant compact subset X_β of $\Sigma^{\mathbb{N}}$ by

$$X_\beta = \{\omega \in \Sigma^{\mathbb{N}} \mid \sigma^m(\omega) \le \xi_\beta, m = 0, 1, 2, \ldots\}.$$

Definition 2.1 (see [27, 30]). The subshift (X_β, σ) is called the β-shift.

Example 2.2. $\beta = N \in \mathbb{N}$ with $N > 1$. As $\xi_\beta = (N-1)(N-1)\ldots$, the subshift

$$X_N = \{(x_n)_{n\in\mathbb{N}} \in \{0, 1, \ldots, N-1\}^{\mathbb{N}} \mid x_n = 0, 1, \ldots, N-1\}$$

is the full N-shift.

Example 2.3. $\beta = (1+\sqrt{5})/2$. As $N = 2$ and $d(1,\beta) = 1100\ldots$, $\xi_\beta = 10101010\ldots$,

$$X_{(1+\sqrt{5})/2} = \{(x_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \mid \text{‘11’ does not appear in } (x_n)_{n\in\mathbb{N}}\}.$$

This is a shift of finite type X_A determined by the matrix $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Example 2.4. $\beta = 2+\sqrt{3}$. As $N = 4$ and $d(1,\beta) = \xi_\beta = 3\dot{2}$,

$$X_{2+\sqrt{3}} = \{(x_n)_{n\in\mathbb{N}} \in \{0,1,2,3\}^{\mathbb{N}} \mid (x_{n+m})_{n\in\mathbb{N}} \le 3\dot{2} \text{ for all } m = 0, 1, 2, \ldots\}.$$

This is a sofic shift but not a shift of finite type.

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EXAMPLE 2.5. $\beta = \frac{3}{2}$. As $N = 2$ and $\xi_\beta = 101000001\ldots$,

$$X_{3/2} = \{(x_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \mid (x_{n+m})_{n\in\mathbb{N}} \le 101000001\ldots, m = 0,1,2,\ldots\}.$$

This is not a sofic shift (and hence not a shift of finite type).

A finite sequence $\mu = (\mu_1,\ldots,\mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word. We denote by $|\mu|$ the length k of μ. Set, for $k \in \mathbb{N}$,

$$B_k(X_\beta) = \{\mu \mid \text{a block with length } k \text{ appearing in some } x \in X_\beta\}$$

and $B_*(X_\beta) = \bigcup_{k=0}^{\infty} B_k(X_\beta)$, where $B_0(X_\beta)$ denotes the empty word $\emptyset$.

In [13], a family $\mathcal{O}_\beta$, $1 < \beta \in \mathbb{R}$ of simple purely infinite C^*-algebras has been introduced as the C^*-algebras associated with β-shifts (X_β, σ). We will review the construction of the C^*-algebra $\mathcal{O}_\beta$ for a fixed $1 < \beta \in \mathbb{R}$. Let $\{e_0,\ldots,e_{N-1}\}$ be an orthonormal basis of the N-dimensional Hilbert space $\mathbb{C}^N$. We put

$$\mathcal{H}_\beta^0 = \mathbb{C}\Omega \quad (\Omega: \text{ vacuum vector}),$$

$$\mathcal{H}_\beta^k = \text{the Hilbert space spanned by the vectors } e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k},$$
$$\mu = (\mu_1,\ldots,\mu_k) \in B_k(X_\beta).$$

Let us denote by $\mathcal{H}_\beta$ the Hilbert space of the direct sum $\oplus_{k=0}^{\infty}\mathcal{H}_\beta^k$. We denote by T_ν, $\nu \in B_*(X_\beta)$ the creation operator on $\mathcal{H}_\beta$ of e_ν, which is a partial isometry defined by

$$T_\nu\Omega = e_\nu \quad \text{and} \quad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu & \text{if } \nu\mu \in B_*(X_\beta), \\ 0 & \text{otherwise.} \end{cases}$$

We put $T_\emptyset = 1$ for the empty word $\emptyset$. Let $\mathbb{P}_0$ be the rank-one projection of $\mathcal{H}_\beta$ onto the vacuum vector Ω. It immediately follows that $\sum_{i=0}^{N-1} T_iT_i^* + \mathbb{P}_0 = 1$. For $\mu,\nu \in B_*(X_\beta)$, the operator $T_\mu\mathbb{P}_0T_\nu^*$ is the rank-one partial isometry from the vector e_ν to e_μ, so that the C^*-algebra generated by the elements of the form $T_\mu\mathbb{P}_0T_\nu^*, \mu,\nu \in B_*(X_\beta)$ is nothing but the C^*-algebra $\mathcal{K}(\mathcal{H}_\beta)$ of all compact operators on $\mathcal{H}_\beta$. Let $\mathcal{T}_\beta$ be the C^*-algebra on $\mathcal{H}_\beta$ generated by the elements T_ν, $\nu \in B_*(X_\beta)$.

DEFINITION 2.6 [13]. The C^*-algebra $\mathcal{O}_\beta$ associated with the β-shift is defined as the quotient C^*-algebra $\mathcal{T}_\beta/\mathcal{K}(\mathcal{H}_\beta)$ of $\mathcal{T}_\beta$ by $\mathcal{K}(\mathcal{H}_\beta)$.

We denote by S_i, $i = 0,1,\ldots,N-1$ and S_μ, $\mu \in B_*(X_\beta)$ the quotient images of the operators T_i and T_μ, respectively. Since $S_\mu = S_{\mu_1}\cdots S_{\mu_l}$ for $\mu = (\mu_1,\ldots,\mu_l) \in B_l(X_\beta)$, the C^*-algebra $\mathcal{O}_\beta$ is generated by $N-1$ isometries $S_0,\ldots,S_{N-2}$ and one partial isometry S_{N-1} with the relation $\sum_{i=0}^{N-1} S_iS_i^* = 1$. For $\beta = N \in \mathbb{N}$, the C^*-algebra is isomorphic to the Cuntz algebra $\mathcal{O}_N$. Hence, the family $\mathcal{O}_\beta$, $1 < \beta \in \mathbb{R}$ is regarded as an interpolation of the Cuntz algebras $\mathcal{O}_N$, $1 < N \in \mathbb{N}$.

We put $a_\mu = S_\mu^*S_\mu$ for $\mu \in B_*(X_\beta)$ and define C^*-subalgebras of $\mathcal{O}_\beta$:

$$\mathcal{A}_l = \text{the } C^*\text{-subalgebra of } \mathcal{O}_\beta \text{ generated by } S_\mu^*S_\mu, \mu \in B_l(X_\beta),$$

$$\mathcal{A}_\beta = \text{the } C^*\text{-subalgebra of } \mathcal{O}_\beta \text{ generated by } S_\mu^*S_\mu, \mu \in B_*(X_\beta),$$

$$\mathcal{D}_\beta = \text{the } C^*\text{-subalgebra of } \mathcal{O}_\beta \text{ generated by } S_\mu aS_\mu^*, \mu \in B_*(X_\beta), a \in \mathcal{A}_\beta,$$

$$\mathcal{F}_\beta = \text{the } C^*\text{-subalgebra of } \mathcal{O}_\beta \text{ generated by } S_\mu aS_\nu^*, \mu,\nu \in B_k(X_\beta), k \in \mathbb{Z}_+, a \in \mathcal{A}_\beta.$$

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As $S_\mu^* S_\mu = S_\mu^* S_0^* S_0 S_\mu$, the algebra $\mathcal{A}_l$ is naturally embedded into $\mathcal{A}_{l+1}$. It is commutative and finite dimensional so that the algebras $\mathcal{A}_\beta$, $\mathcal{D}_\beta$ and $\mathcal{F}_\beta$ are all AF (approximately finite dimensional)-algebras; in particular, $\mathcal{A}_\beta$ and $\mathcal{D}_\beta$ are both commutative. Put $\rho_j(x) = S_j^* x S_j$ for $x \in \mathcal{A}_\beta$, $j = 0, 1, \ldots, N-1$. Then the C^*-algebra $\mathcal{O}_\beta$ has a universal property subject to the relations (see [15])

$$\sum_{j=0}^{N-1} S_j S_j^* = 1, \quad \rho_j(x) = S_j^* x S_j \quad \text{for } x \in \mathcal{A}_\beta,\ j = 0, 1, \ldots, N-1.$$

For $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the correspondence $S_j \longrightarrow e^{2\pi\sqrt{-1}t} S_j$, $j = 0, 1, \ldots, N-1$ yields an automorphism of $\mathcal{O}_\beta$, which gives rise to an action on $\mathcal{O}_\beta$ of $\mathbb{T}$ called the gauge action written $\hat{\rho}$. The gauge action has a unique KMS state denoted by φ on $\mathcal{O}_\beta$ at inverse temperature $\log\beta$. For the projections $a_{\xi_1\cdots\xi_n} = S_{\xi_1\cdots\xi_n}^* S_{\xi_1\cdots\xi_n} \in \mathcal{A}_\beta, n = 1, 2, \ldots,$ the values $\varphi(a_{\xi_1\cdots\xi_n})$ are computed as

$$\varphi(a_{\xi_1\cdots\xi_n}) = \beta^n - \xi_1\beta^{n-1} - \cdots - \xi_{n-1}\beta - \xi_n = \sum_{i=1}^{\infty} \frac{\xi_{i+n}}{\beta^i}, \quad n = 1, 2, \ldots ([13]).$$

Let $m(l)$ denote the dimension $\dim\mathcal{A}_l$ of $\mathcal{A}_l$. Denote by $E_1^l, \ldots, E_{m(l)}^l$ the set of minimal projections of $\mathcal{A}_l$. As in [13, Lemma 3.3], the projection $E_i^l, i = 1, \ldots, m(l)$ is of the form $E_i^l = a_{\xi_1\cdots\xi_{p_i}} - a_{\xi_1\cdots\xi_{q_i}}$ for some $p_i, q_i = 0, 1, \ldots$. The projections $a_{\xi_1\cdots\xi_n}, n \in \mathbb{Z}_+$ are totally ordered by the value $\varphi(a_{\xi_1\cdots\xi_n})$. We order $E_1^l, \ldots, E_{m(l)}^l$ following the order $\varphi(a_{\xi_1\cdots\xi_{p_1}}) < \cdots < \varphi(a_{\xi_1\cdots\xi_{p_{m(l)}}})$ in $\mathbb{R}$.

Some basic subclasses of β-shifts are characterized in terms of the β-expansion $d(1,\beta)$ of 1 and the projections $a_{\xi_1\cdots\xi_n}$ in the following way.

Lemma 2.7 ([27], see [13, Proposition 3.8]). *The following are equivalent:*

(i) (X_β, σ) *is a shift of finite type;*
(ii) $d(1,\beta)$ *is finite, that is,* $d(1,\beta) = \xi_1\xi_2\cdots\xi_k 000\cdots$ *for some* k*;*
(iii) $a_{\xi_1\cdots\xi_k} = 1$ *for some* k.

We call (X_β, σ) an SFT β-shift if (X_β, σ) is a shift of finite type.

Lemma 2.8 ([1], see [13, Proposition 3.8]). *The following are equivalent:*

(i) (X_β, σ) *is a sofic shift;*
(ii) $d(1,\beta)$ *is ultimately periodic, that is,* $d(1,\beta) = \xi_1\cdots\xi_l\dot{\xi}_{l+1}\cdots\dot{\xi}_{k+1}$ *for some* $l \le k$*;*
(iii) $a_{\xi_1\cdots\xi_l} = a_{\xi_1\cdots\xi_{k+1}}$ *for some* $l \le k$.

We call (X_β, σ) a sofic β-shift if (X_β, σ) is a sofic shift.

The K-groups of the C^*-algebra $\mathcal{O}_\beta$ have been computed in the following way.

Lemma 2.9 [13].

$$K_0(\mathcal{O}_\beta) = \begin{cases} \mathbb{Z}/(\xi_1 + \xi_2 + \cdots + \xi_k - 1)\mathbb{Z} & \text{if } d(1,\beta) = \xi_1\xi_2\cdots\xi_k 000\ldots, \\ \mathbb{Z}/(\xi_{l+1} + \cdots + \xi_{k+1})\mathbb{Z} & \text{if } d(1,\beta) = \xi_1\cdots\xi_l\dot{\xi}_{l+1}\cdots\dot{\xi}_{k+1}, \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

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The position $[1]$ *of the unit of* $\mathcal{O}_\beta$ *in* $K_0(\mathcal{O}_\beta)$ *corresponds to the class* $[1]$ *of* $1 \in \mathbb{Z}$ *in the first two cases, and to* $1 \in \mathbb{Z}$ *in the third case, and*

$$K_1(\mathcal{O}_\beta) = 0 \quad \textit{for any } \beta > 1.$$

3. Topological full groups of the groupoid G_β

The C^*-algebra $\mathcal{O}_\beta$, $1 < \beta \in \mathbb{R}$ has been originally constructed as the C^*-algebra associated with the subshift (X_β, σ), $1 < \beta \in \mathbb{R}$. It is regarded as the C^*-algebra $C_r^*(G_\beta)$ of a certain essentially principal étale groupoid G_β as in [15, Section 2]. We will review the construction of the groupoid G_β in the following way. We denote by $\Omega_l = \{v_1^l, \ldots, v_{m(l)}^l\}$ the finite set with its discrete topology corresponding to the set of the minimal projections $E_1^l, \ldots, E_{m(l)}^l$ of the commutative algebra $\mathcal{A}_l$, so that $\mathcal{A}_l = C(\Omega_l)$. If $E_j^{l+1} \le E_i^l$, we write $\iota_{l,l+1}(v_j^{l+1}) = v_i^l$. We define an edge e labeled $\alpha \in \{0, 1, \ldots, N-1\}$ from v_i^l to v_j^{l+1} if $S_\alpha^* E_i^l S_\alpha \ge E_j^{l+1}$. Denote by $E_{l,l+1}$ such labeled edges. Let Ω_β be the compact Hausdorff space of the projective limit of the system $\iota_{l,l+1} : \Omega_{l+1} \longrightarrow \Omega_l$, $l \in \mathbb{Z}_+$:

$$\Omega_\beta = \Big\{ (v^l)_{l \in \mathbb{Z}_+} \in \prod_{l \in \mathbb{Z}_+} \Omega_l \;\Big|\; \iota_{l,l+1}(v^{l+1}) = v^l, l \in \mathbb{Z}_+ \Big\}.$$

Let $\mathcal{G}_\beta$ be the set of triplets $(u, \alpha, v) \in \Omega_\beta \times \{0, 1, \ldots, N-1\} \times \Omega_\beta$ such that for each $l \in \mathbb{Z}_+$ there exists $e_{l,l+1} \in E_{l,l+1}$ whose source is u^l, terminal is v^{l+1} and label is α, where $u = (u^l)_{l \in \mathbb{Z}_+}$ and $v = (v^l)_{l \in \mathbb{Z}_+}$. Then $\mathcal{G}_\beta$ becomes a zero-dimensional continuous graph in the sense of Deaconu [10]. Consider the set $G_\beta^{(0)}$ of one-sided paths of the graph $\mathcal{G}_\beta$:

$$G_\beta^{(0)} = \Big\{ (\alpha_i, u_i)_{i=1}^\infty \in \prod_{i=1}^\infty (\{0, 1, \ldots, N-1\} \times \Omega_\beta) \;\Big|\; (u_i, \alpha_{i+1}, u_{i+1}) \in \mathcal{G}_\beta \text{ for all } i \in \mathbb{N} \text{ and } (u_0, \alpha_1, u_1) \in \mathcal{G}_\beta \text{ for some } u_0 \in \Omega_\beta \Big\}.$$

The set $G_\beta^{(0)}$ has the relative topology from the infinite product topology of $\{0, 1, \ldots, N-1\} \times \Omega_\beta$. It is a zero-dimensional compact Hausdorff space such that the C^*-algebra $C(G_\beta^{(0)})$ of complex-valued continuous functions on $G_\beta^{(0)}$ is canonically isomorphic to the C^*-subalgebra $\mathcal{D}_\beta$ of $\mathcal{O}_\beta$, which is called the canonical Cartan subalgebra of $\mathcal{O}_\beta$. The shift map $\sigma_\beta : (\alpha_i, u_i)_{i=1}^\infty \in G_\beta^{(0)} \to (\alpha_{i+1}, u_{i+1})_{i=1}^\infty \in G_\beta^{(0)}$ is a surjective local homeomorphism.

Definition 3.1. The groupoid G_β with unit space $G_\beta^{(0)}$ is defined by the étale groupoid associated with the surjective local homeomorphism σ_β on $G_\beta^{(0)}$ in the following way:

$$G_\beta = \{(x, k-l, y) \in G_\beta^{(0)} \times \mathbb{Z} \times G_\beta^{(0)} \mid \sigma_\beta^k(x) = \sigma_\beta^l(y) \text{ for some } k, l \in \mathbb{Z}_+\}.$$

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For an étale groupoid G, we let $G^{(0)}$ denote the unit space of G and let s and r denote the source map and the range map, respectively. For $x \in G^{(0)}$, the set $G(x) = r(Gx)$ is called the G-orbit of x. If every G-orbit is dense in $G^{(0)}$, G is said to be minimal [24, 28].

LEMMA 3.2. *For $1 < \beta \in \mathbb{R}$, the groupoid G_β is an essentially principal, minimal groupoid.*

PROOF. The C^*-subalgebra $\mathcal{F}_\beta$ of $\mathcal{O}_\beta$ is the C^*-algebra $C_r^*(H_\beta)$ of an AF-subgroupoid H_β of G_β, which is defined by

$$H_\beta = \{(x, 0, y) \in G_\beta^{(0)} \times \mathbb{Z} \times G_\beta^{(0)} \mid \sigma_\beta^k(x) = \sigma_\beta^k(y) \text{ for some } k \in \mathbb{Z}_+\}.$$

As the algebra $\mathcal{F}_\beta$ is simple [13, Proposition 3.5], the groupoid H_β is minimal, so that G_β is minimal. □

A subset $U \subset G$ is called a G-set if $r|_U, s|_U$ are injective. The homeomorphism $r \circ (s|_U)^{-1}$ from $s(U)$ to $r(U)$ is denoted by π_U. Following [24], G is said to be purely infinite if for every clopen set $A \subset G^{(0)}$ there exist clopen G-sets $U, V \subset G$ such that $s(U) = s(V) = A, r(U) \cup r(V) \subset A, r(U) \cap r(V) = \emptyset$.

LEMMA 3.3. *For $1 < \beta \in \mathbb{R}$, the groupoid G_β is purely infinite.*

PROOF. As the C^*-algebra $\mathcal{D}_\beta$ is isomorphic to the algebra $C(G_\beta^{(0)})$ of continuous functions on $G_\beta^{(0)}$, we may identify the projections of $\mathcal{D}_\beta$ with the clopen sets of $G_\beta^{(0)}$. Hence, a clopen set of $G_\beta^{(0)}$ may be considered as a finite sum of the form $P = S_\mu E_i^l S_\mu^*$ for some $\mu \in B_k(X_\beta)$ with $k \le l$ such that $S_\mu^* S_\mu \ge E_i^l$. It is enough to consider $P = S_\mu E_i^l S_\mu^*$ for simplicity. The minimal projection $E_i^l \in \mathcal{A}_l$ is of the form $E_i^l = a_{\xi_1\cdots\xi_{p_i}} - a_{\xi_1\cdots\xi_{q_i}}$ for some $1 \le p_i, q_i \le l$ with $a_{\xi_1\cdots\xi_{p_i}} > a_{\xi_1\cdots\xi_{q_i}}$. Note that

$$S_\mu^* S_\mu \ge a_{\xi_1\cdots\xi_{p_i}}. \tag{3.1}$$

There exists $\gamma = (\gamma_1, \ldots, \gamma_r) \in B_*(X_\beta)$ such that

$$(\xi_1, \ldots, \xi_{p_i}, \gamma_1, \ldots, \gamma_r) \in B_*(X_\beta), \quad (\xi_1, \ldots, \xi_{q_i}, \gamma_1, \ldots, \gamma_r) \notin B_*(X_\beta).$$

Define the words

$$\zeta_1(m) = (\overbrace{0, \ldots, 0}^{m}), \quad \zeta_2(m) = (\overbrace{0, \ldots, 0, 1}^{m}).$$

By [13, Corollary 3.2], there exists $m \in \mathbb{N}$ such that

$$a_{\xi_1\cdots\xi_{p_i}\gamma\zeta_1(m)} = a_{\xi_1\cdots\xi_{p_i}\gamma\zeta_2(m)} = 1. \tag{3.2}$$

Put $\zeta_1 = \zeta_1(m), \zeta_2 = \zeta_2(m)$. By (3.1) and (3.2),

$$a_{\mu\gamma\zeta_1} \ge S_{\gamma\zeta_1}^* a_{\xi_1\cdots\xi_{p_i}} S_{\gamma\zeta_1} = a_{\xi_1\cdots\xi_{p_i}\gamma\zeta_1} = 1,$$

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so that $a_{\mu\gamma\zeta_1} = 1$ and similarly $a_{\mu\gamma\zeta_2} = 1$. We set

$$U = S_{\mu\gamma\zeta_1} E_i^l S_\mu^*, \quad V = S_{\mu\gamma\zeta_2} E_i^l S_\mu^*,$$

which correspond to certain clopen G-sets in G_β. It then follows that

$$U^*U = S_\mu E_i^l a_{\mu\gamma\zeta_1} E_i^l S_\mu^* = S_\mu E_i^l S_\mu^* = P \quad \text{and similarly} \quad V^*V = P,$$

so that

$$UU^* + VV^* = S_{\mu\gamma\zeta_1} E_i^l S_{\mu\gamma\zeta_1}^* + S_{\mu\gamma\zeta_2} E_i^l S_{\mu\gamma\zeta_2}^*.$$

As

$$\begin{aligned} S_{\gamma\zeta_1}^* E_i^l S_{\gamma\zeta_1} &= S_{\zeta_1}^* S_\gamma^* (a_{\xi_1\cdots\xi_{p_i}} - a_{\xi_1\cdots\xi_{q_i}}) S_\gamma S_{\zeta_1} = S_{\zeta_1}^* a_{\xi_1\cdots\xi_{p_i}\gamma} S_{\zeta_1} = 1, \\ PS_{\mu\gamma\zeta_1} E_i^l S_{\mu\gamma\zeta_1}^* &= S_\mu E_i^l S_{\gamma\zeta_1} E_i^l S_{\mu\gamma\zeta_1}^* = S_{\mu\gamma\zeta_1} S_{\gamma\zeta_1}^* E_i^l S_{\gamma\zeta_1} E_i^l S_{\mu\gamma\zeta_1}^*, \end{aligned}$$

so that $PS_{\mu\gamma\zeta_1} E_i^l S_{\mu\gamma\zeta_1}^* = S_{\mu\gamma\zeta_1} E_i^l S_{\mu\gamma\zeta_1}^*$. This implies $UU^* \le P$ and similarly $VV^* \le P$. Since $S_{\mu\gamma\zeta_1} E_i^l S_{\mu\gamma\zeta_1}^* \cdot S_{\mu\gamma\zeta_2} E_i^l S_{\mu\gamma\zeta_2}^* = 0$, we have $UU^* + VV^* \le P$. □

Therefore, we have the following proposition.

Proposition 3.4. *For $1 < \beta \in \mathbb{R}$, the groupoid G_β is an essentially principal, purely infinite, minimal, étale groupoid.*

We will next compute the homology groups $H_i(G_\beta)$ for the étale groupoid G_β. The homology theory for étale groupoids has been studied in [6]. In [22], the homology groups H_i for the groupoids coming from shifts of finite type have been computed such that the groups H_i are isomorphic to the K-groups K_i of the associated Cuntz–Krieger algebra for $i = 0, 1$, and $H_i = 0$ for $i \ge 2$. By following the argument of the proof of [22, Theorem 4.14], we have the following proposition.

Proposition 3.5. *For each $1 < \beta \in \mathbb{R}$, the homology groups $H_i(G_\beta)$ are computed as*

$$H_i(G_\beta) \cong \begin{cases} K_i(\mathcal{O}_\beta) & \text{if } i = 0, 1, \\ 0 & \text{if } i \ge 2. \end{cases} \tag{3.3}$$

Proof. For each $1 < \beta \in \mathbb{R}$, the map $\rho_\beta : (x, n, y) \in G_\beta \longrightarrow n \in \mathbb{Z}$ gives rise to a groupoid homomorphism such that the skew product $G_\beta \times_{\rho_\beta} \mathbb{Z}$ is homologically similar to the AF-groupoid H_β (see [22, Lemma 4.13]). We know that the groupoid C^*-algebra $C_r^*(G_\beta \times_{\rho_\beta} \mathbb{Z})$ is stably isomorphic to the crossed product $\mathcal{O}_\beta \times_{\hat\rho} \mathbb{T}$ of $\mathcal{O}_\beta$ by the gauge action, which is stably isomorphic to the AF-algebra $C_r^*(H_\beta)$. Since the $\mathbb{Z}$-module structure on $H_0(G_\beta \times_{\rho_\beta} \mathbb{Z})$ is given by the induced action $\hat{\hat\rho}_*$ on $K_0(\mathcal{O}_\beta \times_{\hat\rho} \mathbb{T})$ of the bidual action $\hat{\hat\rho}$ on $\mathcal{O}_\beta \times_{\hat\rho} \mathbb{T}$, we get (3.3) by the same argument as [22, Theorem 4.14]. □

In [22], the notion of topological full groups for étale groupoids has been introduced. We will study the topological full groups of the groupoid G_β for the β-shift (X_β, σ).

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DEFINITION 3.6 [22, Definition 2.3]. The topological full group $[[G_\beta]]$ of the groupoid G_β is defined by the group of all homeomorphisms α of $G_\beta^{(0)}$ such that $\alpha = \pi_U$ for some compact open G_β-set U.

In what follows, we denote the topological full group $[[G_\beta]]$ by Γ_β. By [22, Proposition 5.6], there exists a short exact sequence

$$1 \longrightarrow U(C(G_\beta^{(0)})) \longrightarrow N(C(G_\beta^{(0)}), C_r^*(G_\beta)) \longrightarrow \Gamma_\beta \longrightarrow 1,$$

where $U(C(G_\beta^{(0)}))$ denotes the group of unitaries in $C(G_\beta^{(0)})$ and $N(C(G_\beta^{(0)}), C_r^*(G_\beta))$ denotes the group of unitaries in $C_r^*(G_\beta)$ which normalize $C(G_\beta^{(0)})$.

Consider the full n-shift (X_n, σ) and its groupoid G_n (see [24, 28]). The groupoid C^*-algebra $C_r^*(G_n)$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$ of order n. Nekrashevych [25] has shown that the Higman–Thompson group V_n is identified with a certain subgroup of the unitary group of $\mathcal{O}_n$. The identification gives rise to an isomorphism between the Higman–Thompson group V_n and the topological full group Γ_n (see also [24, Remark 6.3]). Hence, our groups $\Gamma_\beta, 1 < \beta \in \mathbb{R}$ are considered as an interpolation of the Higman–Thompson groups $V_n, 1 < n \in \mathbb{N}$. It is well known that the groups $V_n, 1 < n \in \mathbb{N}$ are nonamenable and their commutator subgroups $D(V_n)$ are all simple. Proposition 3.4 says that the groupoid G_β is an essentially principal, purely infinite, minimal groupoid for every $1 < \beta \in \mathbb{R}$. By [24, Proposition 4.10 and Theorem 4.16], we have the following generalization of the above fact for $V_n, 1 < n \in \mathbb{N}$.

THEOREM 3.7. *Let $1 < \beta \in \mathbb{R}$ be a real number. Then the group Γ_β is a countably infinite, discrete, nonamenable group such that its commutator subgroup $D(\Gamma_\beta)$ is simple.*

4. Realization of $\mathcal{O}_\beta$ on $L^2([0, 1])$

The Higman–Thompson group $V_n, 1 < n \in \mathbb{N}$ is represented as the group of right-continuous PL bijective functions $f : [0, 1) \longrightarrow [0, 1)$ having finitely many singularities such that all singularities of f are in $\mathbb{Z}[1/n]$, the derivative of f at any nonsingular point is n^k for some $k \in \mathbb{Z}$ and $f(\mathbb{Z}[1/n] \cap [0, 1)) = \mathbb{Z}[1/n] \cap [0, 1)$. In order to represent our group Γ_β as a group of PL functions on $[0, 1)$, we will represent the algebra $\mathcal{O}_\beta$ on $L^2([0, 1])$ in the following way.

We denote by H the Hilbert space $L^2([0, 1])$ of the square-integrable functions on $[0, 1]$ with respect to the Lebesgue measure. The essentially bounded measurable functions $L^\infty([0, 1])$ act on H by multiplication. We define the sequence

$$\beta_n = \beta^n - \xi_1\beta^{n-1} - \cdots - \xi_{n-1}\beta - \xi_n = \sum_{i=1}^{\infty} \frac{\xi_{i+n}}{\beta^i}, \quad n = 1, 2, \ldots.$$

Consider the functions $g_0, g_1, \ldots, g_{N-1}$ defined by

$$g_i(x) = \frac{1}{\beta}(x + i) \quad \text{for } i = 0, 1, \ldots, N-2, \quad x \in [0, 1],$$

$$g_{N-1}(x) = \frac{1}{\beta}(x + N - 1) \quad \text{for } x \in [0, \beta_1].$$

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They satisfy the following equalities:

$$\bigcup_{i=0}^{N-2} g_i([0,1]) \cup g_{N-1}([0,\ \beta_1]) = [0,1],$$
$$f_\beta(g_i(x)) = x \quad \text{for } i = 0,1,\dots,N-2, \quad x \in [0,1],$$
$$f_\beta(g_{N-1}(x)) = x \quad \text{for } x \in [0,\ \beta_1].$$

For a measurable subset U of $[0,1]$, denote by χ_U the multiplication operator on H of the characteristic function of U. Define the bounded linear operators T_{f_β}, $T_{g_i}, i = 0,1,\dots,N-2$ on H by

$$(T_{f_\beta}\xi)(x) = \xi(f_\beta(x)) \quad \text{for } \xi \in H, x \in [0,1],$$
$$(T_{g_i}\xi)(x) = \xi(g_i(x)) \quad \text{for } \xi \in H, x \in [0,1],\ i = 0,1,\dots,N-2.$$

For the function g_{N-1} on $[0,\ \beta_1]$, define the operator $T_{g_{N-1}}$ by

$$(T_{g_{N-1}}\xi)(x) = \begin{cases} \xi(g_{N-1}(x)) & \text{for } x \in [0,\ \beta_1], \\ 0 & \text{for } x \in (\beta_1, 1]. \end{cases}$$

The following lemma is straightforward.

Lemma 4.1. *Keep the above notation. We have*

(i) $T_{f_\beta}^* = (1/\beta)\sum_{i=0}^{N-1} T_{g_i}$.

(ii) $T_{f_\beta}^* T_{f_\beta} = (N-1)/\beta + (1/\beta)\chi_{[0,\beta_1]}$.

(iii) $T_{g_i}^* T_{g_i} = \begin{cases} \beta\chi_{[i/\beta,(i+1)/\beta)} & \textit{for } i = 0,1,\dots,N-2, \\ \beta\chi_{[(N-1)/\beta,1)} & \textit{for } i = N-1. \end{cases}$

(iv) $T_{g_i} T_{g_i}^* = \begin{cases} \beta 1 & \textit{for } i = 0,1,\dots,N-2, \\ \beta\chi_{[0,\beta_1]} & \textit{for } i = N-1. \end{cases}$

We define the operators $s_i, i = 0,\dots,N-1$ on H by setting

$$s_i = \frac{1}{\sqrt{\beta}} T_{g_i}^*, \quad i = 0,1,\dots,N-1.$$

By the above lemma, we have the following proposition.

Proposition 4.2. *The operators $s_i, i = 0,\dots,N-1$ are partial isometries such that*

$$s_i^* s_i = \begin{cases} 1 & \textit{for } i = 0,1,\dots,N-2, \\ \chi_{[0,\beta_1]} & \textit{for } i = N-1, \end{cases}$$

$$s_i s_i^* = \begin{cases} \chi_{[i/\beta,(i+1)/\beta)} & \textit{for } i = 0,1,\dots,N-2, \\ \chi_{[(N-1)/\beta,1)} & \textit{for } i = N-1 \end{cases} \quad \textit{and hence} \quad \sum_{i=0}^{N-1} s_i s_i^* = 1.$$

The natural ordering of $\Sigma = \{0, 1, \ldots, N-1\}$ induces the lexicographical order on $B_*(X_\beta)$, which means that for $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_\beta)$ and $\nu = (\nu_1, \ldots, \nu_m) \in B_m(X_\beta)$, the order $\mu \prec \nu$ is defined if $\mu_1 < \nu_1$ or $\mu_i = \nu_i$ for $i = 1, \ldots, k-1$ for some $k \leq m, n$ and $\mu_k < \nu_k$. For a word $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_\beta)$, denote by $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_n) \in B_n(X_\beta)$ the least word in $B_n(X_\beta)$ satisfying $(\mu_1, \ldots, \mu_n) \prec (\tilde{\mu}_1, \ldots, \tilde{\mu}_n)$. If $\mu = (\mu_1, \ldots, \mu_n)$ is maximal in $B_n(X_\beta)$, we set $\tilde{\mu} = \emptyset$. We will use the following notation for $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_\beta)$:

$$l(\mu) := \frac{\mu_1}{\beta} + \frac{\mu_2}{\beta^2} + \cdots + \frac{\mu_n}{\beta^n}, \quad r(\mu) := \frac{\tilde{\mu}_1}{\beta} + \frac{\tilde{\mu}_2}{\beta^2} + \cdots + \frac{\tilde{\mu}_n}{\beta^n}.$$

If $\tilde{\mu} = \emptyset$, we set $r(\mu) = 1$. For $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_\beta)$, we set $s_\mu = s_{\mu_1} \cdots s_{\mu_n}$.

Lemma 4.3. *For $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_\beta)$,*

$$s_\mu s_\mu^* = \chi_{[l(\mu), r(\mu))}. \tag{4.1}$$

Proof. For $n = 1$, the equality (4.1) holds by the above proposition. Suppose that the equality (4.1) holds for a fixed $n = k$. It then follows that for $j = 0, \ldots, N-1$ and $\xi, \eta \in H$,

$$\langle s_j s_{\mu_1} \cdots s_{\mu_k} s_{\mu_k}^* \cdots s_{\mu_1}^* s_j^* \xi \mid \eta \rangle = \frac{1}{\beta} \int_0^1 \chi_{[l(\mu), r(\mu))} \xi(g_j(x)) \overline{\eta(g_j(x))}\, dx. \tag{4.2}$$

For $j = 0, 1, \ldots, N-2$, put $y = g_j(x) \in [j/\beta, (j+1)/\beta]$, so that $x = f_\beta(y) = \beta y - j$. The above equation (4.2) becomes

$$\int_0^1 \chi_{[j/\beta,(j+1)/\beta)}(y) \chi_{[l(\mu),r(\mu))}(f_\beta(y)) \xi(y) \overline{\eta(y)}\, dy = \langle \chi_{[j/\beta,(j+1)/\beta) \cap f_\beta^{-1}([l(\mu),r(\mu)))} \xi \mid \eta \rangle.$$

As

$$\begin{aligned}&\Big[\frac{j}{\beta}, \frac{j+1}{\beta}\Big) \cap f_\beta^{-1}([l(\mu), r(\mu)))\\ &\quad = \Big[\frac{j}{\beta} + \frac{\mu_1}{\beta^2} + \frac{\mu_2}{\beta^3} + \cdots + \frac{\mu_k}{\beta^{k+1}}, \frac{j}{\beta} + \frac{\tilde{\mu}_1}{\beta^2} + \frac{\tilde{\mu}_2}{\beta^3} + \cdots + \frac{\tilde{\mu}_k}{\beta^{k+1}}\Big),\end{aligned}$$

$$s_j s_{\mu_1} \cdots s_{\mu_k} s_{\mu_k}^* \cdots s_{\mu_1}^* s_j^* = \chi_{[j/\beta+\mu_1/\beta^2+\mu_2/\beta^3+\cdots+\mu_k/\beta^{k+1}, j/\beta+\tilde{\mu}_1/\beta^2+\tilde{\mu}_2/\beta^3+\cdots+\tilde{\mu}_k/\beta^{k+1})}.$$

Since $(j, \tilde{\mu}_1, \ldots, \tilde{\mu}_k)$ is minimal in $B_{k+1}(X_\beta)$ satisfying $(j, \mu_1, \ldots, \mu_k) \prec (j, \tilde{\mu}_1, \ldots, \tilde{\mu}_k)$, the desired equality holds for $k+1$ and $j = 0, \ldots, N-2$. For $j = N-1$, we may similarly show the equality (4.1). □

The following lemma is straightforward.

Lemma 4.4. *For a measurable subset $F \subset [0, 1]$, we have $s_j^* \chi_F s_j = \chi_{g_j^{-1}(F)}$ for $j = 0, 1, \ldots, N-1$.*

We then have the following lemmas.

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LEMMA 4.5. *For the maximal element* $\xi_\beta = (\xi_1, \xi_2, \dots) \in X_\beta$,

$$s^*_{\xi_1\xi_2\cdots\xi_n} s_{\xi_1\xi_2\cdots\xi_n} = \chi_{[0,\beta_n]}, \qquad n \in \mathbb{N}. \tag{4.3}$$

PROOF. The equality (4.3) holds for $n = 1$. Suppose that the equality (4.3) holds for $n = k$. It then follows that

$$s^*_{\xi_1\xi_2\cdots\xi_{k+1}} s_{\xi_1\xi_2\cdots\xi_{k+1}} = s^*_{\xi_{k+1}}\chi_{[0,\beta_k]} s_{\xi_{k+1}} = \chi_{g^{-1}_{\xi_{k+1}}([0,\beta_k])}.$$

Since

$$g^{-1}_{\xi_{k+1}}([0,\ \beta_k]) = \left\{x \in [0,1] \,\middle|\, \frac{1}{\beta}x + \frac{\xi_{k+1}}{\beta} \le \beta^k - \xi_1\beta^{k-1} - \cdots - \xi_k\right\} = [0,\ \beta_{k+1}],$$

the equality (4.3) holds for $n = k + 1$. □

LEMMA 4.6. *For* $n \in \mathbb{N}$ *and* $j = 0, 1, \dots, N-1$,

$$s^*_{\xi_1\xi_2\cdots\xi_n j} s_{\xi_1\xi_2\cdots\xi_n j} = \begin{cases} 0 & \text{for } j > \xi_{n+1}, \\ \chi_{[0,\beta_{n+1}]} & \text{for } j = \xi_{n+1}, \\ 1 & \text{for } j < \xi_{n+1}. \end{cases}$$

PROOF. We have

$$s^*_{\xi_1\xi_2\cdots\xi_n j} s_{\xi_1\xi_2\cdots\xi_n j} = s^*_{\xi_j}\chi_{[0,\beta_n]} s_{\xi_j} = \chi_{g_j^{-1}([0,\beta_n])}$$

and

$$g_j^{-1}([0,\ \beta_n]) = \left\{x \in [0,1] \,\middle|\, \frac{1}{\beta}x + \frac{j}{\beta} \le \beta_n\right\} = [0,\ \beta\beta_n - j].$$

Since $\beta\beta_n - j = \beta_{n+1} + \xi_{n+1} - j$,

$$s^*_{\xi_1\xi_2\cdots\xi_n j} s_{\xi_1\xi_2\cdots\xi_n j} = \chi_{[0,\beta_{n+1}+\xi_{n+1}-j]}.$$

If $\xi_{n+1} = j$, the equality $s^*_{\xi_1\xi_2\cdots\xi_n j} s_{\xi_1\xi_2\cdots\xi_n j} = \chi_{[0,\beta_{n+1}]}$ holds. If $\xi_{n+1} < j$, we have $\xi_{n+1} - j \le -1$ and hence $\beta_{n+1} + \xi_{n+1} - j \le 0$, so that $[0,\ \beta_{n+1} + \xi_{n+1} - j] = \{0\}$ or $\emptyset$, which shows that $s^*_{\xi_1\xi_2\cdots\xi_n j} s_{\xi_1\xi_2\cdots\xi_n j} = 0$. If $\xi_{n+1} > j$, we have $\xi_{n+1} - j \ge 1$ and hence $\beta_{n+1} + \xi_{n+1} - j \ge 1$, so that $[0,\ \beta_{n+1} + \xi_{n+1} - j] = [0,1]$, which shows that $s^*_{\xi_1\xi_2\cdots\xi_n j} s_{\xi_1\xi_2\cdots\xi_n j} = \chi_{[0,1]} = 1$. □

Therefore, we have the following theorem.

THEOREM 4.7. *The correspondence* $S_j \longrightarrow s_j$ *for* $j = 0, 1, \dots, N-1$ *gives rise to an isomorphism from* $\mathcal{O}_\beta$ *to the* C^**-algebra* $C^*(s_0, s_1, \dots, s_{N-1})$ *on* $L^2([0,1])$ *generated by the partial isometries* $s_0, s_1, \dots, s_{N-1}$.

PROOF. Let us denote by $\mathcal{A}_{[0,1],l}$ the C^*-algebra on $L^2([0,1])$ generated by the projections $s^*_\mu s_\mu, \mu \in B_l(X_\beta)$, and $\mathcal{A}_{[0,1],\beta}$ the C^*-algebra generated by $\bigcup_{l\in\mathbb{N}} \mathcal{A}_{[0,1],l}$. By the previous lemma and [13, Corollary 3.2], the C^*-algebra $\mathcal{A}_{[0,1],l}$ is isomorphic to the C^*-subalgebra $\mathcal{A}_l$ of $\mathcal{O}_\beta$, so that $\mathcal{A}_{[0,1],\beta}$ is isomorphic to $\mathcal{A}_\beta$ through the correspondence $S^*_\mu S_\mu \longleftrightarrow s^*_\mu s_\mu$ for $\mu \in B_*(X_\beta)$. The isomorphism from $\mathcal{A}_\beta$ to $\mathcal{A}_{[0,1],\beta}$ is denoted by π. Put $\rho_j(x) = S^*_j x S_j$ for $x \in \mathcal{A}_\beta$, $j = 0, 1, \dots, N-1$. Then the relations

$$\pi(\rho_j(x)) = s^*_j \pi(x) s_j, \qquad x \in \mathcal{A}_\beta,\ j = 0, 1, \dots, N-1 \tag{4.4}$$

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hold by the previous lemma. Since the C^*-algebra $\mathcal{O}_\beta$ has the universal property subject to the relation (4.4) (see [15]), there exists a surjective $*$-homomorphism $\tilde{\pi}$ from $\mathcal{O}_\beta$ to $C^*(s_0, s_1, \ldots, s_{N-1})$ such that $\tilde{\pi}(S_j) = s_j$, $j = 0, 1, \ldots, N-1$ and $\tilde{\pi}(x) = \pi(x)$, $x \in \mathcal{A}_\beta$. As the C^*-algebra $\mathcal{O}_\beta$ is simple, the $*$-homomorphism $\tilde{\pi}$ is actually an isomorphism. □

In what follows, we may identify the C^*-algebra $\mathcal{O}_\beta$ with the C^*-algebra $C^*(s_0, s_1, \ldots, s_{N-1})$ through the identification of the generating partial isometries S_j and s_j, $j = 0, 1, \ldots, N-1$.

5. PL functions for SFT β-shifts

In this section, we will realize the group Γ_β for an SFT β-shift as PL functions on $[0, 1)$. For a word $\mu = (\mu_1, \ldots, \mu_n) \in B_n(X_\beta)$, denote by $U_\mu \subset X_\beta$ the cylinder set

$$U_\mu = \{(x_i)_{i\in\mathbb{N}} \in X_\beta \mid x_1 = \mu_1, \ldots, x_n = \mu_n\}.$$

We put

$$\Gamma^+(\mu) = \{(x_i)_{i\in\mathbb{N}} \in X_\beta \mid (\mu_1, \ldots, \mu_n, x_1, x_2, \ldots) \in X_\beta\}$$

for the set of followers of μ. Recall that φ stands for the unique KMS state for the gauge action on the C^*-algebra $\mathcal{O}_\beta$. We note that the value $\varphi(a_{\mu_1\cdots\mu_n})$ is computed inductively in the following way. For $n = 1$,

$$\varphi(a_{\mu_1}) = \begin{cases} 1 & \text{if } \mu_1 < \xi_1, \\ \beta - \xi_1 & \text{if } \mu_1 = \xi_1, \\ 0 & \text{if } \mu_1 > \xi_1. \end{cases}$$

Suppose that the value $\varphi(a_{\mu_1\cdots\mu_k})$ is computed for all $\mu = (\mu_1, \ldots, \mu_k) \in B_k(X_\beta)$ with $k < n$. If $(\mu_1, \ldots, \mu_n)$ is the maximal element $(\xi_1, \ldots, \xi_n)$ in $B_n(X_\beta)$, then

$$\varphi(a_{\mu_1\cdots\mu_n}) = \beta^n - \xi_1\beta^{n-1} - \cdots - \xi_{n-1}\beta - \xi_n. \tag{5.1}$$

If $(\mu_1, \ldots, \mu_n) \neq (\xi_1, \ldots, \xi_n)$, then there exists $k \leq n$ such that $\mu_k < \xi_k$. If $k = n$, then $\varphi(a_{\mu_1\cdots\mu_n}) = 1$. If $k < n$, we see that $a_{\mu_1\cdots\mu_k} = 1$, so that

$$a_{\mu_1\cdots\mu_n} = S^*_{\mu_n} \cdots S^*_{\mu_{k+1}} S_{\mu_{k+1}} \cdots S_{\mu_n} = a_{\mu_{k+1}\cdots\mu_n}.$$

Hence,

$$\varphi(a_{\mu_1\cdots\mu_n}) = \varphi(a_{\mu_{k+1}\cdots\mu_n}).$$

Since $|(\mu_{k+1}, \ldots, \mu_n)| < n$, the value $\varphi(a_{\mu_{k+1}\cdots\mu_n})$ is computed. Therefore, the value $\varphi(a_{\mu_1\cdots\mu_n})$ is computed for all $(\mu_1, \ldots, \mu_n) \in B_n(X_\beta)$. The following lemma is clear from Lemma 4.5 and (5.1).

LEMMA 5.1. *Assume that the generating partial isometries $S_0, S_1, \ldots, S_{N-1}$ are represented on $L^2([0, 1])$. For a word $\mu \in B_*(X_\beta)$, the projection $S^*_\mu S_\mu$ is identified with the characteristic function $\chi_{[0,\varphi(a_\mu))}$ of the interval* $[0, \varphi(a_\mu))$.

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Recall that for a word $\nu = (\nu_1, \ldots, \nu_n) \in B_n(X_\beta)$, the notation

$$l(\nu) = \frac{\nu_1}{\beta} + \frac{\nu_2}{\beta^2} + \cdots + \frac{\nu_n}{\beta^n}, \quad r(\nu) = \frac{\tilde{\nu}_1}{\beta} + \frac{\tilde{\nu}_2}{\beta^2} + \cdots + \frac{\tilde{\nu}_n}{\beta^n}$$

is introduced in Section 4, where $\tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_n)$ is the smallest word in $B_n(X_\beta)$ satisfying $\nu \prec \tilde{\nu}$. If ν is the maximum word in $B_n(X_\beta)$, we set $r(\nu) = 1$. The following two lemmas are crucial.

Lemma 5.2. *For $\mu, \nu \in B_*(X_\beta)$, we have $\Gamma^+(\mu) = \Gamma^+(\nu)$ if and only if*

$$\frac{r(\mu) - l(\mu)}{r(\nu) - l(\nu)} = \beta^{|\nu|-|\mu|}.$$

Proof. We note that $\Gamma^+(\mu) = \Gamma^+(\nu)$ if and only if $S_\mu^* S_\mu = S_\nu^* S_\nu$. By the above lemma, we have $\Gamma^+(\mu) = \Gamma^+(\nu)$ if and only if $\varphi(a_\mu) = \varphi(a_\nu)$. Since $\varphi(S_\mu^* S_\mu) = \beta^{|\mu|}\varphi(S_\mu S_\mu^*)$ and $\varphi(S_\mu S_\mu^*) = r(\mu) - l(\mu)$, we have $\varphi(a_\mu) = \varphi(a_\nu)$ if and only if $\beta^{|\mu|}(r(\mu) - l(\mu)) = \beta^{|\nu|}(r(\nu) - l(\nu))$. □

We note that the above lemma holds for any real number $\beta > 1$ even if (X_β, σ) is not a shift of finite type.

Lemma 5.3. *For $\tau \in \Gamma_\beta$, there exists $u_\tau \in N(\mathcal{D}_\beta, O_\beta)$ such that there exist $\mu(i), \nu(i) \in B_*(X_\beta), i = 1, 2, \ldots, m$ satisfying*

(1) $u_\tau = \sum_{i=1}^m S_{\mu(i)} S_{\nu(i)}^*$ *such that*

  (a) $S_{\nu(i)}^* S_{\nu(i)} = S_{\mu(i)}^* S_{\mu(i)},\ i = 1, 2, \ldots, m,$

  (b) $\sum_{i=1}^m S_{\nu(i)} S_{\nu(i)}^* = \sum_{i=1}^m S_{\mu(i)} S_{\mu(i)}^* = 1.$

(2) $f \circ \tau^{-1} = u_\tau f u_\tau^*$ *for* $f \in \mathcal{D}_\beta$.

Proof. Since (X_β, σ) is SFT, there exist continuous functions $k, l : X_\beta \longrightarrow \mathbb{Z}_+$ for $\tau \in \Gamma_\beta$ such that $\sigma^{l(x)}(\tau(x)) = \sigma^{k(x)}(x), x \in X_\beta$. Hence, there exists a family of cylinder sets $U_{\nu(1)}, \ldots, U_{\nu(m)}, U_{\mu(1)}, \ldots, U_{\mu(m)}$ such that

$$\Gamma^+(\nu(i)) = \Gamma^+(\mu(i)), \quad i = 1, \ldots, m,$$

$$X_\beta = \bigsqcup_{i=1}^m U_{\nu(i)} = \bigsqcup_{i=1}^m U_{\mu(i)}$$

and

$$\tau(x_1, x_2, \ldots) = (\mu(i)_1, \ldots, \mu(i)_{l_i}, x_{k_i+1}, x_{k_i+2}, \ldots) \text{ for } (x_n)_{n\in\mathbb{N}} \in U_{\nu(i)},$$

where $l_i = |\mu(i)|, k_i = |\nu(i)|$ and $\mu(i) = (\mu(i)_1, \ldots, \mu(i)_{l_i})$. Hence,

$$\sum_{i=1}^m S_{\nu(i)} S_{\nu(i)}^* = \sum_{i=1}^m S_{\mu(i)} S_{\mu(i)}^* = 1, \quad S_{\nu(i)}^* S_{\nu(i)} = S_{\mu(i)}^* S_{\mu(i)}, \quad i = 1, 2, \ldots, m.$$

By putting $u_\tau = \sum_{i=1}^m S_{\mu(i)} S_{\nu(i)}^*$, we see that u_τ belongs to $N(\mathcal{D}_\beta, O_\beta)$ and satisfies $\chi_{U_\eta} \circ \tau^{-1} = u_\tau S_\eta S_\eta^* u_\tau^*$ for all $\eta \in B_*(X_\beta)$, so that $f \circ \tau^{-1} = u_\tau f u_\tau^*$ for $f \in \mathcal{D}_\beta$. □

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Following Nekrashevych [25], we will introduce a notation of tables in order to represent elements of Γ_β.

DEFINITION 5.4. A *β-adic table* for an SFT β-shift is a matrix

$$\begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$$

for $\nu(i), \mu(i) \in B_*(X_\beta), i = 1, 2, \ldots, m$ such that

(a) $\Gamma^+(\nu(i)) = \Gamma^+(\mu(i)), i = 1, 2, \ldots, m,$

(b) $X_\beta = \sqcup_{i=1}^m U_{\nu(i)} = \sqcup_{i=1}^m U_{\mu(i)}$ are disjoint unions.

We may assume that $\nu(1) \prec \nu(2) \prec \cdots \prec \nu(m)$. Since the above two conditions (a), (b) are equivalent to the conditions (a), (b) in Lemma 5.3(1), respectively, we have the following lemma.

LEMMA 5.5. *For an element $\tau \in \Gamma_\beta$ with its unitary $u_\tau = \sum_{i=1}^m S_{\mu(i)} S_{\nu(i)}^* \in N(\mathcal{D}_\beta, \mathcal{O}_\beta)$ as in Lemma 5.3, the matrix*

$$T_\tau = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$$

is a β-adic table for an SFT β-shift.

DEFINITION 5.6.

(i) An interval $[x_1, x_2]$ in $[0, 1]$ is said to be a *β-adic interval* for the word $\nu \in B_*(X_\beta)$ if $x_1 = l(\nu)$ and $x_2 = r(\nu)$.

(ii) A rectangle $I \times J$ in $[0, 1] \times [0, 1]$ is said to be a *β-adic rectangle* if both I, J are β-adic intervals for words $\nu \in B_n(X_\beta), \mu \in B_m(X_\beta)$ such that $I = [l(\nu), r(\nu))$, $J = [l(\mu), r(\mu))$ and

$$\frac{r(\mu) - l(\mu)}{r(\nu) - l(\nu)} = \beta^{n-m}.$$

(iii) For two partitions $0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1$ and $0 = y_0 < y_1 < \cdots < y_{m-1} < y_m = 1$ of $[0, 1]$, put $I_p = [x_{p-1}, x_p), J_p = [y_{p-1}, y_p), p = 1, 2, \ldots, m$. The partition $I_p \times J_q, p, q = 1, 2, \ldots, m$ of $[0, 1) \times [0, 1)$ is said to be a *β-adic pattern of rectangles* for an SFT β-shift if there exists a permutation σ on $\{1, 2, \ldots, m\}$ such that the rectangles $I_p \times J_{\sigma(p)}$ are β-adic rectangles for all $p = 1, 2, \ldots, m$.

For a β-adic pattern of rectangles above, the slopes of diagonals $s_p = (y_{\sigma(p)} - y_{\sigma(p)-1})/(x_p - x_{p-1}), p = 1, 2, \ldots, m$ are said to be rectangle slopes. We then have the following lemma.

LEMMA 5.7. *For a β-adic table*

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$$

there exists a β-adic pattern of rectangles whose rectangle slopes are

$$\beta^{|\nu(1)|-|\mu(1)|}, \beta^{|\nu(2)|-|\mu(2)|}, \ldots, \beta^{|\nu(m)|-|\mu(m)|}.$$

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Proof. We are assuming the ordering such as $\nu(1) \prec \cdots \prec \nu(m)$. Since $X_\beta = \sqcup_{j=1}^m U_{\mu(j)}$ is a disjoint union, there exists a permutation σ_0 on $\{1, 2, \ldots, m\}$ such that $\mu(\sigma_0(1)) \prec \mu(\sigma_0(2)) \prec \cdots \prec \mu(\sigma_0(m))$. Put

$$x_i = l(\nu(i+1)), \quad y_i = l(\mu(\sigma_0(i+1))), \quad i = 0, 1, \ldots, m-1$$

and

$$I_p = [x_{p-1}, x_p), \quad J_p = [y_{p-1}, y_p), \quad p = 1, 2, \ldots, m.$$

Define the permutation $\sigma := \sigma_0^{-1}$ on $\{1, 2, \ldots, m\}$. We note that $r(\nu(i)) = l(\nu(i+1))$, $r(\mu(\sigma_0(i))) = l(\mu(\sigma_0(i+1)))$ for $i = 1, 2, \ldots, m-1$. Then the rectangles $I_p \times J_{\sigma(p)}$, $p = 1, 2, \ldots, m$ are β-adic rectangles such that

$$\frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}} = \frac{r(\mu(p)) - l(\mu(p))}{r(\nu(p)) - l(\nu(p))}.$$

Since $r(\zeta) - l(\zeta) = \varphi(S_\zeta S_\zeta^*) = \frac{1}{\beta^{|\zeta|}}\varphi(S_\zeta^* S_\zeta)$ for $\zeta \in B_*(X_\beta)$,

$$r(\nu(p)) - l(\nu(p)) = \frac{1}{\beta^{|\nu(p)|}}\varphi(S_{\nu(p)}^* S_{\nu(p)}),$$

$$r(\mu(p)) - l(\mu(p)) = \frac{1}{\beta^{|\mu(p)|}}\varphi(S_{\mu(p)}^* S_{\mu(p)}).$$

As the condition $\Gamma^+(\nu(p)) = \Gamma^+(\mu(p))$ implies $S_{\nu(p)}^* S_{\nu(p)} = S_{\mu(p)}^* S_{\mu(p)}$,

$$\frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}} = \beta^{|\nu(p)|-|\mu(p)|}, \quad p = 1, 2, \ldots, m. \qquad \square$$

We define a β-adic version of PL functions on $[0, 1)$ in the following way.

Definition 5.8. A PL function f on $[0, 1)$ is called a *β-adic PL function* for an SFT β-shift if f is a right-continuous bijection on $[0, 1)$ such that there exists a β-adic pattern of rectangles $I_p \times J_p$, $p = 1, 2, \ldots, m$, where $I_p = [x_{p-1}, x_p)$, $J_p = [y_{p-1}, y_p)$, $p = 1, 2, \ldots, m$, with a permutation σ on $\{1, 2, \ldots, m\}$ such that

$$f(x_{p-1}) = y_{\sigma(p)-1}, \quad f_-(x_p) = y_{\sigma(p-1)+1}, \quad p = 1, 2, \ldots, m,$$

where $f_-(x_p) = \lim_{h\to 0+} f(x_p - h)$ and f is linear on $[x_{p-1}, x_p)$ with slope $(y_{\sigma(p)} - y_{\sigma(p)-1})/(x_p - x_{p-1})$ for $p = 1, 2, \ldots, m$.

The following proposition is immediate from the definition of β-adic PL functions.

Proposition 5.9. *A β-adic PL function for an SFT β-shift naturally gives rise to a β-adic pattern of rectangles for an SFT β-shift.*

We may directly construct a β-adic PL function f_T from a β-adic table $T = \left[\begin{smallmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{smallmatrix}\right]$ as follows. Put $x_i = l(\nu(i+1)), \hat{y}_i = l(\mu(i+1)), i = 0, 1, \ldots, m-1$. Define f_T by $f_T(x_i) = \hat{y}_i, i = 0, 1, \ldots, m-1$ and f_T is linear on $[x_{i-1}, x_i), i = 1, 2, \ldots, m$ with slope $(\hat{y}_i - \hat{y}_{i-1})/(x_i - x_{i-1}) = (r(\mu(i)) - l(\mu(i)))/(r(\nu(i)) - l(\nu(i))) = \beta^{|\nu(i)|-|\mu(i)|}$. Hence, the function f_T yields a β-adic PL function.

It is straightforward to see that the composition of two β-adic PL functions is also a β-adic PL function. Hence, the set of β-adic PL functions forms a group under compositions. We reach the following theorem.

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THEOREM 5.10. *The topological full group Γ_β for an SFT β-shift (X_β, σ) is realized as the group of all β-adic PL functions for an SFT β-shift.*

6. PL functions for sofic β-shifts

In this section, we will represent the topological full group Γ_β for sofic β-shifts as PL functions on $[0, 1)$. Throughout this section, we assume that (X_β, σ) is sofic. By Lemma 2.7, the algebra $\mathcal{A}_\beta$ is finite dimensional. We set $K_\beta = \dim\mathcal{A}_\beta$. Let $E_1, \dots, E_{K_\beta}$ be the minimal projections of $\mathcal{A}_\beta$ so that $\sum_{i=1}^{K_\beta} E_i = 1$. Then any minimal projection E_i is of the form $E_i = a_{\xi_1\cdots\xi_{p_i}} - a_{\xi_1\cdots\xi_{q_i}}$ for some $p_i, q_i \in \mathbb{Z}_+$. We order $E_1, \dots, E_{K_\beta}$ following the order $\varphi(a_{\xi_1\cdots\xi_{p_1}}) < \cdots < \varphi(a_{\xi_1\cdots\xi_{p_{K_\beta}}})$ in $\mathbb{R}$, where φ is the unique KMS state on $\mathcal{O}_\beta$ for the gauge action. Recall that $\hat{\rho}_t \in \mathrm{Aut}(\mathcal{O}_\beta), t \in \mathbb{R}/\mathbb{Z}$ denotes the gauge action on $\mathcal{O}_\beta$ and $N(\mathcal{D}_\beta, \mathcal{O}_\beta)$ denotes the normalizer group of $\mathcal{D}_\beta \subset \mathcal{O}_\beta$. Fix $u \in N(\mathcal{D}_\beta, \mathcal{O}_\beta)$ for a while. For $m \in \mathbb{Z}$ and $\mu \in B_n(X_\beta), n \in \mathbb{N}$, put

$$u_m = \int_{\mathbb{T}} \hat{\rho}_t(u) e^{-2\pi\sqrt{-1}mt}\, dt \quad \text{and} \quad u_\mu = S_\mu^* u_n, \quad u_{-\mu} = u_{-n} S_\mu.$$

It is straightforward to see the following lemma.

LEMMA 6.1. *The operators $u_\mu, u_{-\mu}$ for $\mu \in B_n(X_\beta)$ and u_0 are partial isometries in $\mathcal{F}_\beta$ such that u is decomposed as the following finite sum:*

$$u = \sum_{n \text{ finite}} \sum_{\mu\in B_n(X_\beta)} S_\mu v_\mu + u_0 + \sum_{n \text{ finite}} \sum_{\mu\in B_n(X_\beta)} u_{-\mu} S_\mu^*$$

such that $u_\mu \mathcal{D}_\beta u_\mu^*, u_\mu^* \mathcal{D}_\beta u_\mu, u_{-\mu}\mathcal{D}_\beta u_{-\mu}^*$ and $u_{-\mu}^*\mathcal{D}_\beta u_{-\mu}$ are contained in $\mathcal{D}_\beta$.

Define a subalgebra $\mathcal{F}_\beta^k$ of $\mathcal{F}_\beta$ for $k \in \mathbb{Z}_+$ by

$$\mathcal{F}_\beta^k = C^*(S_\xi E_i S_\eta^* \mid \xi, \eta \in B_k(X_\beta), i = 1, 2, \dots, K_\beta).$$

We set

$$\mathrm{supp}_+(u) = \{\mu \in B_*(X_\beta) \mid u_\mu \neq 0\}, \quad \mathrm{supp}_-(u) = \{\mu \in B_*(X_\beta) \mid u_{-\mu} \neq 0\}.$$

Both of them are finite sets. For $\mu \in \mathrm{supp}_+(u)$, there exists $k_+(\mu) \in \mathbb{Z}_+$ such that $u_\mu \in \mathcal{F}_\beta^{k_+(\mu)}$. For $\mu \in \mathrm{supp}_-(u)$, there exists $k_-(\mu) \in \mathbb{Z}_+$ such that $u_{-\mu} \in \mathcal{F}_\beta^{k_-(\mu)}$. There exists $k_0 \in \mathbb{Z}_+$ such that $u_0 \in \mathcal{F}_\beta^{k_0}$. We then have the following lemma.

LEMMA 6.2. *Keep the above notation.*

(i) *For $\mu \in \mathrm{supp}_+(u)$ and $\eta \in B_{k_+(\mu)}(X_\beta), i = 1, 2, \dots, K_\beta$ such that $u_\mu^* u_\mu \geq S_\eta E_i S_\eta^* \neq 0$, there uniquely exists $\xi \in B_{k_+(\mu)}(X_\beta)$ such that $u_\mu u_\mu^* \geq S_\xi E_i S_\xi^* \neq 0$ and*

$$Ad(u_\mu)(S_\eta E_i S_\eta^*) = S_\xi E_i S_\xi^*.$$

(ii) *For $\mu \in \mathrm{supp}\_(u)$ and $\eta \in B_{k_-(\mu)}(X_\beta), i = 1, 2, \ldots, K_\beta$ such that $u^*_{-\mu}u_{-\mu} \geq S_\eta E_i S^*_\eta \neq 0$, there uniquely exists $\xi \in B_{k_-(\mu)}(X_\beta)$ such that $u_{-\mu}u^*_{-\mu} \geq S_\xi E_i S^*_\xi \neq 0$ and*

$$Ad(u_{-\mu})(S_\eta E_i S^*_\eta) = S_\xi E_i S^*_\xi.$$

(iii) *For $\eta \in B_{k_0}(X_\beta), i = 1, 2, \ldots, K_\beta$ such that $u^*_0 u_0 \geq S_\eta E_i S^*_\eta \neq 0$, there uniquely exists $\xi \in B_{k_0}(X_\beta)$ such that $u_0 u^*_0 \geq S_\xi E_i S^*_\xi \neq 0$ and*

$$Ad(u_0)(S_\eta E_i S^*_\eta) = S_\xi E_i S^*_\xi.$$

Proof. (i) As $u_\mu \in \mathcal{F}^{k_+(\mu)}_\beta$, it is written $u_\mu = \sum_{\xi,\eta' \in B_{k_+(\mu)}(X_\beta)} S_\xi a_{\xi,\eta'} S^*_{\eta'}$ for some $a_{\xi,\eta'} \in \mathcal{A}_\beta$. Suppose that $u^*_\mu u_\mu \geq S_\eta E_i S^*_\eta \neq 0$. Hence, $S^*_\eta S_\eta \geq E_i$. It then follows that

$$\begin{aligned} Ad(u_\mu)(S_\eta E_i S^*_\eta) &= u_\mu S_\eta E_i S^*_\eta u^*_\mu \\ &= \sum_{\xi,\xi' \in B_{k_+(\mu)}(X_\beta)} S_\xi a_{\xi,\eta} S^*_\eta S_\eta E_i S^*_\eta S_\eta a^*_{\xi',\eta} S^*_{\xi'} \\ &= \sum_{\xi,\xi' \in B_{k_+(\mu)}(X_\beta)} S_\xi a_{\xi,\eta} E_i a^*_{\xi',\eta} S^*_{\xi'}. \end{aligned}$$

Since $Ad(u_\mu)(S_\eta E_i S^*_\eta)$ belongs to $\mathcal{D}_\beta$, we have, for $\xi \neq \xi'$,

$$0 = S_\xi S^*_\xi Ad(u_\mu)(S_\eta E_i S^*_\eta) S_{\xi'} S^*_{\xi'} = S_\xi a_{\xi,\eta} E_i a^*_{\xi',\eta} S^*_{\xi'},$$

so that

$$Ad(u_\mu)(S_\eta E_i S^*_\eta) = \sum_{\xi \in B_{k_+(\mu)}(X_\beta)} S_\xi a_{\xi,\eta} E_i a^*_{\xi,\eta} S^*_\xi.$$

Since $u_\mu u^*_\mu = \sum_{\xi,\zeta \in B_{k_+(\mu)}(X_\beta)} S_\xi a_{\xi,\zeta} a^*_{\xi,\zeta} S^*_\xi$ is a projection, the operators $a_{\xi,\eta} a^*_{\xi,\eta}$ are projections in $\mathcal{A}_\beta$ for all $\xi \in B_{k_+(\mu)}(X_\beta)$. As $S^*_\xi S_\xi a_{\xi,\eta} E_i a^*_{\xi',\eta} S^*_{\xi'} S_{\xi'} = a_{\xi,\eta} E_i a^*_{\xi',\eta}$, we have $a_{\xi,\eta} a^*_{\xi,\eta} \cdot a_{\xi',\eta} a^*_{\xi',\eta} = 0$ for $\xi \neq \xi'$, so that there uniquely exists $\xi \in B_{k_+(\mu)}(X_\beta)$ such that $a_{\xi,\eta} a^*_{\xi,\eta} E_i = E_i$ for the word η and i. By the identity $a_{\xi,\eta} E_i a^*_{\xi,\eta} = a_{\xi,\eta} a^*_{\xi,\eta} E_i$,

$$Ad(u_\mu)(S_\eta E_i S^*_\eta) = S_\xi E_i S^*_\xi.$$

(ii) and (iii) are similar to (i). □

Proposition 6.3. *For a unitary $u \in N(\mathcal{D}_\beta, \mathcal{O}_\beta)$, there exists a finite family of partial isometries $u_\mu, u_0, u_{-\mu}$ in $\mathcal{F}_\beta$ such that u is decomposed in the following way:*

$$u = \sum_{n finite} \sum_{\mu \in B_n(X_\beta)} S_\mu u_\mu + u_0 + \sum_{n finite} \sum_{\mu \in B_n(X_\beta)} u_{-\mu} S^*_\mu$$

such that

(1) *For any $\eta \in B_{k_+(\mu)}(X_\beta)$ with $S_\eta E_i S^*_\eta \leq u^*_\mu u_\mu$, the equality*

$$Ad(S_\mu u_\mu)(S_\eta E_i S^*_\eta) = S_\mu S_\xi E_i S^*_\xi S^*_\mu$$

holds for some $\xi \in B_{k_+(\mu)}(X_\beta)$.

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(2) *For any* $\eta \in B_{k_0}(X_\beta)$ *with* $S_\eta E_i S_\eta^* \le u_0^* u_0$, *the equality*

$$Ad(u_0)(S_\eta E_i S_\eta^*) = S_\xi E_i S_\xi^*$$

holds for some $\xi \in B_{k_0}(X_\beta)$.

(3) *For any* $\eta \in B_{k_-(\mu)}(X_\beta)$ *with* $S_\eta E_i S_\eta^* \le u_{-\mu}^* u_{-\mu}$, *the equality*

$$Ad(u_{-\mu} S_\mu^*)(S_\mu S_\eta E_i S_\eta^* S_\mu^*) = S_\xi E_i S_\xi^*$$

holds for some $\xi \in B_{k_-(\mu)}(X_\beta)$.

Therefore, we have the following lemma.

Lemma 6.4. *For* $\tau \in \Gamma_\beta$, *there exists* $u_\tau \in N(\mathcal{D}_\beta, \mathcal{O}_\beta)$ *such that there exists a family* $S_{\nu(j)} E_{i_j} S_{\nu(j)}^*, S_{\mu(j)} E_{i_j} S_{\mu(j)}^*$, $j = 1, 2, \ldots, m$ *of projections satisfying*

(1) $u_\tau = \sum_{j=1}^m S_{\mu(j)} E_{i_j} S_{\nu(j)}^*$ *such that*

  (a) $S_{\nu(j)}^* S_{\nu(j)}, S_{\mu(j)}^* S_{\mu(j)} \ge E_{i_j}, \quad j = 1, 2, \ldots, m,$

  (b) $\sum_{j=1}^m S_{\nu(j)} E_{i_j} S_{\nu(j)}^* = \sum_{j=1}^m S_{\mu(j)} E_{i_j} S_{\mu(j)}^* = 1.$

(2) $f \circ \tau^{-1} = u_\tau f u_\tau^*$ *for* $f \in \mathcal{D}_\beta$.

For $i = 1, 2, \ldots, K_\beta$, put

$$\Gamma_n^-(i) = \{\mu \in B_n(X_\beta) \mid S_\mu^* S_\mu \ge E_i\}, \quad \Gamma_*^-(i) = \bigcup_{n=0}^\infty \Gamma_n^-(i).$$

For $\nu = (\nu_1, \ldots, \nu_n) \in \Gamma_n^-(i)$ and $i = 1, 2, \ldots, K_\beta$, define the projection in $\mathcal{D}_\beta$ by

$$\nu_{[i]} := S_\nu E_i S_\nu^*$$

and define

$$\begin{aligned} r(\nu_{[i]}) &= l(\nu) + \frac{1}{\beta^n}\varphi(a_{\xi_1\cdots\xi_{p_i}}) \\ &= \frac{\nu_1}{\beta} + \frac{\nu_2}{\beta^2} + \cdots + \frac{\nu_n}{\beta^n} + \frac{\xi_{p_i+1}}{\beta^{n+1}} + \frac{\xi_{p_i+2}}{\beta^{n+2}} + \ldots, \\ l(\nu_{[i]}) &= l(\nu) + \frac{1}{\beta^n}\varphi(a_{\xi_1\cdots\xi_{q_i}}) \\ &= \frac{\nu_1}{\beta} + \frac{\nu_2}{\beta^2} + \cdots + \frac{\nu_n}{\beta^n} + \frac{\xi_{q_i+1}}{\beta^{n+1}} + \frac{\xi_{q_i+2}}{\beta^{n+2}} + \cdots, \end{aligned}$$

where $E_i = a_{\xi_1\cdots\xi_{p_i}} - a_{\xi_1\cdots\xi_{q_i}}$. The following lemma is obvious.

Lemma 6.5. *Assume that the generating partial isometries* $S_0, S_1, \ldots, S_{N-1}$ *are represented on* $L^2([0,1])$. *For* $\nu \in \Gamma_n^-(i)$, *the projection* $S_\nu E_i S_\nu^*$ *is identified with the characteristic function* $\chi_{[l(\nu_{[i]}), r(\nu_{[i]}))}$ *of the interval* $[l(\nu_{[i]}), r(\nu_{[i]}))$.

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For $\nu \in \Gamma_*^-(i)$ and $\mu \in \Gamma_*^-(j)$ with $S_\nu E_i S_\nu^* \cdot S_\mu E_j S_\mu^* = 0$, define

$$\nu_{[i]} < \mu_{[j]} \text{ if } r(\nu_{[i]}) \le l(\mu_{[j]}).$$

Note that under the condition $S_\nu E_i S_\nu^* \cdot S_\mu E_j S_\mu^* = 0$, the intervals $[l(\nu_{[i]}), r(\nu_{[i]}))$ and $[l(\mu_{[j]}), r(\mu_{[j]}))$ are disjoint. Hence, the condition $\nu_{[i]} < \mu_{[j]}$ implies that the interval $[l(\nu_{[i]}), r(\nu_{[i]}))$ is located on the left-hand side of $[l(\mu_{[j]}), r(\mu_{[j]}))$.

Lemma 6.6. *Keep the above notation.*

(i) *For $\nu \in \Gamma_n^-(i)$ and $\mu \in \Gamma_k^-(j)$, we have $S_\nu E_i S_\nu^* \cdot S_\mu E_j S_\mu^* = 0$ if and only if $[l(\nu_{[i]}), r(\nu_{[i]})) \cap [l(\mu_{[j]}), r(\mu_{[j]})) = \emptyset$.*

(ii) *For $\nu(j) \in \Gamma_{n_j}^-(i_j)$, $j = 1, 2, \ldots, m$, we have $\sum_{j=1}^m S_{\nu(j)} E_{i_j} S_{\nu(j)}^* = 1$ if and only if $[0, 1) = \sqcup_{j=1}^m [l(\nu(j)_{[i_j]}), r(\nu(j)_{[i_j]}))$ is a disjoint union.*

(iii) *For $\nu(j) \in \Gamma_{n_j}^-(i_j)$, $j = 1, 2, \ldots, m$ such that $\sum_{j=1}^m S_{\nu(j)} E_{i_j} S_{\nu(j)}^* = 1$ and $\nu(1)_{[i_1]} < \nu(2)_{[i_2]} < \cdots < \nu(m)_{[i_m]}$,*

$$r(\nu(j)_{[i_j]}) = l(\nu(j+1)_{[i_{j+1}]}), \quad j = 1, 2, \ldots, m.$$

Definition 6.7. A *β-adic table* for a sofic β-shift is a matrix

$$T = \begin{bmatrix} \mu(1)_{[i_1]} & \mu(2)_{[i_2]} & \cdots & \mu(m)_{[i_m]} \\ \nu(1)_{[i_1]} & \nu(2)_{[i_2]} & \cdots & \nu(m)_{[i_m]} \end{bmatrix}$$

such that

(a) $\nu(j) \in \Gamma_*^-(i_j), \quad \mu(j) \in \Gamma_*^-(i_j) \quad$ for $j = 1, 2, \ldots, m$.

(b) $\sqcup_{j=1}^m [l(\nu(j)_{[i_j]}), r(\nu(j)_{[i_j]})) = \bigcup_{j=1}^m [l(\mu(j)_{[i_j]}), r(\mu(j)_{[i_j]})) = [0, 1)$.

We may assume that

$$\nu(1)_{[i_1]} < \nu(2)_{[i_2]} < \cdots < \nu(m)_{[i_m]}.$$

Therefore, we have the following lemma.

Lemma 6.8. *For an element $\tau \in \Gamma_\beta$ with its unitary $u_\tau = \sum_{j=1}^m S_{\mu(j)} E_{i_j} S_{\nu(j)}^* \in N(\mathcal{D}_\beta, \mathcal{O}_\beta)$ as in Lemma 6.4, the matrix*

$$T_\tau = \begin{bmatrix} \mu(1)_{[i_1]} & \mu(2)_{[i_2]} & \cdots & \mu(m)_{[i_m]} \\ \nu(1)_{[i_1]} & \nu(2)_{[i_2]} & \cdots & \nu(m)_{[i_m]} \end{bmatrix}$$

is a β-adic table for a sofic β-shift.

Definition 6.9. (i) An interval $[x_1, x_2)$ in $[0, 1]$ is said to be a *β-adic interval* for the word $\nu_{[i]}$ if $x_1 = l(\nu_{[i]})$ and $x_2 = r(\nu_{[i]})$ for some $\nu \in B_*(X_\beta)$ and $i = 1, 2, \ldots, K_\beta$.

(ii) A rectangle $I \times J$ in $[0, 1] \times [0, 1]$ is said to be a *β-adic rectangle* if both I, J are β-adic intervals for words $\nu_{[i]}, \mu_{[i]}$ such that $I = [l(\nu_{[i]}), r(\nu_{[i]}))$ and $J = [l(\mu_{[i]}), r(\mu_{[i]}))$ and

$$\frac{r(\mu_{[i]}) - l(\mu_{[i]})}{r(\nu_{[i]}) - l(\nu_{[i]})} = \beta^{|\nu| - |\mu|}.$$

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(iii) For two partitions $0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1$ and $0 = y_0 < y_1 < \cdots < y_{m-1} < y_m = 1$ of $[0, 1]$, put $I_p = [x_{p-1}, x_p), J_p = [y_{p-1}, y_p), p = 1, 2, \ldots, m$. The partition $I_p \times J_q, p, q = 1, 2, \ldots, m$ of $[0, 1) \times [0, 1)$ is said to be a *β-adic pattern of rectangles* for a sofic β-shift if there exists a permutation σ on $\{1, 2, \ldots, m\}$ such that the rectangles $I_p \times J_{\sigma(p)}$ are β-adic rectangles for all $p = 1, 2, \ldots, m$.

For a β-adic pattern of rectangles above, the slopes of diagonals $s_p = (y_{\sigma(p)} - y_{\sigma(p)-1})/(x_p - x_{p-1}), p = 1, 2, \ldots, m$ are said to be rectangle slopes. Similarly to Lemma 5.7 for an SFT β-shift, we have the following lemma.

LEMMA 6.10. *For a β-adic table for a sofic β-shift*

$$T = \begin{bmatrix} \mu(1)_{[i_1]} & \mu(2)_{[i_2]} & \cdots & \mu(m)_{[i_m]} \\ \nu(1)_{[i_1]} & \nu(2)_{[i_2]} & \cdots & \nu(m)_{[i_m]} \end{bmatrix},$$

there exists a β-adic pattern of rectangles for a sofic β-shift whose rectangle slopes are

$$\beta^{|\nu(1)|-|\mu(1)|}, \beta^{|\nu(2)|-|\mu(2)|}, \ldots, \beta^{|\nu(m)|-|\mu(m)|}.$$

Similarly to the preceding section, we will define a β-adic version of PL functions on $[0, 1)$ for a sofic β-shift in the following way.

DEFINITION 6.11. A PL function f on $[0, 1)$ is called a *β-adic PL function* for a sofic β-shift if f is a right-continuous bijection on $[0, 1)$ such that there exists a β-adic pattern of rectangles $I_p \times J_p, p = 1, 2, \ldots, m$, where $I_p = [x_{p-1}, x_p), J_p = [y_{p-1}, y_p), p = 1, 2, \ldots, m$ with a permutation σ on $\{1, 2, \ldots, m\}$ such that

$$f(x_{p-1}) = y_{\sigma(p)-1}, \quad f_-(x_p) = y_{\sigma(p-1)+1}, \quad p = 1, 2, \ldots, m,$$

where $f_-(x_p) = \lim_{h\to 0+} f(x_p - h)$ and f is linear on $[x_{p-1}, x_p)$ with slope $(y_{\sigma(p)} - y_{\sigma(p)-1})/(x_p - x_{p-1})$ for $p = 1, 2, \ldots, m$.

Similarly to the preceding section, we have the following proposition.

PROPOSITION 6.12. *A β-adic PL function for a sofic β-shift naturally gives rise to a β-adic pattern of rectangles for a sofic β-shift.*

We may directly construct a β-adic PL function f_T for a sofic β-shift from a β-adic table for a sofic β-shift $T = \left[\begin{smallmatrix} \nu(1)_{[i_1]} & \nu(2)_{[i_2]} & \cdots & \nu(m)_{[i_m]} \\ \mu(1)_{[i_1]} & \mu(2)_{[i_2]} & \cdots & \mu(m)_{[i_m]} \end{smallmatrix}\right]$ as follows. Put $x_j = l(\nu(j+1)_{[i_j]}), \hat{y}_j = l(\mu(j+1)_{[i_j]}), j = 0, 1, \ldots, m-1$. Define f_T by $f_T(x_j) = \hat{y}_j$, $j = 0, 1, \ldots, m-1$ and f_T is linear on $[x_{j-1}, x_j), j = 1, 2, \ldots, m$ with slope $(r(\mu(j)) - l(\mu(j)))/(r(\nu(j)) - l(\nu(j))) = \beta^{|\nu(j)|-|\mu(j)|}$. The function f_T yields a β-adic PL function for a sofic β-shift.

It is straightforward to see that the composition of two β-adic PL functions for a sofic β-shift is also a β-adic PL function for a sofic β-shift. Hence, the set of β-adic PL functions for a sofic β-shift forms a group under compositions. We reach the following theorem.

THEOREM 6.13. *The topological full group Γ_β for a sofic β-shift (X_β, σ) is realized as the group of all β-adic PL functions for a sofic β-shift.*

7. Classification of the topological full groups Γ_β

In this section, we will classify the groups Γ_β for SFT β-shifts and sofic β-shifts. We will first classify Γ_β for SFT β-shifts.

1. SFT case:

PROPOSITION 7.1. *Suppose that the β-shift (X_β, σ) is a shift of finite type such that the β-expansion of* 1 *is* $1 = \eta_1/\beta + \eta_2/\beta^2 + \cdots + \eta_n/\beta^n$. *Set*

$$\begin{aligned}
T_i &= S_{i-1} \quad \textit{for } i = 1, \ldots, \eta_1, \\
T_{\eta_1+i} &= S_{\eta_1} S_{i-1} \quad \textit{for } i = 1, \ldots, \eta_2, \\
T_{\eta_1+\eta_2+i} &= S_{\eta_1} S_{\eta_2} S_{i-1} \quad \textit{for } i = 1, \ldots, \eta_3, \\
&\vdots \\
T_{\eta_1+\eta_2+\cdots+\eta_{n-1}+i} &= S_{\eta_1} S_{\eta_2} \cdots S_{\eta_{n-1}} S_{i-1} \quad \textit{for } i = 1, \ldots, \eta_n.
\end{aligned}$$

Define the C^*-subalgebras $\widehat{O}_\beta, \widehat{\mathcal{D}}_\beta$ of O_β by

$$\begin{aligned}
\widehat{O}_\beta &= C^*(T_i; i = 1, 2, \ldots, \eta_1 + \eta_2 + \cdots + \eta_n), \\
\widehat{\mathcal{D}}_\beta &= C^*(T_\mu T_\mu^*; \mu = (\mu_1, \ldots, \mu_m), \mu_i = 1, 2, \ldots, \eta_1 + \eta_2 + \cdots + \eta_n).
\end{aligned}$$

Then the C^*-algebras $\widehat{O}_\beta$ and $\widehat{\mathcal{D}}_\beta$ coincide with O_β and $\mathcal{D}_\beta$, respectively, and are isomorphic to the Cuntz algebra $O_{\eta_1+\eta_2+\cdots+\eta_n}$ and the canonical Cartan subalgebra $\mathcal{D}_{\eta_1+\eta_2+\cdots+\eta_n}$, respectively, that is,

$$\widehat{O}_\beta = O_\beta = O_{\eta_1+\eta_2+\cdots+\eta_n}, \quad \widehat{\mathcal{D}}_\beta = \mathcal{D}_\beta = \mathcal{D}_{\eta_1+\eta_2+\cdots+\eta_n}.$$

PROOF. It is direct to see that the operators $T_1, T_2, \ldots, T_{\eta_1+\eta_2+\cdots+\eta_n}$ are all isometries. Then

$$\begin{aligned}
\sum_{i=1}^{\eta_1} T_i T_i^* &= \sum_{j=0}^{\eta_1-1} S_j S_j^* = 1 - S_{\eta_1} S_{\eta_1}^*, \\
\sum_{i=\eta_1+1}^{\eta_1+\eta_2} T_i T_i^* &= \sum_{j=0}^{\eta_2-1} S_{\eta_1} S_j S_j^* S_{\eta_1}^* = S_{\eta_1}(1 - S_{\eta_2} S_{\eta_2}^*) S_{\eta_1}^*, \\
&\vdots \\
\sum_{i=\eta_1+\eta_2+\cdots+\eta_{n-1}+1}^{\eta_1+\eta_2+\cdots+\eta_n} T_i T_i^* &= \sum_{j=0}^{\eta_n-1} S_{\eta_1} S_{\eta_2} \cdots S_{\eta_{n-1}} S_j S_j^* S_{\eta_{n-1}}^* S_{\eta_1}^* \cdots S_{\eta_2}^* S_{\eta_1}^* \\
&= S_{\eta_1} S_{\eta_2} \cdots S_{\eta_{n-1}} (1 - S_{\eta_n} S_{\eta_n}^*) S_{\eta_{n-1}}^* \cdots S_{\eta_2}^* S_{\eta_1}^* \\
&= S_{\eta_1} S_{\eta_2} \cdots S_{\eta_{n-1}} S_{\eta_{n-1}}^* \cdots S_{\eta_2}^* S_{\eta_1}^*.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{\eta_1+\eta_2+\cdots+\eta_n} T_i T_i^* \\
&= \sum_{i=1}^{\eta_1} T_i T_i^* + \sum_{i=\eta_1+1}^{\eta_1+\eta_2} T_i T_i^* + \cdots + \sum_{i=\eta_1+\eta_2+\cdots+\eta_{n-1}+1}^{\eta_1+\eta_2+\cdots+\eta_n} T_i T_i^* \\
&= 1 - S_{\eta_1} S_{\eta_1}^* + S_{\eta_1}(1 - S_{\eta_2} S_{\eta_2}^*) S_{\eta_1}^* + \cdots + S_{\eta_1} S_{\eta_2} \cdots S_{\eta_{n-1}} S_{\eta_{n-1}}^* \cdots S_{\eta_2}^* S_{\eta_1}^* = 1.
\end{aligned}$$

Hence, the C^*-algebra $\widehat{O}_\beta$ is isomorphic to the Cuntz algebra $O_{\eta_1+\eta_2+\cdots+\eta_n}$. The inclusion relation $\widehat{O}_\beta \subset O_\beta$ is clear. To show the converse inclusion relation $O_\beta \subset \widehat{O}_\beta$, it suffices to prove that the partial isometry S_{η_1} belongs to the algebra $\widehat{O}_\beta$. By the equality

$$\varphi(S_{\eta_1}^* S_{\eta_1}) = \beta - \eta_1 = \frac{\eta_2}{\beta} + \frac{\eta_3}{\beta^2} + \cdots + \frac{\eta_n}{\beta^{n-1}},$$

$$\begin{aligned}
S_{\eta_1}^* S_{\eta_1} = &\sum_{j=0}^{\eta_2-1} S_j S_j^* + \sum_{j=0}^{\eta_3-1} S_{\eta_2} S_j S_j^* S_{\eta_2}^* + \cdots \\
&+ \sum_{j=0}^{\eta_n-1} S_{\eta_2} S_{\eta_3} \cdots S_{\eta_{n-1}} S_j S_j^* S_{\eta_n-1}^* \cdots S_{\eta_3}^* S_{\eta_2}^*,
\end{aligned}$$

so that

$$\begin{aligned}
S_{\eta_1} &= S_{\eta_1} S_{\eta_1}^* S_{\eta_1} \\
&= \sum_{j=0}^{\eta_2-1} T_{\eta_1+j+1} S_j^* + \sum_{j=0}^{\eta_3-1} T_{\eta_1+\eta_2+j+1} (S_{\eta_2} S_j)^* + \cdots \\
&\quad + \sum_{j=0}^{\eta_n-1} T_{\eta_1+\eta_2+\cdots+\eta_{n-1}+j+1} (S_{\eta_2} S_{\eta_3} \cdots S_{\eta_{n-1}} S_j)^*.
\end{aligned}$$

Denote by η_0 the empty word. The following set W_β of the words

$$W_\beta = \{(\eta_2, \eta_3, \ldots, \eta_{m-1}, i) \mid i = 0, 1, \ldots, \eta_m - 1, m = 1, 2, \ldots, n\}$$

are all admissible words of X_β. By cutting a word in the subwords beginning with η_1, one easily sees that any admissible word of X_β is decomposed into a product of some of the words of the following set:

$$C_\beta = \{(\eta_1, \eta_2, \ldots, \eta_{m-1}, j) \mid j = 0, 1, \ldots, \eta_m - 1, m = 1, 2, \ldots, n\}.$$

Hence, any word of W_β is a product of some of the words of C_β. This implies that the operators

$$S_{\eta_2} S_{\eta_3} \cdots S_{\eta_{m-1}} S_j, \quad j = 0, 1, \ldots, \eta_m - 1, m = 1, 2, \ldots, n$$

are products of some of $T_1, T_2, \ldots, T_{\eta_1+\eta_2+\cdots+\eta_n}$. Therefore, S_{η_1} is written as a product of $T_i, T_i^*, i = 1, 2, \ldots, \eta_1 + \eta_2 + \cdots + \eta_n$. This shows that $O_\beta \subset \widehat{O}_\beta$. The equality $\mathcal{D}_\beta = \widehat{\mathcal{D}}_\beta$ is direct. □

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The above proposition implies that the SFT β-shift (X_β, σ) is continuously orbit equivalent to the full $(\eta_1 + \eta_2 + \cdots + \eta_n)$-shift $(X_{\eta_1+\eta_2+\cdots+\eta_n}, \sigma)$ [16, 22, 29]. Therefore, we have the following theorem.

THEOREM 7.2. *If the β-expansion of* 1 *is finite such that*

$$1 = \frac{\eta_1}{\beta} + \frac{\eta_2}{\beta^2} + \cdots + \frac{\eta_n}{\beta^n},$$

then the group Γ_β is isomorphic to the Higman–Thompson group $V_{\eta_1+\eta_2+\cdots+\eta_n}$.

COROLLARY 7.3. *Let (X_β, σ) and $(X_{\beta'}, \sigma)$ be SFT β-shifts such that their finite β-expansions of* 1 *are*

$$1 = \frac{\eta_1}{\beta} + \frac{\eta_2}{\beta^2} + \cdots + \frac{\eta_n}{\beta^n} = \frac{\eta'_1}{\beta'} + \frac{\eta'_2}{\beta'^2} + \cdots + \frac{\eta'_{n'}}{\beta'^{n'}},$$

respectively. Then the following are equivalent:

(i) *the groups Γ_β and $\Gamma_{\beta'}$ are isomorphic;*
(ii) *the Cuntz algebras $\mathcal{O}_{\eta_1+\eta_2+\cdots+\eta_n}$ and $\mathcal{O}_{\eta'_1+\eta'_2+\cdots+\eta'_{n'}}$ are isomorphic;*
(iii) $\eta_1 + \eta_2 + \cdots + \eta_n = \eta'_1 + \eta'_2 + \cdots + \eta'_{n'}$.

PROOF. The implication (iii) implies that (ii) is trivial, and its converse (ii) implies that (iii) is well known [7, 8]. Assume that the groups Γ_β and $\Gamma_{\beta'}$ are isomorphic. By [19] or more generally [24], the C^*-algebras $C^*_r(G_\beta)$ and $C^*_r(G_{\beta'})$ of the groupoids G_β and $G_{\beta'}$ associated with their respective shifts (X_β, σ) and $(X_{\beta'}, \sigma)$ of finite type are isomorphic. Since $C^*_r(G_\beta) = \mathcal{O}_\beta$ and $C^*_r(G_{\beta'}) = \mathcal{O}_{\beta'}$, Proposition 7.1 implies (ii), so that the implication (i) implies that (ii) holds. The implication (iii) implies that (i) is a direct consequence of the above theorem. □

2. Sofic case:

Assume that the β-shift X_β is sofic. Put

$$k_\beta = \min\{k \in \mathbb{N} \mid \mathcal{A}_k = \mathcal{A}_{k+1}\}, \qquad K_\beta = k_\beta + 1.$$

Hence, $\mathcal{A}_{k_\beta} = \mathcal{A}_{k_\beta+1} = \cdots = \mathcal{A}_\beta$ and $\dim \mathcal{A}_\beta = K_\beta$. There exists $l \in \mathbb{N}$ with $0 < l \le k_\beta$ such that

$$a_{\xi_1\cdots\xi_{K_\beta}} = a_{\xi_1\cdots\xi_l} \quad \text{and hence} \quad d(1,\beta) = \xi_1 \cdots \xi_l \dot{\xi}_{l+1} \cdots \dot{\xi}_{K_\beta}. \tag{7.1}$$

Let $E_1, \ldots, E_{K_\beta}$ be the minimal projections of $\mathcal{A}_\beta$ as in the preceding section, so that

$$\mathcal{A}_\beta = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_{K_\beta}. \tag{7.2}$$

Define a labeled graph $\mathcal{G}_\beta$ over $\Sigma = \{0, 1, \ldots, N-1\}$ with vertex set $\{v_1, v_2, \ldots, v_{K_\beta}\}$ corresponding to the minimal projections $E_1, \ldots, E_{K_\beta}$ in the following way. Define a labeled edge from v_i to v_j labeled $\alpha \in \Sigma$ if $S^*_\alpha E_i S_\alpha \ge E_j$. We denote by $\mathcal{E}_\beta$ the edge set of the labeled graph $\mathcal{G}_\beta$ with labeling map $\lambda : \mathcal{E}_\beta \longrightarrow \Sigma$. The vertex set $\{v_1, v_2, \ldots, v_{K_\beta}\}$ is denoted by $\mathcal{V}_\beta$. Let $\mathcal{M}_\beta$ be the $K_\beta \times K_\beta$ symbolic matrix of $\mathcal{G}_\beta$ and M_β the $K_\beta \times K_\beta$ nonnegative matrix obtained from $\mathcal{M}_\beta$ by putting all the symbols equal to 1. For an

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edge $e \in \mathcal{E}_\beta$, denote by $\lambda(e) \in \Sigma$ and $s(e), t(e) \in \{1, 2, \ldots, K_\beta\}$ the letter of the label of e and the number of the source vertex $v_{s(e)}$ of e and that of the terminal vertex $v_{t(e)}$ of e, respectively. Define a partial isometry $s_e = S_{\lambda(e)} E_{t(e)}$ for an edge $e \in \mathcal{E}_\beta$ in the C^*-algebra $\mathcal{O}_\beta$. Define the $|\mathcal{E}_\beta| \times |\mathcal{E}_\beta|$ matrix $B_\beta = [B_\beta(e, f)]_{e, f \in \mathcal{E}_\beta}$ with entries in $\{0, 1\}$ by

$$B_\beta(e, f) = \begin{cases} 1 & \text{if } t(e) = s(f), \\ 0 & \text{if } t(e) \neq s(f). \end{cases}$$

We have the following lemma (see [31, Section 4]).

Lemma 7.4. *The partial isometries $s_e, e \in \mathcal{E}_\beta$ satisfy the following relations:*

$$\sum_{e \in \mathcal{E}_\beta} s_e s_e^* = 1, \qquad s_e^* s_e = \sum_{f \in \mathcal{E}_\beta} B_\beta(e, f) s_f s_f^*.$$

Hence, the C^*-algebra $C^*(s_e; e \in \mathcal{E}_\beta)$ generated by $s_e, e \in \mathcal{E}_\beta$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{B_\beta}$.

Proof. We see the identities

$$1 = \sum_{i=1}^{K_\beta} E_i = \sum_{i=1}^{K_\beta} \sum_{\alpha=0}^{N-1} S_\alpha S_\alpha^* E_i S_\alpha S_\alpha^*.$$

The projection $S_\alpha^* E_i S_\alpha$ is not zero if and only if there exists $e \in \mathcal{E}_\beta$ such that $\alpha = \lambda(e)$ and $i = s(e)$. Hence,

$$S_\alpha^* E_i S_\alpha = \sum_{\substack{e \in \mathcal{E}_\beta, \\ \alpha = \lambda(e),\, i = s(e)}} E_{t(e)},$$

so that

$$1 = \sum_{i=1}^{K_\beta} \sum_{\alpha=0}^{N-1} \sum_{\substack{e \in \mathcal{E}_\beta, \\ \alpha = \lambda(e),\, i = s(e)}} S_\alpha E_{t(e)} S_\alpha^* = \sum_{e \in \mathcal{E}_\beta} s_e s_e^*.$$

For an edge $e \in \mathcal{E}_\beta$,

$$\begin{aligned} s_e^* s_e &= E_{t(e)} S_{\lambda(e)}^* S_{\lambda(e)} E_{t(e)} = E_{t(e)} \\ &= \sum_{\alpha=0}^{N-1} S_\alpha S_\alpha^* E_{t(e)} S_\alpha S_\alpha^* \\ &= \sum_{\alpha=0}^{N-1} S_\alpha \cdot \sum_{\substack{f \in \mathcal{E}_\beta, \\ \alpha = \lambda(f),\, t(e) = s(f)}} E_{t(f)} \cdot S_\alpha^* = \sum_{f \in \mathcal{E}_\beta} B_\beta(e, f) s_f s_f^*. \end{aligned}$$

□

Denote by $\mathcal{D}_{B_\beta}$ the canonical Cartan subalgebra of $\mathcal{O}_{B_\beta}$, which is a C^*-subalgebra of $\mathcal{O}_{B_\beta}$ generated by the projections $s_{e_1} \cdots s_{e_n} s_{e_n}^* \cdots s_{e_1}^*, e_1, \ldots, e_n \in \mathcal{E}_\beta$.

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LEMMA 7.5. *$\mathcal{O}_\beta = \mathcal{O}_{B_\beta}$ and $\mathcal{D}_\beta = \mathcal{D}_{B_\beta}$.*

PROOF. Since $s_e = S_{\lambda(e)} E_{t(e)}, e \in \mathcal{E}_\beta$, we have $s_e \in \mathcal{O}_\beta$, so that the inclusion $\mathcal{O}_{B_\beta} \subset \mathcal{O}_\beta$ is obvious. For $\alpha \in \Sigma = \{0, 1, \ldots, N-1\}$, $i = 1, 2, \ldots, K_\beta$, we know that $S_\alpha E_i \neq 0$ if and only if $S_\alpha^* S_\alpha \geq E_i$, which is equivalent to the condition that there exists an edge $e \in \mathcal{E}_\beta$ such that $\alpha = \lambda(e), i = t(e)$. For $i = 1, 2, \ldots, K_\beta$, take $e \in \mathcal{E}_\beta$ such that $\alpha = \lambda(e), i = t(e)$. We then have $s_e^* s_e = E_{t(e)} = E_i$, so that $E_i \in \mathcal{O}_{B_\beta}$. For $\alpha \in \Sigma$,

$$S_\alpha = \sum_{i=1}^{K_\beta} S_\alpha E_i = \sum_{e \in \mathcal{E}_\beta, \alpha = \lambda(e)} S_{\lambda(e)} E_{t(e)} = \sum_{e \in \mathcal{E}_\beta, \alpha = \lambda(e)} s_e,$$

so that $S_\alpha \in \mathcal{O}_{B_\beta}$. We thus have the inclusion $\mathcal{O}_\beta \subset \mathcal{O}_{B_\beta}$ and hence $\mathcal{O}_\beta = \mathcal{O}_{B_\beta}$.

We will next show that $\mathcal{D}_\beta = \mathcal{D}_{B_\beta}$. We have $s_e s_e^* = S_{\lambda(e)} E_{t(e)} S_{\lambda(e)}^* \in \mathcal{D}_\beta$. Suppose that $s_{e_1} \cdots s_{e_n} s_{e_n}^* \cdots s_{e_1}^* \in \mathcal{D}_\beta$. By the equality

$$s_{e_0} s_{e_1} \cdots s_{e_n} s_{e_n}^* \cdots s_{e_1}^* s_{e_0}^* = S_{\lambda(e_0)} E_{t(e_0)} s_{e_1} \cdots s_{e_n} s_{e_n}^* \cdots s_{e_1}^* E_{t(e_0)} S_{\lambda(e_0)}^* \in \mathcal{D}_\beta,$$

the inclusion relation $\mathcal{D}_{B_\beta} \subset \mathcal{D}_\beta$ holds by induction. Conversely, suppose that $S_\alpha E_i S_\alpha^*$ is not zero. Take $e \in \mathcal{E}_\beta$ such that $\alpha = \lambda(e), i = t(e)$, so that

$$S_\alpha E_i S_\alpha^* = S_{\lambda(e)} E_{t(e)} S_{\lambda(e)}^* = s_e s_e^*$$

belongs to $\mathcal{D}_{B_\beta}$. Suppose next that $S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^*$ belongs to $\mathcal{D}_{B_\beta}$ and $S_{\mu_0} S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^* S_{\mu_0}^*$ is not zero. The labeled graph $\mathcal{G}_\beta$ is left-resolving, which means that there uniquely exists a finite sequence of edges $e_1, e_2, \ldots, e_n \in \mathcal{E}_\beta$ for the vertex v_i such that

$$\lambda(e_p) = \mu_p,\ t(e_p) = s(e_{p+1}) \quad \text{for } p = 1, 2, \ldots, n-1,$$
$$\lambda(e_n) = \mu_n,\ t(e_n) = i.$$

Put $j = s(e_1)$, so that

$$E_j \geq S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^*, \quad E_j S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^* E_j = S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^*.$$

Take a unique edge $e_0 \in \mathcal{E}_\beta$ such that $\lambda(e_0) = \mu_0, t(e_0) = j$. Hence, $S_{\mu_0} E_j = s_{e_0}$. It then follows that

$$\begin{aligned} S_{\mu_0} S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^* S_{\mu_0}^* &= S_{\mu_0} E_j S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^* E_j S_{\mu_0}^* \\ &= s_{e_0} S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^* s_{e_0}^*. \end{aligned}$$

As $S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^* \in \mathcal{D}_{B_\beta}$, we have $s_{e_0} S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^* s_{e_0}^* \in \mathcal{D}_{B_\beta}$. Thus, the element $S_{\mu_0} S_{\mu_1 \cdots \mu_n} E_i S_{\mu_1 \cdots \mu_n}^* S_{\mu_0}^*$ belongs to $\mathcal{D}_{B_\beta}$. By induction, we have $\mathcal{D}_\beta \subset \mathcal{D}_{B_\beta}$ and hence $\mathcal{D}_\beta = \mathcal{D}_{B_\beta}$. □

A nonnegative square matrix B is said to be elementary equivalent to a nonnegative square matrix M if there exist nonnegative rectangular matrices R and S such that $B = RS$ and $M = SR$ (see [14]).

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LEMMA 7.6. *The matrix B_β is elementary equivalent to the matrix M_β. Hence,*

$$\det(1 - B_\beta) = \det(1 - M_\beta).$$

PROOF. Note that $\dim \mathcal{A}_\beta = |\mathcal{V}_\beta| = K_\beta$. Define a $|\mathcal{E}_\beta| \times |\mathcal{V}_\beta|$ matrix R_β and a $|\mathcal{V}_\beta| \times |\mathcal{E}_\beta|$ matrix S_β as follows.

$$R_\beta(e, i) = \begin{cases} 1 & \text{if } t(e) = v_i, \\ 0 & \text{otherwise}, \end{cases} \qquad S_\beta(j, f) = \begin{cases} 1 & \text{if } s(f) = v_j, \\ 0 & \text{otherwise} \end{cases}$$

for $e, f \in \mathcal{E}_\beta, v_i, v_j \in \mathcal{V}_\beta$ and $i, j = 1, 2 \ldots, K_\beta$. It is direct to see that

$$B_\beta = R_\beta S_\beta, \quad M_\beta = S_\beta R_\beta$$

and $\det(1 - B_\beta) = \det(1 - M_\beta)$. □

Recall that φ stands for the unique KMS state on the C^*-algebra O_β under the gauge action. It satisfies the identities

$$\varphi(a_{\xi_1 \cdots \xi_j}) = \beta^j - \xi_1 \beta^{j-1} - \xi_2 \beta^{j-2} - \cdots - \xi_{j-1}\beta - \xi_j, \quad j = 1, \ldots, K_\beta.$$

By (7.2), the K_0-group $K_0(\mathcal{A}_{k_\beta})$ of the algebra $\mathcal{A}_{k_\beta}$ is generated by the classes of the minimal projections $E_1, \ldots, E_{K_\beta}$ of $\mathcal{A}_{k_\beta}(= \mathcal{A}_\beta)$, so that $K_0(\mathcal{A}_{k_\beta})$ is isomorphic to $\mathbb{Z}^{K_\beta}$. Since a minimal projection E_i is of the form $a_{\xi_1 \cdots \xi_{p_i}} - a_{\xi_1 \cdots \xi_{q_i}}$, the following correspondence:

$$\begin{aligned} [1] \in K_0(\mathcal{A}_{k_\beta}) &\longrightarrow (1, 0, 0, \ldots, 0) \in \mathbb{Z} \oplus \beta\mathbb{Z} \oplus \cdots \oplus \beta^{k_\beta}\mathbb{Z}, \\ [a_{\xi_1}] \in K_0(\mathcal{A}_{k_\beta}) &\longrightarrow (-\xi_1, 1, 0, \ldots, 0) \in \mathbb{Z} \oplus \beta\mathbb{Z} \oplus \cdots \oplus \beta^{k_\beta}\mathbb{Z}, \\ [a_{\xi_1 \cdots \xi_j}] \in K_0(\mathcal{A}_{k_\beta}) &\longrightarrow (-\xi_j, -\xi_{j-1}, \ldots, -\xi_2, -\xi_1, 1, 0, \ldots, 0) \in \mathbb{Z} \oplus \beta\mathbb{Z} \oplus \cdots \oplus \beta^{k_\beta}\mathbb{Z} \end{aligned}$$

for $j = 1, \ldots, K_\beta$ yields an isomorphism from $K_0(\mathcal{A}_{k_\beta})$ to $\mathbb{Z} \oplus \beta\mathbb{Z} \oplus \cdots \oplus \beta^{k_\beta}\mathbb{Z}$ as a group, which we denote by Φ. By (7.1),

$$\begin{aligned} &\beta^{k_\beta+1} - \xi_1 \beta^{k_\beta} - \xi_2 \beta^{k_\beta-1} - \cdots - \xi_{k_\beta}\beta - \xi_{k_\beta+1} \\ &\quad = \beta^l - \xi_1 \beta^{l-1} - \xi_2 \beta^{l-2} - \cdots - \xi_{l-1}\beta - \xi_l, \end{aligned}$$

so that β is a solution of a monic polynomial of degree K_β. We denote this polynomial by

$$\beta^{k_\beta+1} - \eta_1 \beta^{k_\beta} - \eta_2 \beta^{k_\beta-1} - \cdots - \eta_{k_\beta}\beta - \eta_{k_\beta+1} = 0.$$

Then

$$\eta_1 + \eta_2 + \cdots + \eta_{k_\beta} + \eta_{k_\beta+1} = \xi_{l+1} + \xi_{l+2} + \cdots + \xi_{k_\beta+1} + 1. \tag{7.3}$$

LEMMA 7.7 [13, Lemma 4.8]. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{Z}^{K_\beta} & \xrightarrow{\quad M_\beta \quad} & \mathbb{Z}^{K_\beta} \\ \| & & \| \\ K_0(\mathcal{A}_{k_\beta}) & \xrightarrow{\quad \lambda_{\beta*} \quad} & K_0(\mathcal{A}_{k_\beta}) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathbb{Z} \oplus \beta\mathbb{Z} \oplus \cdots \oplus \beta^{k_\beta}\mathbb{Z} & \xrightarrow{\ \tau\ } & \mathbb{Z} \oplus \beta\mathbb{Z} \oplus \cdots \oplus \beta^{k_\beta}\mathbb{Z} \end{array}$$

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where $\lambda_{\beta*}$ is the endomorphism of $K_0(\mathcal{A}_\beta)$ induced from the map $\lambda_\beta : \mathcal{A}_{k_\beta} \to \mathcal{A}_{k_\beta+1}$ $(= \mathcal{A}_\beta)$ defined by

$$\lambda_\beta(a) = \sum_{\alpha=0}^{N-1} S_\alpha^* a S_\alpha \quad \text{for } a \in \mathcal{A}_\beta$$

and τ is an endomorphism of $\mathbb{Z} \oplus \beta\mathbb{Z} \oplus \cdots \oplus \beta^{k_\beta}\mathbb{Z}$ defined by

$$\tau(m_0, m_1, \ldots, m_{k_\beta-1}, 0) = (0, m_0, m_1, \ldots, m_{k_\beta-1}), \quad m_i \in \mathbb{Z},$$
$$\tau(0, \ldots, 0, 1) = (\eta_{k_\beta+1}, \eta_{k_\beta}, \ldots, \eta_2, \eta_1).$$

Define the $(k_\beta + 1) \times (k_\beta + 1)$ matrix

$$L_\beta = \begin{bmatrix} & & & \eta_{k_\beta+1} \\ 1 & & & \eta_{k_\beta} \\ & \ddots & & \vdots \\ & & 1 & \eta_1 \end{bmatrix}$$

where the blanks denote zeros. The matrix L_β acts from the left-hand side of the transpose $(m_0, m_1, \ldots, m_{k_\beta})^t$ of $(m_0, m_1, \ldots, m_{k_\beta})$, so that it represents the homomorphism τ. The characteristic polynomial of L_β is

$$\det(t - L_\beta) = t^{k_\beta+1} - \eta_1 t^{k_\beta} - \eta_2 t^{k_\beta-1} - \cdots - \eta_{k_\beta} t - \eta_{k_\beta+1}$$

and the number β is one of the eigenvalues of the transpose of L_β with eigenvector $[1, \beta, \beta^2, \ldots, \beta^{k_\beta}]$. Hence, we have the following corollary.

Corollary 7.8. $\det(1 - B_\beta) = \det(1 - L_\beta) = 1 - \eta_1 - \eta_2 - \cdots - \eta_{k_\beta} - \eta_{k_\beta+1} < 0$.

Proposition 7.9. *There exists an isomorphism Φ from the Cuntz–Krieger algebra $\mathcal{O}_{B_\beta}$ onto the Cuntz algebra $\mathcal{O}_{\xi_1+\cdots+\xi_{k_\beta+1}+1}$ such that $\Phi(\mathcal{D}_{B_\beta}) = \mathcal{D}_{\xi_1+\cdots+\xi_{k_\beta+1}+1}$. Therefore, their topological full groups Γ_{B_β} and $\Gamma_{\xi_1+\cdots+\xi_{k_\beta+1}+1}$ are isomorphic.*

Proof. We have already shown that $\mathcal{O}_\beta$ is isomorphic to $\mathcal{O}_{\xi_1+\cdots+\xi_{k_\beta+1}+1}$ by [13]. By the preceding lemma, we know that $\mathcal{O}_\beta = \mathcal{O}_{B_\beta}$ and $\mathcal{D}_\beta = \mathcal{D}_{B_\beta}$, so that $\mathcal{O}_{B_\beta}$ is isomorphic to $\mathcal{O}_{\xi_1+\cdots+\xi_{k_\beta+1}+1}$. By the preceding lemma with (7.3),

$$\begin{aligned} \det(1 - B_\beta) &= 1 - \eta_1 - \eta_2 - \cdots - \eta_{k_\beta} - \eta_{k_\beta+1} \\ &= 1 - (\xi_{l+1} + \cdots + \xi_{k_\beta+1} + 1). \end{aligned}$$

Hence, the topological Markov shift (X_{B_β}, σ) is continuously orbit equivalent to the full shift $(X_{\xi_{l+1}+\cdots+\xi_{k_\beta+1}+1}, \sigma)$ by [17] (see [20]). Thus, their topological full groups Γ_{B_β} and $\Gamma_{\xi_{l+1}+\cdots+\xi_{k_\beta+1}+1}$ are isomorphic. □

Theorem 7.10. *Suppose that (X_β, σ) is sofic such that the β-expansion of 1 is*

$$d(1, \beta) = \xi_1 \cdots \xi_l \dot{\xi}_{l+1} \cdots \dot{\xi}_{k+1}.$$

Then there exists an isomorphism Φ *from* O_β *to* $O_{\xi_{l+1}+\cdots+\xi_{k+1}+1}$ *such that* $\Phi(\mathcal{D}_\beta) = \mathcal{D}_{\xi_{l+1}+\cdots+\xi_{k+1}+1}$. *Therefore, their topological full groups* Γ_β *and* $\Gamma_{\xi_{l+1}+\cdots+\xi_{k+1}+1}$ *are isomorphic. This implies that the group* Γ_β *is isomorphic to the Higman–Thompson group* $V_{\xi_{l+1}+\cdots+\xi_{k+1}+1}$.

3. Nonsofic case:

THEOREM 7.11. *If* $1 < \beta \in \mathbb{R}$ *is not ultimately periodic, then the group* Γ_β *is not isomorphic to any of the Higman–Thompson groups* V_n, $1 < n \in \mathbb{N}$.

PROOF. By Proposition 3.4, the groupoid G_β is an essentially principal, purely infinite, minimal groupoid. Suppose that Γ_β is isomorphic to one of the Higman–Thompson groups V_n for some $n \in \mathbb{N}$. Since V_n is isomorphic to the topological full group Γ_n of the groupoid G_n for the full n-shift, by Matui [24], the groupoid G_β is isomorphic to G_n. By Renault [28, Theorem 4.11], there exists an isomorphism Φ from $C^*_r(G_\beta)$ to $C^*_r(G_n)$. The C^*-algebra $C^*_r(G_\beta)$ is isomorphic to O_β, and $C^*_r(G_n)$ is isomorphic to the Cuntz algebra O_n. Since β is not ultimately periodic, we know that $K_0(O_\beta) = \mathbb{Z}$ by [13, Theorem 4.12], which is a contradiction to the fact that $K_0(O_n) = \mathbb{Z}/(1 - n)\mathbb{Z}$. □

References

[1] A. Bertrand-Mathis, 'Développement en base θ, répartition modulo un de la suite $(x\theta^n)_{n\geq 0}$, languages codés et θ-shift', *Bull. Soc. Math. France* **114** (1986), 271–323.

[2] F. Blanchard, 'β-expansions and symbolic dynamics', *Theoret. Comput. Sci.* **65** (1989), 131–141.

[3] K. S. Brown, 'Finiteness properties of groups', *J. Pure Appl. Algebra* **44** (1987), 45–75.

[4] J. W. Cannon, W. J. Floyd and W. R. Parry, 'Introductory notes on Richard Thompson's groups', *Enseign. Math.* **42**(2) (1996), 215–256.

[5] S. Cleary, 'Regular subdivision in $\mathbb{Z}[(1 + \sqrt{5})/2]$', *Illinois J. Math.* **44** (2000), 453–464.

[6] M. Crainic and I. Moerdijk, 'A homology theory for étale groupoids', *J. reine angew. Math.* **521** (2000), 25–46.

[7] J. Cuntz, 'Simple C^*-algebras generated by isometries', *Comm. Math. Phys.* **57** (1977), 173–185.

[8] J. Cuntz, 'K-theory for certain C^*-algebras', *Ann. of Math.* (2) **113** (1981), 181–197.

[9] J. Cuntz and W. Krieger, 'A class of C^*-algebras and topological Markov chains', *Invent. Math.* **56** (1980), 251–268.

[10] V. Deaconu, 'Groupoids associated with endomorphisms', *Trans. Amer. Math. Soc.* **347** (1995), 1779–1786.

[11] C. Frougny and B. Solomyak, 'Finite β-expansions', *Ergodic Theory Dynam. Systems* **12** (1992), 713–723.

[12] G. Higman, 'Finitely presented infinite simple groups', *Notes on Pure Mathematics*, Vol. 8 (Australian National University, Canberra, 1974).

[13] Y. Katayama, K. Matsumoto and Y. Watatani, 'Simple C^*-algebras arising from β-expansion of real numbers', *Ergodic Theory Dynam. Systems* **18** (1998), 937–962.

[14] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding* (Cambridge University Press, Cambridge, 1995).

[15] K. Matsumoto, 'C^*-algebras associated with presentations of subshifts', *Doc. Math.* **7** (2002), 1–30.

[16] K. Matsumoto, 'Orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras', *Pacific J. Math.* **246** (2010), 199–225.

[17] K. Matsumoto, 'Classification of Cuntz–Krieger algebras by orbit equivalence of topological Markov shifts', *Proc. Amer. Math. Soc.* **141** (2013), 2329–2342.

https://doi.org/10.1017/S1446788714000214 Published online by Cambridge University Press

[18] K. Matsumoto, ‘K-groups of the full group actions on one-sided topological Markov shifts’, *Discrete Contin. Dyn. Syst.* **33** (2013), 3753–3765.
[19] K. Matsumoto, Full groups of one-sided topological Markov shifts, *Israel J. Math.*, to appear.
[20] K. Matsumoto and H. Matui, Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras, *Kyoto J. Math.*, to appear.
[21] H. Matui, ‘Some remarks on topological full groups of Cantor minimal systems’, *Internat. J. Math.* **17** (2006), 231–251.
[22] H. Matui, ‘Homology and topological full groups of étale groupoids on totally disconnected spaces’, *Proc. Lond. Math. Soc.* **104** (2012), 27–56.
[23] H. Matui, ‘Some remarks on topological full groups of Cantor minimal systems II’, *Ergodic Theory Dynam. Systems* **33** (2013), 1542–1549.
[24] H. Matui, Topological full groups of one-sided shifts of finite type, *J. reine angew Math.*, to appear.
[25] V. V. Nekrashevych, ‘Cuntz–Pimsner algebras of group actions’, *J. Operator Theory* **52** (2004), 223–249.
[26] E. Pardo, ‘The isomorphism problem for Higman–Thompson groups’, *J. Algebra* **344** (2011), 172–183.
[27] W. Parry, ‘On the β-expansion of real numbers’, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.
[28] J. Renault, *A Groupod Approach to C^*-Algebras*, Lecture Notes in Mathematics, 793 (Springer, Berlin–Heidelberg–New York, 1980).
[29] J. Renault, ‘Cartan subalgebras in C^*-algebras’, *Irish Math. Soc. Bull.* **61** (2008), 29–63.
[30] A. Rényi, ‘Representations for real numbers and their ergodic properties’, *Acta Math. Acad. Sci. Hungar.* **8** (1957), 477–493.
[31] M. Rørdam, ‘Classification of Cuntz–Krieger algebras’, *K-Theory* **9** (1995), 31–58.
[32] R. J. Thompson, ‘Embeddings into finitely generated simple groups which preserve the word problem’, in: *Word Problems II*, Studies in Logic and the Foundations of Mathematics, 95 (eds. S. I. Adian, W. W. Boone and G. Higman) (North-Holland, Amsterdam, 1980), 401–441.

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