NUMERICAL INTEGRATION OF PRECESSION AND NUTATION OF THE RIGID EARTH

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Abstract: A numerical solution for the luni-solar precession and nutation of the rigid Earth is obtained and compared with the result from the analytical theories which are the basis of the current IAU precession and nutation formulas. We have developed a new scheme of numerical calculation by modifying the equations of motion, which enables us to avoid the numerical integration with a small step. Some errors are found in the long periodic region of nutation in the current IAU theory.

Keywords: Precession and nutation, numerical integration, astronomical ephemeris.

1. Introduction

The present value of precession in the astronomical ephemerides is that described in the paper of Lieske et al (1977), which is fundamentally based on the theories by Newcomb (1894, 1906) and Andoyer (1911). As for nutation, the authority is the 1980 IAU nutation theory (IAU, 1982) which was developed by Wahr (1981) for a non-rigid Earth using as the basis the nutation theory of the rigid Earth obtained by Kinoshita et al. (1979). This theory for the rigid Earth is no more than a thorough recomputation of the preceding work of Woolard (1953).

All these theories on precession and nutatioin for the rigid Earth are analytical. The precision of the theory of precession is believed to be better than 0.1mas(milli-arcsecond) except for the obliquity of ecliptic at the epoch and the coefficient of the linear term in the precession in longitude which must be determined by observation. The nutation series for the rigid Earth, which is the basis of the 1980 IAU theory, contains all the terms with the amplitude greater than 0.05mas. However each term has an error of 0.05mas at most since the expression of its amplitude is truncated to be integrals of 0.1mas. The number of terms in the series exceeds 100, thus the precision of the whole series is considered to be several 0.1mas. This comes from the fact that the target precision of Kinoshita and his coworkers was 1mas.

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Rather curiously, no numerical treatment has been attempted for precession and nutation. One of the reasons may be a great rapidity of the rotational motion of the Earth, i.e. one rotation in a day. This makes one feel at the first glance that the step in numerical integration of the equations of motion must be very small. Of course, another reason may be full confidence in the analytical theories.

Having a slight doubt about the precision of the current theories and introducing a method which enables to avoid numerical integration with a very small step, the present authors develop a numerical solution to luni-solar precession and nutation.

The present work is in the nature of a pilot study. A more complete treatment such as changing the theory of Sun and Moon from the old ones (Newcomb and Brown) to the latest ones or considering the higher order luni-solar torques, planetary torques, geodesic rotation and other minute effects should be made later.

2. Equation of motion

We describe the equations of motion for the rotation of rigid Earth in a fixed coordinate system. The ecliptic and mean equinox of J2000.0 are adopted as this fundamental reference frame. The precession thus obtained can be compared directly with the expressions given by Lieske et al., but the result for the nutation must be reduced to the ecliptic and mean equinox of date before comparison because the nutation in astronomical ephemeris is referred to this frame.

Eulerian angles ψ, θ and ϕ shown in Figure 1 are used as the dependent variables. The obliquity of ecliptic ε used in the precession and nutation theory is just the same as θ. We must note that ψ in Figure 1 is different from that used in the precession and nutation theory, which is measured on the ecliptic westward from the X-axis to the above mentioned node. Hence it is equal to $180°-\psi$, ψ being one of our Eulerian angles.

In order to formulate the equation of motion, we first write down L, the classic Lagrangean of the axially symmetric rigid Earth rotating around its center of mass under external forces. It is given by

$$2L = A(\dot{\theta}^2+\dot{\psi}^2\sin^2\theta) + C(\dot{\phi}+\dot{\psi}\cos\theta)^2 + 2U(\psi, \theta; t) \tag{1}$$

where A and C are the moments of inertia with respect to an axis in the equatorial plane and the axis perpendicular to the plane, respectively, the latter of which we call the figure axis hereafter. The symbol U is the perturbing function.

From this Lagrangean, we obtain the following equation of motion for the orientation of the figure axis:

$$\begin{aligned} A\ddot{\theta} &= -C\omega\dot{\psi}\sin\theta + A\dot{\psi}^2\sin\theta\cos\theta + \partial U/\partial\theta, \\ A\ddot{\psi}\sin^2\theta &= C\omega\dot{\theta}\sin\theta - A\dot{\psi}\dot{\theta}\sin 2\theta + \partial U/\partial\psi \end{aligned} \tag{2}$$

Here $^{\bullet}$ denotes the time-derivative and $\omega = \dot{\phi} + \dot{\psi}\cos\theta$ is a constant named the siderial angular velocity of the rotation of the Earth.

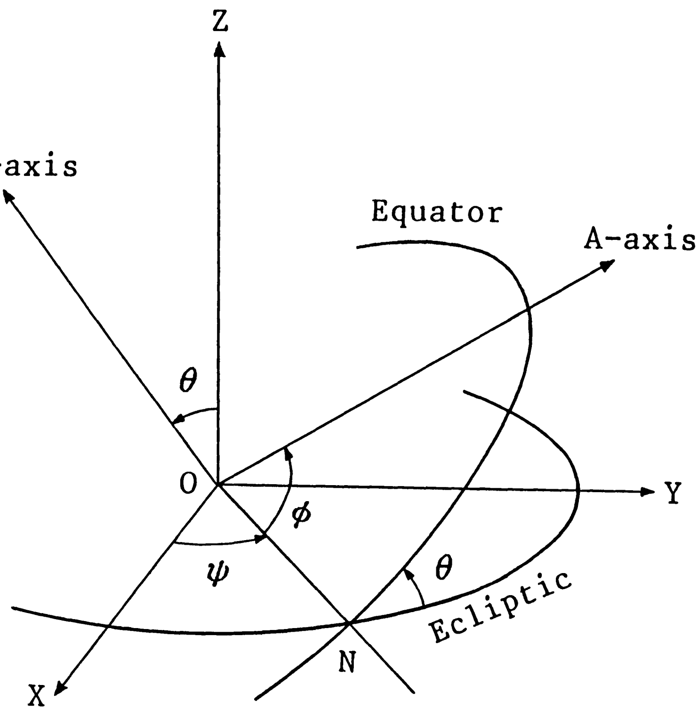

Figure 1 Adopted Eulerian Angles

Usually the above equations are directly numerical integrated as two second order ordinary differential equations. This approach, however, is not practical. The motion of the figure axis contains the well-known Eulerian motion or free nutation which is a circular oscillation with the period of about one day in space. This free nutation is independent of precession and nutation which are a forced motion, therefore it is not taken into account in the computation of precession and nutation. Nevertheless, as long as we stick to the form of equation (2), we would have to solve this motion simultaneously in order to obtain the forced motion of the figure axis of the Earth by performing a numerical integration, in which the step would have to be taken very small because of the rapidity of the motion.

3. Numerical scheme

Solving equation (2) with respect to $\dot{\theta}$ and $\dot{\psi}$, we have a modification of equation of motion as follows:

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$$\frac{d\psi}{dt} = \frac{2(\,\partial U/\partial\theta - A\ddot{\theta}\,)}{C\omega\sin\theta + [(C\omega\sin\theta)^2 - 2A\,\sin2\theta\,(\,\partial U/\partial\theta - A\ddot{\theta}\,)]^{1/2}}$$
$$\frac{d\theta}{dt} = \frac{\partial U/\partial\psi - A\ddot{\psi}\sin^2\theta}{C\omega\sin\theta - A\dot{\psi}\sin2\theta} \qquad (3)$$

The above equation has the form of

$$\dot{y} = f\,(\,y\,,\,\ddot{y}\,,\,t\,) \qquad (4)$$

where y is a vector (ψ, θ). We should note that in our case the contribution of $\ddot{y}$ in f is much smaller than that of y and t in f. If we apply the Picard's method (method of succesive approximation) to this differential equation, we have a following iteration formula:

$$y^{(n+1)} = \int f(\,y^{(n)}(t),\,y^{(n)}(t),\,t\,)\,dt \qquad (\,n = 0,\,1,\,\ldots\,) \qquad (5)$$

where $y^{(0)}(t)$ is a suitable initial guess. It is clear that if the above iteration procedure converges, the limit of $y^{(n)}$ will be a special solution of the differential equation (4). The convergibility of this procedure depends on whether the iteration produces any rapidly oscillating term which destroys the numerical stability. The functional form of equation (3) assures that there comes no rapidly oscillating terms in the above iteration procedures as long as the first guess $y^{(0)}$ includes no rapidly oscillating terms.

In our case, we have already a very well approximated solution of y, i.e. the analytical one. Then only two iterations were actually needed; the one to obtain the difference from the analytical one and the other to assure the convergence.

4. Perturbing function

As for the perturbing function U, we consider only the attractions of the Moon and the Sun. Further we assume both the Moon and the Sun as masspoints and neglect the contribution from the third and higher order multipole moments of the Earth. All these assumptions are adopted in order to make the environment of our computation same as that of Kinoshita et al.. It is easy for our formulation to omit all of the above assumptions since ours is a purely numerical one.

Then U is written as

$$U = U_M + U_S \qquad (6)$$

where two constituents have the same form:

$$U_B = -\,3k^2m(C-A)z^2/(2r^5) \qquad (7)$$

Here the suffix B denotes the disturbing body, i.e. M(Moon) or S(Sun) in our case, k is the Gaussian gravitational constant and m, r and z are respectively the mass, the geocentric distance, the z-coordinate of the disturbing body (Moon or Sun) referred to the equator of the Earth.

The coordinates of the Moon and the Sun are taken from an abridged trigonometric series for them developed by Kubo (1980). This series is a subset of Newcomb's theory for the Sun and Brown's one for the Moon. The error of the series is estimated to be 2" in average and 10" at maximum. However, the effect of this error is small as less than a few 0.01mas in nutation and less than 0.2mas in precession. We remark that Woolard or Kinoshita et al. also made such truncation of the series of coordinates of the Moon and the Sun.

In the following we will state our treatment needed in dealing with the old ephemerides of the Moon and the Sun.

In terms of the ecliptic longitude λ and latitude β of the Moon or the Sun, equation (7) can be expressed as

$$U = -(3/2)C\omega\,[(C-A)/C][k^2m/(\omega a^3)] \times$$
$$(a/r)^3[\cos\theta\sin\beta_0 + \sin\theta\cos\beta_0\sin(\lambda_0-\psi)]^2, \tag{8}$$

where a is a conventional unit in which r of the Moon or the Sun is expressed. The suffix B will be dropped hereafter. The suffix 0 assigned to λ and β means that they are referred to the ecliptic and mean equinox of J2000.0.

We now discuss on the quantity $k^2m/(\omega a^3)$ in the coefficients in equation (8). Here ω is the sidereal mean motion of the rotation of the Earth with the value 1299548.204"/day. In case of the Moon, we take as a the equatorial radius of the Earth a_e. Introduce a_M defined by

$$a_M = [k^2(m_E+m_M)/n_M^2]^{1/3} = 0.002571881428\ \text{AU}, \tag{9}$$

where m_E and m_M are the masses of the Earth and the Moon, respectively, and n_M (= 47434.88963"/day) is the sidereal mean motion of the Moon. Further, we have a relation among a_e, a_M and the mean distance of the Moon a_0:

$$a_M = a_0/F_2 = a_e/(3422.448''F_2) = 60.32291182\ a_e, \tag{10}$$

F_2 being a constant, whose value is 0.999093142. Hence,

$$\begin{aligned} k^2m_M/(\omega a_M^3) &= (a_M/a_e)^3[m_M/(m_E+m_M)][k^2(m_E+m_M)/(\omega a_M^3)] \\ &= (60.32291182)^3 \times 0.01215056777 \times (n_M^2/\omega) \\ &= 4617924.822''/\text{day}. \end{aligned} \tag{11}$$

In case of the Sun, we take 1 AU as a. Introduce a_S defined by

$$a_S = [k^2(m_S+m_E+m_M)/n_S^2]^{1/3} = 1.000000036\ \text{AU}, \tag{12}$$

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where m_S is the mass of the Sun and n_S (= 3548.192807"/day) is the sidereal mean motion of the Sun. Then,

$$k^2 m_S/(\omega\ AU^3) = (a_S/AU)^3[m_S/(m_S+m_E+m_M)][k^2(m_S+m_E+m_M)/(\omega\ a_S^3)]$$

$$= (1.000000036)^3 \times 0.9999969596 \times (n_S^2/\omega)$$

$$= 9.687701648"/\text{day}. \tag{13}$$

Finally, we adopt the following value of common factor in U_M and U_S:

$$(C-A)/C = 0.0032739935. \tag{14}$$

All the numerical values adopted above are coincident with the IAU (1976) system of astronomical constants and are the same as those used in Kinoshita et al..

5. Integration

The integration is carried out by the Simpson's formula for definite integral with a step of 2 hours. In doing this, the perturbing force by the Sun is evaluated at 0h every day and interpolated to every 2 hours, while for the Moon the coordinates are evaluated at 0h every day and interpolated to every 2 hours and then the force is calculated. Differences up to the fourth order are taken into consideration in the interpolation. As for the geodesic precession, we added compulsively a linear term of the amount of 1.92"/cy or 0.0526mas/day to our ψ as Kinoshita et al. did.

The initial values (suffixed i) adopted in the integration are

$$t_i = \text{JD } 2446066.5 \quad \text{or Jan. 1, 1985} \quad 0^h \text{ TD } (T_i = -0.1499931553),$$

$$\theta_i = 23^\circ\ 26'\ 21.448" + 4.849", \text{ and}$$

$$\psi_i = -5038.7784"\ T_i + 1.07259"\ T_i^{\ 2} + 0.001147"\ T_i^{\ 3}$$

$$+ 13.715" + 180^\circ,$$

where T is measured from J2000.0 in the unit of Julian century. The values 4.849" and 13.715" were added to θ_i and ψ_i respectively so that they coincide with the corresponding analytical values for the rigid Earth at the epoch t_i.

It should be noticed that since our formulation obtains the forced oscillation only, a slightly different value for θ_i or ψ_i substantially results a mere constant bias of the same amount to all the values of θ or ψ throughout the period to be integrated.

6. Result and discussion

The integration has been carried out for a period of about 18,000 days. In the following discussion, ε is used in place of θ and $180^\circ - \psi$ in the

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preceding sections is replaced by ψ, according to the conventional notations used in precession and nutation theory.

As mentioned in Section 2, the nutation obtained in the fixed reference frame $\Delta\psi_0$ and $\Delta\varepsilon_0$ must be reduced to that in the frame of the ecliptic and mean equinox of date $\Delta\psi$ and $\Delta\varepsilon$, respectively. The reduction formulas are

$$
\begin{aligned}
\Delta\psi &= \Delta\psi_0 + (\pi \cos\Pi \cot\varepsilon)\Delta\psi_0 + (\pi \sin\Pi / \sin^2\varepsilon)\Delta\varepsilon_0, \\
\Delta\varepsilon &= \Delta\varepsilon_0 - (\pi \sin\Pi)\Delta\psi_0,
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
\pi &= 47.0029'' \, T - 0.03302'' \, T^2 + 0.000060 \, T^3, \\
\Pi &= 5^\circ \, 07' \, 25.018'' - 4168.9695'' \, T + 1.03723'' \, T^2 + 0.001147'' \, T^3.
\end{aligned}
\tag{16}
$$

It should be noted that π must be converted to be measured in radians in equation (15). We first examine the short periodic terms of nutation. We compare our result with the analytical one for a period of 250 days beginning from JD 2446066.5. In Figure 2 the differences between our numerical values (N) and analytical values (A) for $\Delta\psi$ and $\Delta\varepsilon$ are shown. Figure 3 shows their power spectra. The constant biases of some mas in $\Delta\psi_{N-A}$ and $\Delta\varepsilon_{N-A}$ are meaningless because of the reason mentioned in Section 5. As far as the short periodic region of nutation is concerned, the differences between N and A in $\Delta\psi$ and $\Delta\varepsilon$ are reasonable considering the precision of the analytical computations.

When we proceed to precession and long periodic region of nutation, however, we see a fairly different aspect. Figure 4 shows the $\Delta\psi_{N-A}$ and $\Delta\varepsilon_{N-A}$ for a period from JD 2445106.5 to JD 2462706.5. In Figure 4, one dot means the average for 32 days. Figure 5 is their power spectra.

In $\Delta\psi_{N-A}$ all the analytical values of precession and nutation have been subtracted from the numerical solution. Therefore it would be a horizontal straight line if both $\Delta\psi_N$ and $\Delta\psi_A$ were correct. In $\Delta\varepsilon_{N-A}$, however, only the analytical nutation has been subtracted from the numerical result. Therefore from the graph of $\Delta\varepsilon_{N-A}$ should be further subtracted the analytical precession, i.e.,

$$
+ \, 51.27 \, T^2 - 7.726 \, T^3 \quad \text{(in mas)}. \tag{17}
$$

An analysis of $\Delta\psi_{N-A}$ and $\Delta\varepsilon_{N-A}$, where the theoretical precession (17) in $\Delta\varepsilon_N$ has been removed, gives the following expressions for the differences in precession and long periodic terms of nutation:

$$
\begin{aligned}
\Delta\psi_{N-A} &= +\,15.1 \, (\pm 2.5) \, T - 2.2 \, (\pm 14.9) \, T^2 \\
&\quad + 0.6 \sin(\Omega - 26^\circ) + 1.3 \sin(2\Omega - 2^\circ) \quad \text{(in mas)}, \\
\Delta\varepsilon_{N-A} &= -\,0.3 \, (\pm 1.2) \, T - 6.7 \, (\pm 6.2) \, T^2 \\
&\quad + 0.8 \cos(\Omega + 26^\circ) - 0.3 \cos(2\Omega + 37^\circ) \quad \text{(in mas)},
\end{aligned}
\tag{18}
$$

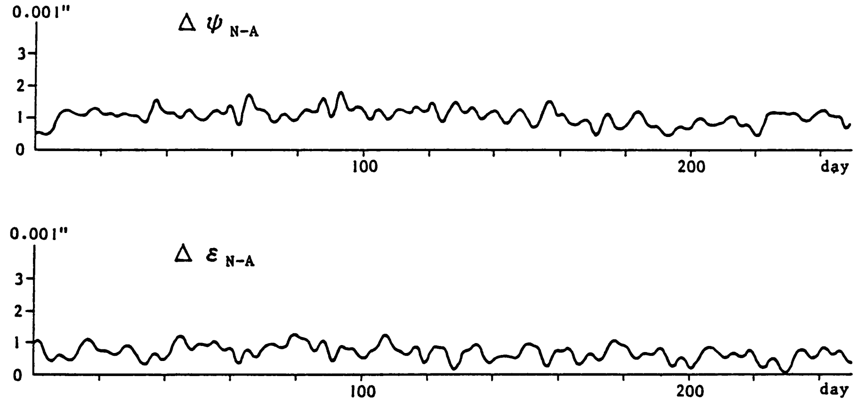

Figure 2 Difference between Numerical(N) and Analytical(A) Nutation for a period of 250days from JD 2446066.5 (Jan. 1, 1985)

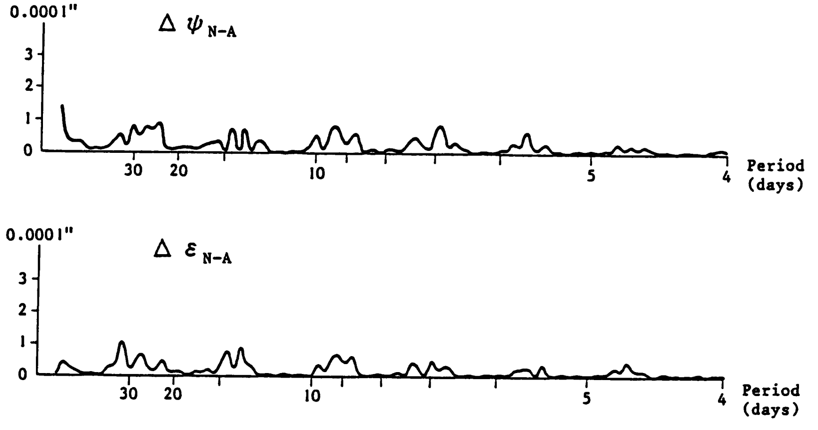

Figure 3 Power Spectra of Differences showed in Figure 2

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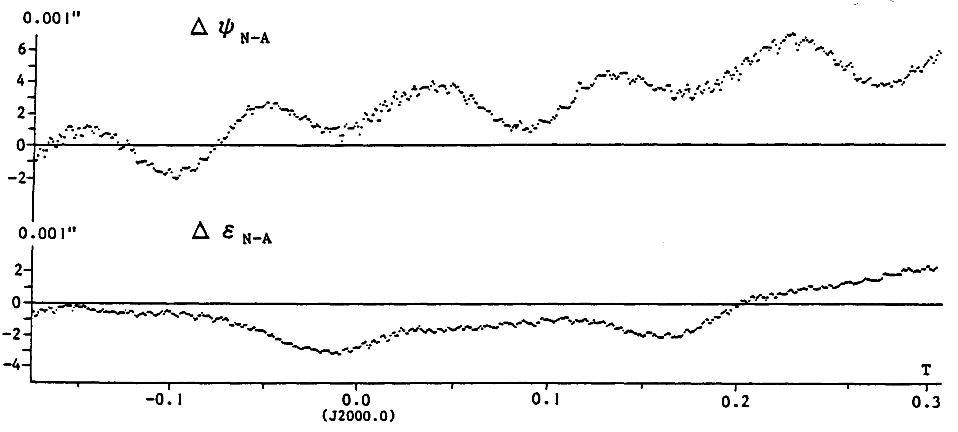

Figure 4 Difference between Numerical(N) and Analytical(A) Nutation for a period of 17,600 days from JD 2445106.5 to JD 2462706.5

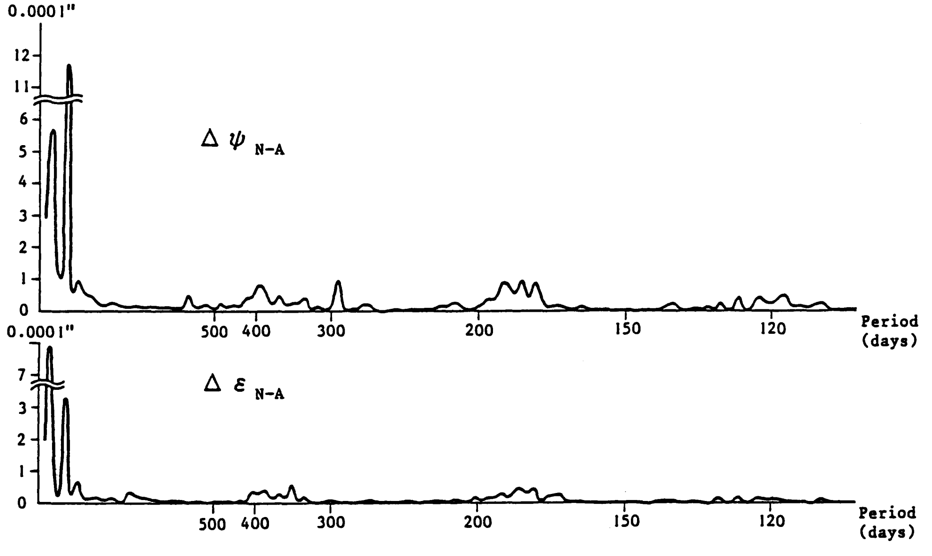

Figure 5 Power Spectra of Differences showed in Figure 4

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the first and the second terms being for precession and the third and the fourth terms for nutation in each equation. The angle Ω is the longitude of the ascending node of the Moon's orbit on the ecliptic.

Among the four terms for precession in (27), only the linear term + 15.1mas T in $\Delta\psi_{N-A}$ is significant judging from the mean errors. Since this term is to be determined by observation, the difference is not important physically. Rather, this indicates that the adopted numerical value of (C-A)/C in equation (14), which was determined by the comparison of analytical theories with the observed amount of luni-solar precession, would be wrong. On the other hand, all the terms for nutation in (27) are significant. Among them the terms with the argument of 2Ω (9.3yr period) are well coincident with the result which Kubo (1982) obtained analytically:

$$\delta(\Delta\psi) = + 1.2 \text{ mas } \sin 2\Omega, \quad \text{and} \quad \delta(\Delta\varepsilon) = - 0.2 \text{ mas } \cos 2\Omega. \tag{19}$$

As for the remaining two terms of nutation + 0.6mas $\sin(\Omega-26^\circ)$ and + 0.8mas $\cos(\Omega+26^\circ)$, as well as phase shifts in the arguments of 9.3yr terms, we can say nothing at present about whether our numerical way or the analytical theory is wrong. The differences arising from the modification of the equations of motion and from the integration in our solution are estimated to be small enough. The largest source of the error in our calculation would be in the low precision of the coordinates of the Moon and the Sun we adopted. Especially, some long periodic terms which are missing because of their smallness in the trigonometric series for the Moon might be questionable, although the effect to our result does not seem so large as 0.2mas.

However, it is desirable to follow the present scheme once again using latest precise ephemerides of the Moon and the Sun and taking into some other minute effects.

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