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Chaotic behavior of the p-adic Potts–Bethe mapping II

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Abstract. The renormalization group method has been developed to investigate p-adic q-state Potts models on the Cayley tree of order k. This method is closely related to the examination of dynamical behavior of the p-adic Potts–Bethe mapping which depends on the parameters q, k. In Mukhamedov and Khakimov [Chaotic behavior of the p-adic Potts–Behte mapping. *Discrete Contin. Dyn. Syst.* **38** (2018), 231–245], we have considered the case when q is not divisible by p and, under some conditions, it was established that the mapping is conjugate to the full shift on κ_p symbols (here κ_p is the greatest common factor of k and $p-1$). The present paper is a continuation of the forementioned paper, but here we investigate the case when q is divisible by p and k is arbitrary. We are able to fully describe the dynamical behavior of the p-adic Potts–Bethe mapping by means of a Markov partition. Moreover, the existence of a Julia set is established, over which the mapping exhibits a chaotic behavior. We point out that a similar result is not known in the case of real numbers (with rigorous proofs).

Key words: p-adic numbers, Potts–Bethe mapping, chaos, shift
2020 Mathematics subject classification: 37B05, 37B10 (Primary); 12J12, 39A70 (Secondary)

1. *Introduction*

The presentpaper is a continuation of [**35**], where we have started to investigate the chaotic behavior of the Potts–Bethe mapping over the p-adic field (here p is some prime number).

Note that the mapping is governed by

$$f_{\theta,q,k}(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2}\right)^k, \tag{1.1}$$

where $k, q \in \mathbb{N}$ and $|\theta - 1|_p < 1$. In [35], we have considered the case when q is not divisible by p, that is, $|q|_p = 1$. In that setting, under some conditions, we were able to prove that $f_{\theta,q,k}$ is conjugate to the full shift on κ_p symbols (here κ_p is the greatest common factor (GCF) of k and $p-1$). In the current paper, we are going to study the same Potts–Bethe mapping when q is divisible by p, that is $|q|_p < 1$. It is known that the thermodynamic behavior of the central site of the Potts model with nearest-neighbor interactions on a Cayley tree is reduced to the recursive system which is given by (1.1). The existence of at least two non-trivial p-adic Gibbs measures indicates that the phase transition may exist. This is closely connected to the chaotic behavior of the associated dynamical system [12, 16, 17, 23, 26, 27]. Therefore, it is important to investigate the chaotic properties of (1.1).

We stress that the Potts–Ising mapping is a particular case of the Potts–Bethe mapping, which can be obtained from (1.1) by putting $q = 2$. Recently, in [30, 34] under some condition, a Julia set of the Potts–Ising mapping was described, and it was shown that restricted to its Julia set, the Potts–Ising mapping is conjugate to a full shift. Therefore, it is natural to consider the Potts–Bethe mapping for $q \geq 3$ with $|q|_p < 1$ and $k \geq 2$. In [43], all fixed points of $f_{\theta,q,k}$ were found when $k = 2$ and $|q|_p < 1$. Then, using these fixed points, the dynamics of (1.1) whenever $k = 2$ and $|q|_p < 1$ was investigated in [11, 31, 32]. Recently in [1, 44], the Potts–Bethe mapping was studied for the case $k = 3$ and $|q|_p < 1$. In the present paper, we are going to consider a more general case, that is, arbitrary $k \geq 2$ and $|q|_p < 1$. To formulate our main result, let us recall some necessary notions.

It is easy to notice that the function (1.1) is defined on $\mathbb{Q}_p \setminus \{x^{(\infty)}\}$, where $x^{(\infty)} = 2 - q - \theta$. For the sake of convenience, we write $\mathrm{Dom}(f_{\theta,q,k}) := \mathbb{Q}_p \setminus \{x^{(\infty)}\}$. Let us denote

$$\mathcal{P}_{x^{(\infty)}} = \bigcup_{n=1}^{\infty} f_{\theta,q,k}^{-n}(x^{(\infty)}).$$

One can see that the set $\mathcal{P}_{x^{(\infty)}}$ is at most countable, and could be empty for some k, q and θ (see §3). If it is not empty, then for any $x_0 \in \mathcal{P}_{x^{(\infty)}}$, there exists an $n \geq 1$ such that after n-times, we will 'lose' that point.

For a given mapping f on $\mathbb{Q}_p$, we denote by $\mathrm{Fix}(f)$ the set of all fixed points of f, that is,

$$\mathrm{Fix}(f) = \{x \in \mathbb{Q}_p : f(x) = x\}.$$

Let f be an analytic function and $x^{(0)} \in \mathrm{Fix}(f)$. We define

$$\lambda = \frac{d}{dx} f(x^{(0)}).$$

The fixed point $x^{(0)}$ is called *attractive* if $0 < |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and *repelling* if $|\lambda|_p > 1$.

For an attractive fixed point $x^{(0)}$ of f, its basin of attraction is defined by

$$A(x^{(0)}) = \{x \in \mathbb{Q}_p : \lim_{n\to\infty} f^n(x) = x^{(0)}\},$$

where $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$.

The main result of the present paper is given in the following theorem.

THEOREM 1.1. *Let $p \geq 3$, $k \geq 2$, $|q|_p < 1$, $|\theta - 1|_p < 1$, and $x_0^* = 1$. Then the dynamical structure of the system $(\mathbb{Q}_p, f_{\theta,q,k})$ is described as follows.*

(A) *If $|k|_p \leq |q + \theta - 1|_p$, then $\operatorname{Fix}(f_{\theta,q,k}) = \{x_0^*\}$ and*

$$A(x_0^*) = \operatorname{Dom}(f_{\theta,q,k}).$$

(B) *Assume that $|k|_p > |q + \theta - 1|_p$ and $|\theta - 1|_p < |q|_p^2$. Then there exists a non-empty set $J_{f_{\theta,q,k}} \subset \operatorname{Dom}(f_{\theta,q,k}) \setminus \mathcal{P}_{x^{(\infty)}}$ which is invariant with respect to $f_{\theta,q,k}$ and*

$$A(x_0^*) = \operatorname{Dom}(f_{\theta,q,k}) \setminus (\mathcal{P}_{x^{(\infty)}} \cup J_{f_{\theta,q,k}}).$$

Moreover, if κ_p is the GCF of k and $p - 1$, then the following hold:

(B1) *if $\kappa_p = 1$, then there exists $x_* \in \operatorname{Fix}(f_{\theta,q,k})$ such that $x_* \neq x_0^*$ and $J_{f_{\theta,q,k}} = \{x_*\}$;*

(B2) *if $\kappa_p \geq 2$, then $(J_{f_{\theta,q,k}}, f_{\theta,q,k}, |\cdot|_p)$ is topologically conjugate to the full shift dynamics on κ_p symbols.*

Remark 1.2. It is worth pointing out that, in the present paper, the condition $|\theta - 1|_p < |q|_p^2$ is assumed to get essential estimations and calculations to prove the main result. The results of a recent paper [1] show that such a condition could be loosened to $|\theta - 1|_p < |q|_p$, but only for the case $k = 3$ where explicit expressions of the fixed points of the function $f_{\theta,q,k}$ have essentially been used to get more exact estimations. However, in this paper, we are able to prove the chaoticity of the Potts–Bethe mapping for arbitrary values of k (under the condition $|\theta - 1|_p < |q|_p^2$) and moreover, we are not even using the existence of the fixed points. Once we have proved that the Potts–Bethe mapping is conjugate to a full shift, then one concludes the existence of the fixed points. Roughly speaking, we are constructing (explicitly) a Markov partition of the mapping (1.1) which allows us to prove the main result of the current paper. However, the results of [1] indicate that the chaoticity of the function (1.1) could be obtained even in the case of $|q|^2 \leq |\theta - 1|_p < |q|_p$, but this will be a topic for another work. Here, it is better to emphasize that the results are valid when $p \geq 3$. The case $p = 2$ is considered pathological in the p-adic analysis (see for example [10]). Indeed, in [1], it was established that when $p = 2$ and $k = 3$, the function (1.1) does not have chaotic behavior. For general values of k, owing to huge calculations and numerous technical issues, this case could be investigated elsewhere.

Remark 1.3. In [41, 42], the authors established that the function (1.1) may have at least one fixed point and, moreover, they found a necessary condition (that is q is divisible by p) for the existence of more than one fixed point. Therefore, the following conjecture was formulated: *Let $k \in \mathbb{N}$, $q \in p\mathbb{N}$, and $|\theta - 1|_p < 1$, then the function (1.1) has at least two fixed points*. The formulated Theorem 1.1(A) shows that the mentioned conjecture is not always true.

We stress that, in the p-adic setting, owing to the lack of a convex structure of the set of p-adic Gibbs measures, it was quite difficult to constitute a phase transition with some features of the set of p-adic Gibbs measures. However, Theorem 1.1(B2) yields that the set of p-adic Gibbs measures is huge which is *a priori* not clear (see [**24**, **42**]). Moreover, the method of the present work allows one to find lots of periodic p-adic Gibbs measures for the p-adic Potts model. Furthermore, Theorem 1.1(B) together with the results of [**29**, **33**] will open new perspectives in investigations of generalized p-adic self-similar sets.

On one hand, our results shed some light on the question of the investigation of dynamics of rational functions in the p-adic analysis, because a global dynamical structure of rational maps on $\mathbb{Q}_p$ remains unclear. Some particular rational functions have been considered in [**4**, **5**, **7**, **8**, **10**, **13**–**15**, **18**, **21**, **39**]. On the other hand, the obtained results may have potential applications in the cryptography to build pseudo-random codes (see [**2**, **3**, **37**, **45**]). We point out that some p-adic chaotic dynamical systems have been studied in [**9**, **45**].

2. *Preliminaries*

2.1. *p-adic numbers.* Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p: $(p, n) = 1$, $(p, m) = 1$. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

This norm is non-Archimedean and satisfies the so-called strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$. Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\mathrm{ord}_p(x)}(x_0 + x_1 p + x_2 p^2 + \cdots), \tag{2.1}$$

where $\mathrm{ord}_p(x) \in \mathbb{Z}$ and the integers x_j satisfy: $0 \leq x_j \leq p - 1$, $x_0 \neq 0$. In this case, $|x|_p = p^{-\mathrm{ord}_p x}$.

Recall that $\mathbb{Q}_p$ is not an ordered field. So, we may compare two p-adic numbers only with respect to their p-adic norms.

In what follows, to simplify our calculations, we are going to introduce new symbols 'O' and 'o' (roughly speaking, these symbols replace the notation 'mod p^k' without noticing the power of k). Namely, for a given p-adic number x, by $O[x]$, we mean a p-adic number with the norm $p^{-\mathrm{ord}_p(x)}$, that is, $|x|_p = |O(x)|_p$. By $o[x]$, we mean a p-adic number with a norm strictly less than $p^{-\mathrm{ord}_p(x)}$, that is, $|o(x)|_p < |x|_p$. For instance, if $x = 1 - p + p^2$, we can write $x - 1 + p = o[p]$, $x - 1 = o[1]$, or $x = O[1]$. The symbols $O[\cdot]$ and $o[\cdot]$

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will make our work easier when we need to calculate the p-adic norm of p-adic numbers. It is easy to see that $y = O[x]$ if and only if $x = O[y]$.

We give some basic properties of $O[\cdot]$ and $o[\cdot]$, which will be used later on.

LEMMA 2.1. *Let $x, y \in \mathbb{Q}_p$. Then the following statements hold.*

(1) $O[x]O[y] = O[xy]$.
(2) $xO[y] = O[xy]$, $O[y]x = O[xy]$.
(3) $O[x]o[y] = o[xy]$.
(4) $o[x]o[y] = o[xy]$.
(5) $xo[y] = o[xy]$, $o[y]x = o[xy]$.
(6) *If $y \neq 0$, then $O[x]/O[y] = O[x/y]$.*
(7) *If $y \neq 0$, then $o[x]/O[y] = o[x/y]$.*

For each $a \in \mathbb{Q}_p$, $r > 0$, we denote

$$B_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$

We recall that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ and $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}$ are the set of all *p-adic integers* and *p-adic units*, respectively.

The following result is well known as Hensel's lemma.

LEMMA 2.2. [6, 22] *Let $F(x)$ be a polynomial whose coefficients are p-adic integers. Let x^* be a p-adic integer such that for some $i \geq 0$,*

$$F(x^*) \equiv 0 \pmod{p^{2i+1}}, \quad F'(x^*) \equiv 0 \pmod{p^i}, \quad F'(x^*) \not\equiv 0 \pmod{p^{i+1}}.$$

Then $F(x)$ has a p-adic integer root x_* such that $x_* \equiv x^* \pmod{p^{i+1}}$.

The *p-adic exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for every $x \in B_{p^{-1/(p-1)}}(0)$. Denote

$$\mathcal{E}_p = \{x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)}\}.$$

This set is the range of the p-adic exponential function. The following fact is well known.

LEMMA 2.3. [40] *The set $\mathcal{E}_p$ has the following properties.*

(a) *$\mathcal{E}_p$ is a group under multiplication.*
(b) *If $a, b \in \mathcal{E}_p$, then the following are true:*

$$|a - b|_p < \begin{cases} \frac{1}{2}, & p = 2, \\ 1, & p \neq 2, \end{cases} \qquad |a + b|_p = \begin{cases} \frac{1}{2}, & p = 2, \\ 1, & p \neq 2. \end{cases}$$

(c) *If $a \in \mathcal{E}_p$, then there is an element $h \in B_{p^{-1/(p-1)}}(0)$ such that $a = \exp_p(h)$.*

LEMMA 2.4. *Let $k \geq 2$ and $\alpha, \beta \in \mathcal{E}_p$. Then there exists a unique $\gamma \in 1 + p\mathbb{Z}_p$ such that*

$$\sum_{j=0}^{k-1} \alpha^{k-j-1}\beta^j = k\gamma. \tag{2.2}$$

Moreover, if $p \neq 2$, then $\gamma \in \mathcal{E}_p$.

Remark 2.5. We notice that Lemma 2.4 has been proved in [35] for $p \neq 2$. The proof of the case $p = 2$ is similar to that one. We notice that this lemma plays a crucial role in our further investigations. Especially, we will often use the fact $\gamma \in \mathcal{E}_p$.

COROLLARY 2.6. *Let $k \in \mathbb{N}$. Then*

$$\alpha^k - \beta^k = k(\alpha - \beta) + o[k(\alpha - \beta)] \quad \text{for all } \alpha, \beta \in \mathcal{E}_p.$$

Proof. Let $\alpha, \beta \in \mathcal{E}_p$. By Lemma 2.4,

$$\sum_{j=0}^{k-1} \alpha^{k-1-j}\beta^j = k + k(\gamma - 1),$$

where $\gamma - 1 = o[1]$.

Hence, Lemma 2.1 implies

$$\begin{aligned}\alpha^k - \beta^k &= (\alpha - \beta)\sum_{j=0}^{k-1} \alpha^{k-j-1}\beta^j \\ &= k(\alpha - \beta) + k(\alpha - \beta)(\gamma - 1) \\ &= k(\alpha - \beta) + O[k(\alpha - \beta)]o[1] \\ &= k(\alpha - \beta) + o[k(\alpha - \beta)],\end{aligned}$$

which is the required relation. □

Remark 2.7. In our further investigations, we mainly use Corollary 2.6 in the following form. Namely, for $k \in \mathbb{N}$,

$$\alpha^k - 1 = k(\alpha - 1) + o[k(\alpha - 1)] \quad \text{for all } \alpha \in \mathcal{E}_p. \tag{2.3}$$

We notice that a monomial equation $x^k = a$ over $\mathbb{Q}_p$ has been studied in [36, 38]. In our further investigations, we only need the following special case of that equation.

THEOREM 2.8. [36] *Let $p \geq 3$ and $a \in \mathcal{E}_p$. Then the following statements hold:*

(i) *if $|k|_p \leq |a - 1|_p$, then the polynomial $x^k - a$ has no root;*
(ii) *if $|k|_p > |a - 1|_p$, then for every $\xi \in \{y \in \mathbb{F}_p : y^k \equiv a(\text{mod } p)\}$, the polynomial $x^k - a$ has a unique root in $B_1(\xi)$.*

Here $\mathbb{F}_p$ stands for the ring of integers modulo p.

Remark 2.9. Thanks to Theorem 2.8, for every $a \in \mathcal{E}_p$ with $|a - 1|_p < |k|_p$, the equation $x^k = a$ has a single root belonging to $\mathcal{E}_p$, which is called *the principal kth root* and denoted

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by $\sqrt[k]{a}$. In what follows, the symbol $\sqrt[k]{a}$ (for $a \in \mathcal{E}_p$) always means the principal kth root of a. Therefore, for $|a-1|_p < |k|_p$, all solutions of the monomial equation $x^k = a$ have the following form: $x_i = \xi_i \sqrt[k]{a}$, where $\xi_i^k = 1$ and $\sqrt[k]{a}$ is a principal kth root of a.

2.2. *p-adic subshift.* Let $f : X \to \mathbb{Q}_p$ be a map from a compact open set X of $\mathbb{Q}_p$ into $\mathbb{Q}_p$. We assume that (i) $f^{-1}(X) \subset X$; (ii) $X = \cup_{j\in I} B_r(a_j)$ can be written as a finite disjoint union of balls of centers a_j and of the same radius r such that for each $j \in I$, there is an integer $\tau_j \in \mathbb{Z}$ such that

$$|f(x) - f(y)|_p = p^{\tau_j} |x-y|_p, \quad x, y \in B_r(a_j). \tag{2.4}$$

For such a map f, define its Julia set by

$$J_f = \bigcap_{n=0}^{\infty} f^{-n}(X). \tag{2.5}$$

It is clear that $f^{-1}(J_f) = J_f$ and then $f(J_f) \subset J_f$. The triple (X, J_f, f) is called a *p-adic weak repeller* if all τ_j in (2.4) are non-negative, but at least one is positive. We call it a *p-adic repeller* if all τ_j in (2.4) are positive. For any $i \in I$, we let

$$I_i := \{j \in I : B_r(a_j) \cap f(B_r(a_i)) \neq \emptyset\} = \{j \in I : B_r(a_j) \subset f(B_r(a_i))\}$$

(the second equality holds because of the expansiveness of the ultrametric property). Then define a matrix $A = (a_{ij})_{I\times I}$, called *incidence matrix*, as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } j \in I_i, \\ 0 & \text{if } j \notin I_i. \end{cases}$$

If A is irreducible, we say that (X, J_f, f) is *transitive*. Here the irreducibility of A means that for any pair $(i, j) \in I \times I$, there is a positive integer m such that $a_{ij}^{(m)} > 0$, where $a_{ij}^{(m)}$ is the entry of the matrix A^m.

Given I and the irreducible incidence matrix A as above, we denote

$$\Sigma_A = \{(x_k)_{k\geq 0} : \ x_k \in I, \ A_{x_k, x_{k+1}} = 1, \ k \geq 0\},$$

which is the corresponding subshift space, and let σ be the shift transformation on Σ_A. We equip Σ_A with a metric d_f depending on the dynamics, which is defined as follows. First, for $i, j \in I, \ i \neq j$, let $\kappa(i, j)$ be the integer such that $|a_i - a_j|_p = p^{-\kappa(i,j)}$. It is clear that $\kappa(i, j) < \log_p(r)$. By the ultrametric inequality,

$$|x-y|_p = |a_i - a_j|_p \quad i \neq j \quad \text{for all } x \in B_r(a_i), \ \text{for all } y \in B_r(a_j).$$

For $x = (x_0, x_1, \ldots, x_n, \ldots) \in \Sigma_A$ and $y = (y_0, y_1, \ldots, y_n, \ldots) \in \Sigma_A$, define

$$d_f(x, y) = \begin{cases} p^{-\tau_{x_0} - \tau_{x_1} - \cdots - \tau_{x_{n-1}} - \kappa(x_n, y_n)} & \text{if } n \neq 0, \\ p^{-\kappa(x_0, y_0)} & \text{if } n = 0 \end{cases}$$

where $n = n(x, y) = \min\{i \geq 0 : x_i \neq y_i\}$. It is clear that d_f defines the same topology as the classical metric which is defined by $d(x, y) = p^{-n(x,y)}$.

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THEOREM 2.10. [9] *Let (X, J_f, f) be a transitive p-adic weak repeller with incidence matrix A. Then the dynamics $(J_f, f, |\cdot|_p)$ is isometrically conjugate to the shift dynamics (Σ_A, σ, d_f).*

3. *Proof of Theorem 1.1: part (A)*

In what follows, we always assume that $p \geq 3$ and $|q|_p < 1$. To prove Theorem 1.1(A), we need the following auxiliary lemma.

LEMMA 3.1. *Let $p \geq 3$ and $k \in \mathbb{N}$. If $a \in \mathcal{E}_p$ and $|a-1|_p \geq |k|_p$, then $|x^k - a|_p \geq |a - 1|_p$ for any $x \in \mathbb{Q}_p$.*

Proof. Take an arbitrary $a \in \mathcal{E}_p$ such that $|a-1|_p \geq |k|_p$. We distinguish three cases.

Case $x \notin \mathbb{Z}_p^*$. Then we immediately get $|x^k - 1|_p \geq 1$. From $|a-1|_p < 1$, using the strong triangle inequality, one has $|x^k - a|_p \geq 1$. This yields that $|x^k - a|_p > |a-1|_p$.

Case $x \in \mathcal{E}_p$. Then noting $|x-1|_p < 1$, owing to Corollary 2.6, we obtain $|x^k - 1|_p < |k|_p$. The last one together with $|a-1|_p \geq |k|_p$ implies $|x^k - a|_p = |a-1|_p$.

Case $x \in \mathbb{Z}_p^* \setminus \mathcal{E}_p$. In this case, x has the following canonical form:

$$x = x_0 + x_1 \cdot p + x_2 \cdot p^2 + \cdots,$$

where $2 \leq x_0 \leq p-1$ and $0 \leq x_i \leq p-1$, $i \geq 1$. Then $(x/x_0) \in \mathcal{E}_p$. According to Remark 2.7,

$$\left(\frac{x}{x_0}\right)^k - 1 = O[k(x - x_0)] = o[k].$$

Consequently, $|x^k - x_0^k|_p < |k|_p$, which yields $|x^k - 1|_p = |x_0^k - 1|_p$. Now we need to check two subcases, $|x_0^k - 1|_p = 1$ and $|x_0^k - 1|_p < 1$, separately.

Suppose that $|x_0^k - 1|_p = 1$. Then, owing to $|a-1|_p < 1$, one has $|x_0^k - a|_p = 1$. Hence, $|x^k - a|_p > |a-1|_p$.

Let us assume that $|x_0^k - 1|_p < 1$. For convenience, let us write $k = mp^s$, where $s \geq 1$ and $(m, p) = 1$. Then noting $x_0^p \equiv x_0 (\mathrm{mod}\ p)$, from $x^{mp^s} \equiv 1 \ (\mathrm{mod}\ p)$, we obtain $|x_0^m - 1|_p < 1$. Thanks to Remark 2.7, one finds

$$x_0^{mp^s} - 1 = p^s(x_0^m - 1) + o[p^s(x_0^m - 1)],$$

which yields $|x_0^k - 1|_p < |k|_p$. Hence, from $|a-1|_p \geq |k|_p$, it follows that $|x_0^k - a|_p = |a-1|_p$. Consequently, $|x^k - a|_p = |a-1|_p$. This completes the proof. □

Remark 3.2. We notice that the set $\mathcal{P}_{x^{(\infty)}}$ is empty if $|k|_p \leq |q + \theta - 1|_p$. Indeed, from $x^{(\infty)} \in \mathcal{E}_p$, where $x^{(\infty)} = 2 - q - \theta$ and $|x^{(\infty)} - 1|_p \geq |k|_p$, owing to Lemma 3.1, we infer that

$$|f_{\theta,q,k}^n(x) - x^{(\infty)}|_p \geq |x^{(\infty)} - 1|_p \quad \text{for all } n \in \mathbb{N}, \text{ for all } x \in \mathrm{Dom}(f_{\theta,q,k}).$$

Hence, noting $x^{(\infty)} \neq 1$, we can conclude that $\mathcal{P}_{x^{(\infty)}} = \varnothing$.

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Let us define

$$g_{\theta,q}(x) = \frac{\theta x + q - 1}{x + \theta + q - 2}. \tag{3.1}$$

In our further investigations, we use the following simple property of the function $g_{\theta,q}$:

$$g_{\theta,q}(x) - 1 = \frac{(\theta - 1)(x - 1)}{x + \theta + q - 2}. \tag{3.2}$$

We notice that $f_{\theta,q,k}(x) = (g_{\theta,q}(x))^k$ for any $x \in \mathrm{Dom}(f_{\theta,q,k})$. It is clear that the function $f_{\theta,q,k}$ has a fixed point $x_0^* = 1$.

Proof of Theorem 1.1: (A). Let $|k|_p \le |q + \theta - 1|_p$ and denote

$$K_1 = \{x \in \mathbb{Q}_p : |x - 1|_p < |q + \theta - 1|_p\},$$

$$K_2 = \{x \in \mathbb{Q}_p : |x - 1|_p = |x - 2 + q + \theta|_p\}.$$

First, we show that $f_{\theta,q,k}(x) \in K_1 \cup K_2$ for any $x \notin K_1 \cup K_2$. Then we prove that $f_{\theta,q,k}(x) \in K_1$ for any $x \in K_2$. Finally, we show that $f^n_{\theta,q,k}(x) \to 1$ for any $x \in K_1$.

Indeed, let $x \notin K_1 \cup K_2$. From $|q + \theta - 1|_p < 1$, owing to Lemma 3.1,

$$|f_{\theta,q,k}(x) - 2 + q + \theta|_p \ge |q + \theta - 1|_p,$$

which is equivalent to either $|f_{\theta,q,k}(x) - 1|_p < |q + \theta - 1|_p$ or $|f_{\theta,q,k}(x) - 1|_p = |f_{\theta,q,k}(x) - 2 + q + \theta|_p$. This yields that $f_{\theta,q,k}(x) \in K_1 \cup K_2$.

Now assume that $x \in K_2$. Then by (3.2),

$$g_{\theta,q}(x) - 1 = (\theta - 1)O[1] = O[\theta - 1] = o[1]$$

which means $g_{\theta,q}(x) \in \mathcal{E}_p$. Then thanks to Remark 2.7,

$$|f_{\theta,q,k}(x) - 1|_p < |k|_p.$$

The last one together with $|k|_p \le |q + \theta - 1|_p$ implies $|f_{\theta,q,k}(x) - 1|_p < |q + \theta - 1|_p$ and hence $f_{\theta,q,k}(x) \in K_1$.

Finally, we suppose that $x \in K_1$. It then follows from (3.2) that

$$g_{\theta,q}(x) - 1 = \frac{(\theta - 1)(x - 1)}{O[q + \theta - 1]} = (\theta - 1)o[1] = o[\theta - 1] = o[1].$$

This again means $g_{\theta,q}(x) \in \mathcal{E}_p$. By Remark 2.7,

$$f_{\theta,q,k}(x) - 1 = O\left[\frac{k(\theta - 1)(x - 1)}{q + \theta - 1}\right].$$

Noting $|q + \theta - 1|_p > |k(\theta - 1)|_p$, from the last one,

$$|f_{\theta,q,k}(x) - 1|_p < |x - 1|_p.$$

Hence,

$$|f^n_{\theta,q,k}(x) - 1|_p \le \frac{1}{p^n}|x - 1|_p,$$

which yields $f^n_{\theta,q,k}(x) \to 1$ as $n \to \infty$. This completes the proof. □

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4. *Proof of Theorem 1.1: the first part of (B)*

In this section, we are going to study the dynamics of $f_{\theta,q,k}$ when $|\theta-1|_p<|q^2|_p$ and $|q|_p<|k|_p$. In what follows, the following auxiliary fact is needed.

PROPOSITION 4.1. *Let $p\geq 3$ and $|\theta-1|_p<|q|_p<|k|_p$. If $x\in \mathrm{Dom}(f_{\theta,q,k})$ with $|x-2+q+\theta|_p>|\theta-1|_p$, then $f^n_{\theta,q,k}(x)\to 1$ as $n\to\infty$.*

Proof. First, we notice that $|x-2+q+\theta|_p>|\theta-1|_p$ implies $|x-1+q|_p>|\theta-1|_p$. Owing to $|\theta-1|_p<|q|_p$, we are going to consider two cases: (i) $|x-1+q|_p\geq|q|_p$ and (ii) $|\theta-1|_p<|x-1+q|_p<|q|_p$.

Case (i). Let $|x-1+q|_p\geq|q|_p$. This means that either $x\in B_{|q|_p}(1)$ or $|x-1+q|_p=|x-1|_p$. First, we show that the condition $|x-1+q|_p=|x-1|_p$ yields $f_{\theta,q,k}(x)\in B_{|q|_p}(1)$. Furthermore, we establish that $f^n_{\theta,q,k}(x)\to 1$ for any $x\in B_{|q|_p}(1)$.

Let us pick $x\in\mathbb{Q}_p$ such that $|x-1+q|_p=|x-1|_p$. Then $|q|_p\leq|x-1|_p$. Keeping in mind $\theta-1=o[q]$, one finds $\theta-1=o[x-1+q]$ and

$$x+\theta+q-2=O[x-1+q]=O[x-1].$$

So, by (3.2),

$$g_{\theta,q}(x)-1=\frac{(\theta-1)(x-1)}{O[x-1]}=o[q]O[1]=o[q].$$

Because $|k|_p\leq 1$, owing to Remark 2.7, we obtain $|f_{\theta,q,k}(x)-1|_p<|q|_p$, which implies $f_{\theta,q,k}(x)\in B_{|q|_p}(1)$.

Now let us suppose that $x\in B_{|q|_p}(1)$. Then by (3.2),

$$g_{\theta,q}(x)-1=\frac{o[q](x-1)}{q+o[q]}=\frac{o[q](x-1)}{O[q]}=o[1](x-1)=o[x-1].$$

Hence, again thanks to Remark 2.7, one has $|f_{\theta,q,k}(x)-1|_p<|x-1|_p$, which yields

$$|f^n_{\theta,q,k}(x)-1|_p\leq\frac{1}{p^n}|x-1|_p\quad\text{for all } n\in\mathbb{N}.$$

So, $f^n_{\theta,q,k}(x)\to 1$ as $n\to\infty$.

Case (ii). Let $|\theta-1|_p<|x-1+q|_p<|q|_p$. Then

$$g_{\theta,q}(x)-1=\frac{o[x-1+q]O[q]}{O[x-1+q]}=o[1]O[q]=o[q].$$

Again, Remark 2.7 yields $|f_{\theta,q,k}(x)-1|_p<|q|_p$. Hence, by (i), we have $f^n_{\theta,q,k}(x)\to 1$ as $n\to\infty$. This completes the proof. □

COROLLARY 4.2. *Let $p\geq 3$ and $|\theta-1|_p<|q|_p<|k|_p$. If $|x-1+q|_p\geq|q|_p$, then $f^n_{\theta,q,k}(x)\to 1$ as $n\to\infty$.*

Proof. Let $|x-1+q|_p\geq|q|_p$. By $|\theta-1|_p<|q|_p$ and the strong triangle inequality, one finds $|x-2+q+\theta|_p>|\theta-1|_p$. Hence, the last one owing to Proposition 4.1 yields $f^n_{\theta,q,k}(x)\to 1$. □

LEMMA 4.3. *Let $p \geq 3$ and $|\theta - 1|_p < |q|_p < |k|_p$. If $|x - 2 + q + \theta|_p < |q(\theta - 1)|_p$, then $f^n_{\theta,q,k}(x) \to 1$ as $n \to \infty$.*

Proof. Take arbitrary $x \in \mathrm{Dom}(f_{\theta,q,k})$ such that $|x - 2 + q + \theta|_p < |q(\theta - 1)|_p$. Then

$$\frac{\theta x + q - 1}{x + q + \theta - 2} = \theta - \frac{(q + \theta - 1)(\theta - 1)}{x + q + \theta - 2} = \theta + \frac{O[q(\theta - 1)]}{o[q(\theta - 1)]} = \frac{O[q(\theta - 1)]}{o[q(\theta - 1)]},$$

which yields $|f_{\theta,q,k}(x)|_p > 1$. Hence, $|f_{\theta,q,k}(x) - 2 + q + \theta|_p > |\theta - 1|_p$. Then by Proposition 4.1, we obtain the desired assertion. □

Our aim is to construct a set $X \subset \mathrm{Dom}(f_{\theta,q,k})$ for which a triple $(X, J_{f_{\theta,q,k}}, f_{\theta,q,k})$ is a transitive p-adic repeller. Thanks to Proposition 4.1 and Lemma 4.3, the required set X should be a subset of the following set:

$$Y = \{x \in U : \ |q(\theta - 1)|_p \leq |f_{\theta,q,k}(x) - 2 + q + \theta|_p \leq |\theta - 1|_p\},$$

where

$$U := \bigcup_{\substack{\eta \in \mathbb{Q}_p: \\ |q|_p \leq |\eta|_p \leq 1}} B_{|q(\theta-1)|_p}(2 - q - \theta + \eta(\theta - 1)).$$

One can see that for $|q|_p \leq |\eta|_p \leq 1$, we have $x_\eta \in Y$ if and only if

$$\begin{cases} x_\eta = 2 - q - \theta + \eta(\theta - 1) + o[q(\theta - 1)], \\ |q(\theta - 1)|_p \leq \left| f_{\theta,q,k}(x_\eta) - 2 + q + \theta \right|_p \leq |\theta - 1|_p. \end{cases} \tag{4.1}$$

Remark 4.4. We notice that if for $|q|_p \leq |\eta|_p \leq 1$ one of the assumptions of (4.1) does not hold, then $f^n_{\theta,q,k}(x_\eta) \to 1$ as $n \to \infty$.

Now we are going to find a necessary condition for $\eta \in \mathbb{Q}_p$ which yields (4.1).

PROPOSITION 4.5. *Let $p \geq 3$, $|k|_p > |q|_p$ and $|\theta - 1|_p < |q^2|_p$. Assume that for $\eta \in \mathbb{Q}_p$ with $|q|_p \leq |\eta|_p \leq 1$, (4.1) holds. Then the following statements are true:*

(1_η) *If $|\eta|_p = |q|_p$, then $((\eta - q)/\eta)^k = 1 + o[1]$;*

(2_η) *If $|\eta|_p > |q|_p$, then $\eta = k - (((k - 1)q)/2) + (((k - 1)(k - 2)q^2)/6k) + o[q]$.*

Proof. Assume that (4.1) holds. Then

$$f_{\theta,q,k}(x_\eta) = 2 - q + \theta + O[\theta - 1] = 1 - q + o[q] = 1 + o[1]. \tag{4.2}$$

(1_η) Let $|\eta|_p = |q|_p$. By (4.2), one finds

$$\left(1 - \frac{q}{\eta} + \frac{(\eta - 1)(\theta - 1)}{\eta}\right)^k = 1 + o[1]. \tag{4.3}$$

Noting $|\theta - 1|_p < |q|_p$, we obtain $((\eta - 1)(\theta - 1))/\eta = o[1]$. Plugging the last one into (4.3),

$$\left(\frac{\eta - q}{\eta} + o[1]\right)^k = 1 + o[1]. \tag{4.4}$$

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Finally, keeping in mind $|k|_p \le 1$, from (4.4),

$$\left(\frac{\eta - q}{\eta}\right)^k = 1 + o[1].$$

(2_η) Let $|\eta|_p > |q|_p$. First, let us assume that $|k|_p \le |\eta - k|_p$. Then, using the strong triangle inequality, we can easily check

$$\frac{k}{\eta} \neq 1 + o[1]. \tag{4.5}$$

From $|\theta - 1|_p < |q^2|_p$ and $|q|_p < |\eta|_p$,

$$g_{\theta,q}(x_\eta) = 1 - \frac{q}{\eta} + o[q].$$

Keeping in mind $f_{\theta,q,k}(x_\eta) = (g_{\theta,q}(x_\eta))^k$, by (2.3),

$$f_{\theta,q,k}(x_\eta) = 1 - \frac{kq}{\eta} + o\left[\frac{kq}{\eta}\right]. \tag{4.6}$$

Plugging (4.5) into (4.6) yields

$$f_{\theta,q,k}(x_\eta) - 1 \neq -q + o[q],$$

but it contracts to $f_{\theta,q,k}(x_\eta) - 1 + q = o[q]$. This means that $|\eta|_p > |q|_p$ and (4.1) hold only for $|\eta - k|_p < |k|_p$.

So, suppose $|\eta - k|_p < |k|_p$, which implies $|\eta|_p = |k|_p$. Now we prove our assertion by contradiction. Suppose in contrary,

$$\left|\eta - k + \frac{(k-1)q}{2} - \frac{(k-1)(k-2)q^2}{6k}\right|_p \ge |q|_p. \tag{4.7}$$

Noting $|q|_p < |k|_p \le 1$, we then can easily check the following:

$$\left|\frac{(k-1)q}{2}\right|_p \le |q|_p,$$
$$\left|\frac{(k-1)(k-2)q^2}{6k}\right|_p \le |q|_p.$$

These inequalities together with (4.7) yield

$$\left|\eta - k + \frac{(k-1)q}{2} - \frac{(k-1)(k-2)q^2}{6k}\right|_p = \max\{|\eta - k|_p, |q|_p\}. \tag{4.8}$$

Owing to $|\theta - 1|_p < |q^2|_p$ and $|\eta|_p = |k|_p$, we have

$$\begin{aligned} f_{\theta,q,k}(x_\eta) &= \left(1 - \frac{q}{\eta} + \frac{(\eta-1)(\theta-1)}{\eta}\right)^k \\ &= \left(1 - \frac{q}{\eta} + o\left[\frac{q^2}{k}\right]\right)^k \end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{q}{\eta}\right)^k + o\left[\frac{q^2}{k}\right] \\
&= \left(1 - \frac{q}{k}\sum_{n=0}^{\infty}\left(\frac{k-\eta}{k}\right)^n\right)^k + o\left[\frac{q^2}{k}\right] \\
&= 1 - q\sum_{n=0}^{\infty}\left(\frac{k-\eta}{k}\right)^n + \frac{(k-1)q^2}{2k} - \frac{(k-1)(k-2)q^3}{6k^2} + o\left[\frac{q^2}{k}\right] \\
&= 1 - q + \frac{q}{k}\left(\eta - k + \frac{(k-1)q}{2} - \frac{(k-1)(k-2)q^2}{6k}\right) \\
&\quad - q\sum_{n=2}^{\infty}\left(\frac{k-\eta}{k}\right)^n + o\left[\frac{q^2}{k}\right].
\end{aligned} \tag{4.9}
$$

We calculate the norm of $q\sum_{n=2}^{\infty}((k-\eta)/k)^n$. Keeping in mind $|\eta - k|_p < |k|_p$, by the strong triangle inequality,

$$
\left| q\sum_{n=2}^{\infty}\left(\frac{k-\eta}{k}\right)^n \right|_p = \left|\frac{q(k-\eta)^2}{k^2}\right|_p. \tag{4.10}
$$

So, we need to calculate the norm of $(q(k-\eta)^2)/k^2$. One can see that

$$
\left|\frac{(k-\eta)^2}{k}\right|_p < |\eta - k|_p \le \max\{|\eta - k|_p, |q|_p\}.
$$

The last inequality together with (4.10) yields

$$
\left| q\sum_{n=2}^{\infty}\left(\frac{k-\eta}{k}\right)^n \right|_p < \left|\frac{q}{k}\right|_p \cdot \max\{|\eta - k|_p, |q|_p\}. \tag{4.11}
$$

Then by (4.8) and (4.11), using the strong triangle inequality, one finds

$$
\begin{aligned}
&\left|\frac{q}{k}\left(\eta - k + \frac{(k-1)q}{2} - \frac{(k-1)(k-2)q^2}{6k}\right) - q\sum_{n=2}^{\infty}\left(\frac{k-\eta}{k}\right)^n\right|_p \\
&\quad = \left|\frac{q}{k}\right|_p \cdot \max\{|\eta - k|_p, |q|_p\}.
\end{aligned}
$$

From the last equality,

$$
\left|\frac{q}{k}\left(\eta - k + \frac{(k-1)q}{2} - \frac{(k-1)(k-2)q^2}{6k}\right) - q\sum_{n=2}^{\infty}\left(\frac{k-\eta}{k}\right)^n\right|_p \ge \frac{|q^2|_p}{|k|_p}. \tag{4.12}
$$

Hence, plugging (4.12) into (4.9) and noting $|k|_p \le 1$, one finds

$$
|f_{\theta,q,k}(x_\eta) - 1 + q|_p \ge |q^2|_p,
$$

which together with $|\theta-1|_p<|q^2|_p$ implies $|f_{\theta,q,k}(x_\eta)-2+q+\theta|_p>|\theta-1|_p$, which contradicts (4.1). This means that if for $|\eta|_p>|q|_p$, (4.1) holds, then

$$\eta=k-\frac{(k-1)q}{2}+\frac{(k-1)(k-2)q^2}{6k}+o[q]. \qquad \square$$

Remark 4.6. One can see that if $|\eta|_p=|q|_p$ and $((\eta-q)/\eta)^k\in\mathcal{E}_p$, then $((\eta-q)/\eta)\in\mathbb{Z}_p^*\setminus\mathcal{E}_p$. This means that there exists a root of unity $\xi\neq 1$ such that $(\eta-q)/\eta=\xi+o[1]$, which yields $\eta=q/(1-\xi)+o[q]$. Without loss of generality for $\xi=1$, we put $\eta=k-(((k-1)q)/2)+(((k-1)(k-2)q^2)/6k)+o[q]$. Consequently, we have found a relation between all roots of unity and all $\eta\in\mathbb{Q}_p$ for which (4.1) holds.

Let us denote

$$\mathrm{Sol}_p(x^k-1)=\{\xi\in\mathbb{Z}_p^*:\xi^k=1\},\quad \kappa_p=\mathrm{card}(\mathrm{Sol}_p(x^k-1)),$$

where $\mathrm{card}(A)$ is the cardinality of a set A.

We point out that κ_p is the number of solutions of the equation $x^k=1$ in $\mathbb{Q}_p$. From the results of [38], we infer that κ_p is the GCF of k and $p-1$. Therefore, it is clear that $1\le\kappa_p\le k$.

For a given $\xi_i\in\mathrm{Sol}_p(x^k-1)$, $i\in\{1,\dots,\kappa_p\}$, we define

$$x_{\xi_i}=\begin{cases}1-q+(k-1)\left(1-\dfrac{q}{2}+\dfrac{(k-2)q^2}{6k}\right)(\theta-1) & \text{if } \xi_i=1,\\[2ex] 2-q-\theta+\dfrac{q}{1-\xi_i}(\theta-1) & \text{if } \xi_i\neq 1,\end{cases} \tag{4.13}$$

and

$$X=\bigcup_{i=1}^{\kappa_p}B_r(x_{\xi_i}),\quad r=|q(\theta-1)|_p. \tag{4.14}$$

By construction, the set X is a subset of $\mathcal{E}_p\setminus\{1\}$.

Thanks to Remark 4.4, as a corollary of Proposition 4.5, we can formulate the following result which describes the basin of attraction of $x_0^*=1$.

PROPOSITION 4.7. *Let $p\ge 3$ and $|k|_p>|q|_p$. If $|\theta-1|_p<|q^2|_p$, then*

$$\lim_{n\to\infty}f^n_{\theta,q,k}(x)=1\quad \text{for all } x\in\mathrm{Dom}(f_{\theta,q,k})\setminus X.$$

The next result shows that the set X (given by (4.14)) consists of disjoint balls.

LEMMA 4.8. *Let $p\ge 3$ and $|\theta-1|_p<|q^2|_p<|k^2|_p$. If x_{ξ_i} is given by (4.13) and $r=|q(\theta-1)|_p$, then $B_r(x_{\xi_i})\cap B_r(x_{\xi_j})=\varnothing$ if $i\neq j$.*

Proof. Let x_{ξ_i} and x_{ξ_j} be given by (4.13), where $i \neq j$. We consider two cases.

Case $\xi_i = 1$ and $\xi_j \neq 1$. Then from (4.13),

$$x_{\xi_i} - x_{\xi_j} = \left(k - \frac{(k-1)q}{2} + \frac{(k-1)(k-2)q^2}{6k} - \frac{q}{1-\xi_j}\right)(\theta - 1)$$
$$= (k + o[k])(\theta - 1)$$
$$= O[k(\theta - 1)],$$

which implies that $|x_{\xi_i} - x_{\xi_j}|_p > |q(\theta - 1)|_p$. Hence, $B_r(x_{\xi_i}) \cap B_r(x_{\xi_j}) = \varnothing$.

Case $\xi_i \neq 1$ and $\xi_j \neq 1$. In this case,

$$x_{\xi_i} - x_{\xi_j} = \frac{(\xi_i - \xi_j)q(\theta - 1)}{(1-\xi_i)(1-\xi_j)} = \frac{O[1]q(\theta-1)}{O[1]} = O[q(\theta - 1)],$$

which means $|x_{\xi_i} - x_{\xi_j}|_p = r$. Hence, we infer that $B_r(x_{\xi_i}) \cap B_r(x_{\xi_j}) = \varnothing$. □

To prove the first part of (B) of Theorem 1.1, we define the following set:

$$J_{f_{\theta,q,k}} = \bigcap_{n=1}^{\infty} f_{\theta,q,k}^{-n}(X). \tag{4.15}$$

Remark 4.9. In [36], we have considered the following function over $\mathbb{Q}_p$ $(p \geq 3)$:

$$f_{b,c,d}(x) = \left(\frac{bx - c}{x - d}\right)^k, \quad b, c, d \in \mathcal{E}_p,\ c \neq bd.$$

It was proved that the mapping $f_{b,c,d}$ has exactly $\kappa_p + 1$ fixed points belonging to $\mathcal{E}_p$ if $d = 1 - b + c$ and $|b-1|_p < |c-1|_p^2 < |k|_p^2$ (see [36, Theorem 4.5]). One can see that if one takes $b = \theta$, $c = 1 - q$, and $d = 2 - q - \theta$, then the function $f_{b,c,d}$ reduces to $f_{\theta,q,k}$. So, as a corollary of the mentioned result and noting that $\mathrm{Fix}(f_{\theta,q,k}) \cap (\mathbb{Q}_p \setminus X) = \{1\}$, we conclude that if $|\theta - 1|_p < |q|_p^2 < |k|_p^2$, then $f_{\theta,q,k}$ has exactly κ_p fixed points belonging to X. This yields $J_{f_{\theta,q,k}} \neq \varnothing$ for $|\theta - 1|_p < |q|_p^2 < |k|_p^2$. Moreover, we may check that for every $i \in \{1, 2, \ldots, \kappa_p\}$, there exists a unique fixed point of $f_{\theta,q,k}$ in $B_r(x_{\xi_i})$ (see Proposition 5.5).

Proof of Theorem 1.1: (B). By Proposition 4.7, the set $\mathcal{P}_{x^{(\infty)}}$ can not belong to $\mathrm{Dom}(f_{\theta,q,k}) \setminus X$. Then $\mathcal{P}_{x^{(\infty)}} \subset X$. According to the construction of $J_{f_{\theta,q,k}}$ (see (4.15)), we conclude that $J_{f_{\theta,q,k}} \cap \mathcal{P}_{x^{(\infty)}} = \varnothing$. However, owing to Remark 4.9, the set $J_{f_{\theta,q,k}}$ is not empty and by the construction, it is invariant with respect to $f_{\theta,q,k}$. Then for any $x \notin J_{f_{\theta,q,k}} \cup \mathcal{P}_{x^{(\infty)}}$, there exists a number $m \geq 1$ such that $f_{\theta,q,k}^m(x) \notin X$. Hence, owing to Proposition 4.7, we infer that $f_{\theta,q,k}^n(x) \to 1$ as $n \to \infty$. The proof is complete. □

5. *Proof of Theorem 1.1: parts (B1) and (B2)*

In the following, we need some auxiliary facts.

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LEMMA 5.1. *Let $p \geq 3$ and $|k|_p > |q|_p$. Then for any $a \in B_{|q^2|_p}(1-q)$, the equation $x^k = a$ has a unique solution x_* on $\mathcal{E}_p$. Moreover, this solution satisfies*

$$x_* - 1 + \frac{q}{k} + \frac{(k-1)q^2}{2k^2} - \frac{(k-1)(k-2)q^3}{6k^3} = o\left[\frac{q^2}{k^2}\right]. \tag{5.1}$$

Proof. Let $|k|_p > |q|_p$ and $a \in B_{|q^2|_p}(1-q)$. For convenience, we use the canonical form of a:

$$a = 1 + a_t p^t + a_{t+1} p^{t+1} + \cdots$$

We note that $|k|_p > p^{-t}$. Let us put $x_t = 1$ and define a sequence $\{x_{n+t-1}\}_{n\geq 1}$ as follows:

$$x_{n+t} = x_{n+t-1} + \frac{a - x_{n+t-1}^k}{k}. \tag{5.2}$$

First, by induction, let us show that $x_{n+t-1} \in \mathcal{E}_p$ for any $n \geq 1$. It is clear that $x_t \in \mathcal{E}_p$ and, therefore, we assume that $x_{n+t-1} \in \mathcal{E}_p$ for some $n \geq 1$. Then, owing to Remark 2.7, we obtain

$$x_{n+t-1}^k - 1 = k(x_{n+t-1} - 1) + o[k(x_{n+t-1} - 1)],$$

which is equivalent to

$$|x_{n+t-1}^k - 1|_p < |k|_p.$$

The last inequality together with $|a-1|_p < |k|_p$ implies that $|x_{n+t} - x_{n+t-1}|_p < 1$. Consequently, from $x_{n+t-1} \in \mathcal{E}_p$, we find $x_{n+t} \in \mathcal{E}_p$. Hence, $x_{n+t} \in \mathcal{E}_p$ for any $n \geq 1$.

Owing to Corollary 2.6, by (5.2), we have

$$\begin{aligned} x_{n+t}^k - x_{n+t-1}^k &= k(x_{n+t} - x_{n+t-1}) + o[k(x_{n+t} - x_{n+t-1})] \\ &= a - x_{n+t-1}^k + o[a - x_{n+t-1}], \end{aligned}$$

which means

$$|x_{n+t}^k - a|_p < |x_{n+t-1}^k - a|_p.$$

Hence, there exists a number $n_0 \geq 1$ such that

$$|x_{n_0+t}^k - a|_p \leq |(a-1)^2|_p.$$

Now, let us consider a polynomial $F(x) = x^k - a$. It is easy to check that

$$|F'(x_{n_0+t-1})|_p = |k|_p, \quad \text{and} \quad |F(x_{n_0+t-1})|_p \leq |(a-1)^2|_p.$$

So by $|k^2|_p > |(a-1)^2|_p$ and Hensel's lemma, we conclude that F has a root x_* such that

$$|x_* - x_{n_0+t-1}|_p \leq |(a-1)^2|_p.$$

From $x_{n_0+t-1} \in \mathcal{E}_p$, we infer that $x_* \in \mathcal{E}_p$. The uniqueness of the solution on $\mathcal{E}_p$ immediately follows from Remark 2.7.

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Suppose that $x_* \in \mathcal{E}_p$ is a solution of $x^k - a = 0$. Let us show that it can be represented by (5.1). It can be checked that x_* has the following form:

$$x_* = 1 - \frac{q}{k} + \alpha_*, \tag{5.3}$$

where $\alpha_* = o[q/k]$. Indeed, because $x_* \in \mathcal{E}_p$, there exists $y_* \in p\mathbb{Z}_p$ such that x_* can be represented as follows: $x_* = 1 + y_* + o[y_*]$. Then by Remark 2.7, we have $x_*^k = 1 + ky_* + o[ky_*]$. By assumption, $a = 1 - q + o[q^2]$. Hence, we obtain the following implications:

$$\begin{aligned} x_*^k - a = 0 \quad &\Longrightarrow \quad ky_* + q = o[q] \quad \Longrightarrow \quad y_* = -\frac{q}{k} + o\left[\frac{q}{k}\right] \\ &\Longrightarrow \quad x_* = 1 - \frac{q}{k} + o\left[\frac{q}{k}\right]. \end{aligned}$$

Furthermore, from (5.3), one finds

$$\begin{aligned} a = x_*^k &= 1 + k\left(-\frac{q}{k} + \alpha_*\right) + \frac{k(k-1)}{2}\left(-\frac{q}{k} + \alpha_*\right)^2 \\ &\quad + \frac{k(k-1)(k-2)}{6}\left(-\frac{q}{k} + \alpha_*\right)^3 + o\left[\frac{q^2}{k}\right] \\ &= 1 - q + k\alpha_* + \frac{(k-1)q^2}{2k} - \frac{(k-1)(k-2)q^3}{6k^2} + o\left[\frac{q^2}{k}\right]. \end{aligned}$$

Plugging $a = 1 - q + o[q^2]$ into the last equality,

$$k\alpha_* + \frac{(k-1)q^2}{2k} - \frac{(k-1)(k-2)q^3}{6k^2} = o\left[\frac{q^2}{k}\right].$$

Hence,

$$\alpha_* = -\frac{(k-1)q^2}{2k^2} + \frac{(k-1)(k-2)q^3}{6k^3} + o\left[\frac{q^2}{k^2}\right].$$

Putting the last one into (5.3) yields (5.1), which completes the proof. □

Remark 5.2. We point out that in [**38**], the existence of solutions of the equation $x^k = a$ on $\mathbb{Z}_p^*$ has been obtained, but an advantage of Lemma 5.1 is that it provides the uniqueness of solution in $\mathcal{E}_p$ with an explicit expression which is essential in our investigation.

On the set X (see (4.14)), the mapping $f_{\theta,q,k}$ has exactly κ_p inverse branches:

$$h_i(x) = \frac{(q+\theta-2)\xi_i \sqrt[k]{x} - q + 1}{\theta - \xi_i \sqrt[k]{x}},$$

where $\xi_i^k = 1$, $i \in \{1, \ldots, \kappa_p\}$ and $\sqrt[k]{x}$, as before, is a principal root of $x \in X$ (see Remark 2.9).

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PROPOSITION 5.3. *Let $p \geq 3$ and $|k|_p > |q|_p$. If $|\theta - 1|_p < |q^2|_p$, then*

$$h_i(X) \subset B_r(x_{\xi_i}). \tag{5.4}$$

Proof. Let $x \in X$. We consider two cases: $\xi_i = 1$ and $\xi_i \neq 1$.

Case $\xi_i = 1$. In this case, we have

$$\begin{aligned}
h_i(x) - x_{\xi_i} \\
&= \frac{(q+\theta-2)\sqrt[k]{x} - q + 1}{\theta - \sqrt[k]{x}} - \bigg(1 - q + (k-1)\bigg(1 - \frac{q}{2} + \frac{(k-2)q^2}{6k}\bigg)(\theta - 1)\bigg) \\
&= \frac{(\theta-1)(q+\theta-1+(k-(((k-1)q)/2)+(((k-1)(k-2)q^2)/6k))(\sqrt[k]{x}-\theta))}{\theta - \sqrt[k]{x}} \\
&= \frac{(\theta-1)(q+(k-(((k-1)q)/2)+(((k-1)(k-2)q^2)/6k))(\sqrt[k]{x}-1)+o[q^2])}{1-\sqrt[k]{x}+o[q^2]}.
\end{aligned} \tag{5.5}$$

However, owing to Lemma 5.1,

$$\sqrt[k]{x} - 1 = -\frac{q}{k} - \frac{(k-1)q^2}{2k^2} + \frac{(k-1)(k-2)q^3}{6k^3} + o\bigg[\frac{q^2}{k^2}\bigg]. \tag{5.6}$$

Furthermore, keeping in mind $|q|_p < |k|_p$, we can easily check the following:

$$q + \bigg(k - \frac{(k-1)q}{2} + \frac{(k-1)(k-2)q^2}{6k}\bigg)(\sqrt[k]{x} - 1) = o\bigg[\frac{q^2}{k}\bigg]. \tag{5.7}$$

Plugging (5.6), (5.7) into (5.5), one has

$$h_i(x) - x_{\xi_i} = \frac{(\theta-1)o[q^2/k]}{O[q/k]} = (\theta-1)o[q] = o[q(\theta-1)].$$

This means $h_i(x) \in B_r(x_{\xi_i})$. The arbitrariness of $x \in X$ yields (5.4).

Case $\xi_i \neq 1$. Then,

$$\begin{aligned}
h_i(x) - x_{\xi_i} &= \frac{(q+\theta-2)\xi_i\sqrt[k]{x} - q + 1}{\theta - \xi_i\sqrt[k]{x}} - \bigg(2 - q - \theta + \frac{q}{1-\xi_i}(\theta-1)\bigg) \\
&= \frac{(\theta-1)(q-(\theta q/(1-\xi_i))+(\xi_i\sqrt[k]{x}q/(1-\xi_i))+\theta-1)}{\theta-1+1-\xi_i\sqrt{x}} \\
&= \frac{(\theta-1)(q-(q/(1-\xi_i))+(\xi_i q/(1-\xi_i))+o[q])}{O[1]} \\
&= \frac{(\theta-1)o[q]}{O[1]} \\
&= o[q(\theta-1)].
\end{aligned}$$

The last one implies $h_i(x) \in B_r(x_{\xi_i})$. Again, owing to the arbitrariness of $x \in X$, we obtain (5.4). □

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COROLLARY 5.4. *Let $p \geq 3$ and $|k|_p > |q|_p$. If $|\theta - 1|_p < |q^2|_p$ and X is the set given by (4.14), then the following statements hold:*

(i) $f^{-1}_{\theta,q,k}(X) \subset X$;

(ii) $B_r(x_{\xi_i}) \subset f_{\theta,q,k}(B_r(x_{\xi_j}))$ *for any* $i, j \in \{1, \ldots, \kappa_p\}$.

PROPOSITION 5.5. *Let $p \geq 3$, $|k|_p > |q|_p$. If $|\theta - 1|_p < |q^2|_p$ and X is the set given by (4.14), then the following statements hold:*

(a) *if* $\xi_i = 1$, *then*

$$|f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p = \frac{|q(x-y)|_p}{|k(\theta-1)|_p} \quad \text{for any } x, y \in B_r(x_{\xi_i}); \tag{5.8}$$

(b) *if* $\xi_i \neq 1$, *then*

$$|f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p = \frac{|k(x-y)|_p}{|q(\theta-1)|_p} \quad \text{for any } x, y \in B_r(x_{\xi_i}). \tag{5.9}$$

Proof. (a) Recall that for $\xi_i = 1$,

$$x_{\xi_i} = 1 - q + (k-1)\bigg(1 - \frac{q}{2} + \frac{(k-2)q^2}{6k}\bigg)(\theta - 1).$$

Thus for $x \in B_r(x_{\xi_i})$, by (3.2), we have

$$g_{\theta,q}(x) - 1 = \frac{(\theta-1)(-q + o[q])}{k(\theta-1) + o[k(\theta-1)]} = O\bigg[\frac{q}{k}\bigg].$$

This means $g_{\theta,q}(x) \in \mathcal{E}_p$. Then, owing to Corollary 2.6, for any $x, y \in B_r(x_{\xi_i})$,

$$|f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p = |k(g_{\theta,q}(x) - g_{\theta,q}(y))|_p. \tag{5.10}$$

However,

$$\begin{aligned} g_{\theta,q}(x) - g_{\theta,q}(y) &= \frac{(\theta-1)(q+\theta-1)(x-y)}{(x-2+q+\theta)(y-2+q+\theta)} \\ &= \frac{O[q(\theta-1)](x-y)}{(k(\theta-1) + o[k(\theta-1)])^2} \\ &= \frac{O[q](x-y)}{O[k^2(\theta-1)]}. \end{aligned}$$

Plugging the last one into (5.10) implies (5.8).

(b) Recall that for $\xi_i \neq 1$,

$$x_{\xi_i} = 2 - q - \theta + \frac{q(\theta-1)}{1-\xi_i}.$$

Then for $x \in B_r(x_{\xi_i})$,

$$g_{\theta,q}(x) = 1 + \frac{(\theta-1)(x-1)}{x-2+q+\theta} = 1 + \frac{(\theta-1)(-q+o[q])}{(q(\theta-1))/(1-\xi_i) + o[q(\theta-1)]} = \xi_i + o[1].$$

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So, $|g_{\theta,q}(x)|_p = 1$. Moreover, $(g_{\theta,q}(x)/g_{\theta,q}(y)) \in \mathcal{E}_p$ for any $x, y \in B_r(x_{\xi_i})$. Then, owing to Corollary 2.6,

$$|f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p = |k(g_{\theta,q}(x) - g_{\theta,q}(y))|_p. \tag{5.11}$$

However, from

$$\begin{aligned} g_{\theta,q}(x) - g_{\theta,q}(y) &= \frac{(\theta-1)(q+\theta-1)(x-y)}{(x-2+q+\theta)(y-2+q+\theta)} \\ &= \frac{O[q(\theta-1)](x-y)}{((q(\theta-1))/(1-\xi_i)+o[q(\theta-1)])^2} \\ &= \frac{(x-y)}{O[q(\theta-1)]} \end{aligned}$$

and (5.11), we arrive at (5.9). □

Proof of Theorem 1.1. (B1) Assume that $x^k = 1$ has only one solution. Then the set X given by (4.14) consists of only one ball $B_r(x_1)$, where

$$x_1 = 1 - q + (k-1)\left(1 - \frac{q}{2}\right)(\theta - 1), \quad r = |q(\theta-1)|_p.$$

By the proof for the case (B),

$$A(x_0^*) = \mathrm{Dom}(f_{\theta,q,k}) \setminus (J_{f_{\theta,q,k}} \cup \mathcal{P}_{x^{(\infty)}}),$$

where $x_0^* = 1$, and $J_{f_{\theta,q,k}}$ is given by (4.15). By Proposition 5.5, for any $x, y \in B_r(x_1)$, we have

$$|f_{\theta,q,k}(x) - f_{\theta,q,k}(y)|_p > p^2|x-y|_p,$$

which implies that $|J_{f_{\theta,q,k}}| \le 1$. Thanks to Remark 4.9, we have $J_{f_{\theta,q,k}} \neq \varnothing$. Because $|J_{f_{\theta,q,k}}| \le 1$, one has $J_{f_{\theta,q,k}} = \{x_*\}$, where $x_* \in \mathrm{Fix}(f_{\theta,q,k}) \cap (\mathcal{E}_p \setminus \{1\})$.

(B2) Assume that $x^k = 1$ has κ_p ($\kappa_p \ge 2$) solutions. Consider the set X defined by (4.14). Then according to Corollary 5.4(i) and Proposition 5.5, the triple $(X, J_{f_{\theta,q,k}}, f_{\theta,q,k})$ is a p-adic repeller. Owing to Corollary 5.4(ii), the corresponding incidence matrix A has dimension $\kappa_p \times \kappa_p$ and can be written as follows:

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

This means that the triple $(X, J_{f_{\theta,q,k}}, f_{\theta,q,k})$ is transitive. Hence, Theorem 2.10 implies that the dynamics $(J_{f_{\theta,q,k}}, f_{\theta,q,k}, |\cdot|_p)$ is topologically conjugate to the full shift dynamics on κ_p symbols. □

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A. *Appendix*

A.1. *p-adic measure.* Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a *p-adic measure* if for any $A_1, \ldots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \varnothing$ $(i \neq j)$, the equality holds

$$\mu\bigg(\bigcup_{j=1}^{n} A_j\bigg) = \sum_{j=1}^{n} \mu(A_j).$$

A p-adic measure is called a *probability measure* if $\mu(X) = 1$. A p-adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. For more detailed information about p-adic measures, we refer to [19–21].

A.2. *Cayley tree.* Let $\Gamma_+^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k + 1$ edges, except for the root x^0, which has k edges). Here, V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y\rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1\rangle, \ldots, \langle x_{d-1}, y\rangle$ is called a *path* from the point x to the point y. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y.

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

The set of direct successors of x is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n.$$

Observe that any vertex x has k direct successors.

A.3. *p-adic quasi-Gibbs measure.* In this section, we recall the definition of p-adic quasi-Gibbs measure (see [25]).

Let $q \geq 2$ and $\Phi = \{1, 2, \ldots, q\}$. Here, Φ is called a state space and is assigned to the vertices of the tree $\Gamma_+^k = (V, \Lambda)$. A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in \Phi$; in a similar manner, one defines configurations σ_n and ω on V_n and W_n. The set of all configurations on V (respectively V_n, W_n) coincides with $\Omega = \Phi^V$ (respectively $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$, we define their concatenation by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x) & \text{if } x \in V_{n-1}, \\ \omega(x) & \text{if } x \in W_n. \end{cases}$$

It is clear that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

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The Hamiltonian of the p-adic Potts model on Ω_{V_n} is

$$H_n(\sigma) = J \sum_{\langle x,y\rangle \in L_n} \delta_{\sigma(x)\sigma(y)}, \tag{A.1}$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant and δ_{ij} is the Kroneker's symbol.

A construction of a generalized p-adic quasi-Gibbs measure corresponding to the model is given below.

Assume that $\mathbf{h} : V \setminus \{x^{(0)}\} \to \mathbb{Q}_p^{\Phi}$ is a mapping, that is, $\mathbf{h}_x = (h_{1,x}, h_{2,x}, \ldots, h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ $(i \in \Phi)$ and $x \in V \setminus \{x^{(0)}\}$. Given $n \in \mathbb{N}$, we consider a p-adic probability measure $\mu^{(n)}_{\mathbf{h},\rho}$ on Ω_{V_n} defined by

$$\mu^{(n)}_{\mathbf{h}}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp_p\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}. \tag{A.2}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor

$$Z_n^{(\mathbf{h})} = \sum_{\sigma \in \Omega_{V_n}} \exp_p\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}. \tag{A.3}$$

We are interested in the construction of an infinite volume distribution with given finite-dimensional distributions. More exactly, we would like to find a p-adic probability measure $\mu_{\mathbf{h}}$ on Ω which is compatible with the given ones $\mu^{(n)}_{\mathbf{h}}$, that is,

$$\mu_{\mathbf{h}}(\{\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n\}) = \mu^{(n)}_{\mathbf{h}}(\sigma_n) \quad \text{for all } \sigma_n \in \Omega_{V_n},\ n \in \mathbb{N}. \tag{A.4}$$

We say that the p-adic probability distributions (A.2) are *compatible* if for all $n \geq 1$ and $\sigma \in \Phi^{V_{n-1}}$:

$$\sum_{\omega \in \Omega_{W_n}} \mu^{(n)}_{\mathbf{h}}(\sigma_{n-1} \vee \omega) = \mu^{(n-1)}_{\mathbf{h}}(\sigma_{n-1}). \tag{A.5}$$

This condition, according to the Kolmogorov extension theorem (see [20]), implies the existence of a unique p-adic measure $\mu_{\mathbf{h}}$ defined on Ω with a required condition (A.4). Such a measure $\mu_{\mathbf{h}}$ is said to be *a generalized p-adic Gibbs measure* corresponding to the model [25, 26]. If one has $h_x \in \mathcal{E}_p$ for all $x \in V \setminus \{x^{(0)}\}$, then the corresponding measure $\mu_{\mathbf{h}}$ is called a *p-adic Gibbs measure* (see [41]).

By $Q\mathcal{G}(H)$, we denote the set of all generalized p-adic Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_x,\ x \in V\}$. If there are at least two distinct generalized p-adic Gibbs measures such that at least one of them is unbounded, then we say that *a phase transition* occurs.

The following statement describes conditions on h_x guaranteeing compatibility of $\mu^{(n)}_{\mathbf{h}}(\sigma)$.

THEOREM A.1. [25] *The measures* $\mu^{(n)}_{\mathbf{h}}$, $n = 1, 2, \ldots$ *(see* (A.2)*) associated with the q-state Potts model* (A.1) *satisfy the compatibility condition* (A.5) *if and only if for any*

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$n \in \mathbb{N}$, the following equation holds:

$$\hat{h}_x = \prod_{y \in S(x)} \mathbf{F}(\hat{\mathbf{h}}_y, \theta). \tag{A.6}$$

Here and below, a vector $\hat{\mathbf{h}} = (\hat{h}_1, \ldots, \hat{h}_{q-1}) \in \mathbb{Q}_p^{q-1}$ is defined by a vector $\mathbf{h} = (h_1, h_2, \ldots, h_q) \in \mathbb{Q}_p^q$ as follows:

$$\hat{h}_i = \frac{h_i}{h_q}, \quad i = 1, 2, \ldots, q-1 \tag{A.7}$$

and the mapping $\mathbf{F} : \mathbb{Q}_p^{q-1} \times \mathbb{Q}_p \to \mathbb{Q}_p^{q-1}$ is defined by $\mathbf{F}(\mathbf{x}; \theta) = (F_1(\mathbf{x}; \theta), \ldots, F_{q-1}(\mathbf{x}; \theta))$ with

$$F_i(\mathbf{x}; \theta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{q-1} x_j + 1}{\sum_{j=1}^{q-1} x_j + \theta}, \quad \mathbf{x} = \{x_i\} \in \mathbb{Q}_p^{q-1}, \quad i = 1, 2, \ldots, q-1. \tag{A.8}$$

Let us first observe that the set $(\underbrace{1, \ldots, 1, h}_{m}, 1, \ldots, 1)$ $(m = 1, \ldots, q-1)$ is invariant for the equation (A.6). Therefore, in what follows, we restrict ourselves to one of such vectors, let us say $(h, 1, \ldots, 1)$.

In [**32**], to establish the phase transition, we considered translation-invariant (that is, $\mathbf{h} = \{\mathbf{h}_x\}_{x \in V \setminus \{x^0\}}$ such that $\mathbf{h}_x = \mathbf{h}_y$ for all x, y) solutions of (A.6). Then the equation (A.6) is reduced to the following one:

$$h = f_{\theta,q,k}(h), \tag{A.9}$$

where $f_{\theta,q,k}$ is given by (1.1).

Hence, to establish the existence of the phase transition, when $k = 2$, we showed in [**41**] that (A.9) has three non-trivial solutions if q is divisible by p. Note that the full description of all solutions of the last equation has been carried out in [**43**] when $k = 2$. Certain periodic points of $f_{\theta,q,k}$ have been carried out in [**1**, **28**, **31**].

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