ON SOME LIMIT THEOREMS INVOLVING THE EMPIRICAL DISTRIBUTION FUNCTION

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1. Summary. Let $X_1, \ldots, X_n$ be mutually independent random variables with a common continuous distribution function $F(t)$. Let $F_n(t)$ be the corresponding empirical distribution function, that is

$$F_n(t) = (\text{number of } X_i \leq t, \ 1 \leq i \leq n)/n.$$

Using a theorem of Manija [4], we proved among others the following statement in [1].

Theorem 3 of [1]. If $0 < b < 1$ and $\lambda > 0$ then

$$(1.1) \qquad \lim_{n\to\infty} P\{\sqrt{n} \sup_{F(t)\leq b} (F_n(t) - F(t)) < \lambda\}$$

$$= \phi(\lambda R(b)) - \exp(-2\lambda^2)\, \phi(-\lambda(1-2b)\, R(b)),$$

where ϕ is the standard normal distribution function and $R(t) = [t(1-t)]^{-1/2}$.

The purpose of this note is to present another proof of this theorem.

2. Proof. Without loss of generality we may assume $F(t) = t$, with t uniformly distributed on $[0, 1]$, and $F_n(t)$ is then the empirical distribution function constructed by selecting a random sample of size n from this uniform distribution. Let

$$\xi_n(t) = \sqrt{n}\,(F_n(t) - t), \quad 0 \leq t \leq 1.$$

According to a theorem of Donsker and Doob ([2] and [3]) we have, for $\lambda > 0$,

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$$(2.1) \qquad \lim_{n\to\infty} P\{\sup_{0\le t\le 1} \xi_n(t) < \lambda\} = P\{\sup_{0\le t\le 1} \xi(t) < \lambda\},$$

where $\xi(t)$ is a Gaussian process with parameter t, $0 \le t \le 1$, mean value function $E(\xi(t)) = 0$ and covariance function $r(s,t) = s(1-t)$, $0 \le s < t < 1$. This $\xi(t)$ process can be converted into the classical Wiener process, or Brownian motion, by the transformation

$$X(t) = (t+1)\xi\left(\frac{t}{t+1}\right), \quad 0 \le t < \infty,$$

which is due to Doob [3]. Then we have $E(X(t)) = 0$, and $r(s,t) = s$ for $0 \le s \le t < \infty$. In our case it is implied by (2.1) that we have to consider

$$(2.2) \qquad \lim_{n\to\infty} P\{\sup_{0\le t\le b} \xi_n(t) < \lambda\}$$

$$= P\{\sup_{0\le t\le b} \xi(t) < \lambda\} = P\{\sup_{0\le t\le \frac{b}{1-b}} [X(t)/(t+1)] < \lambda\}.$$

Using a result of Doob [3], Quade [5] proves the following statement:

Lemma 2.1 of [5]. If $0 < b < 1$ and $\lambda > 0$ then

$$(2.3) \quad P\{\sup_{0\le t\le b} \xi(t) > \lambda\} = \phi(-\lambda R(b)) + \exp(-2\lambda^2)\phi(-\lambda(1-2b)R(b)).$$

This, through (2.2), immediately implies (1.1) of section 1.

3. <u>A remark.</u> It is interesting to note here that the evaluation of the following statement (with $0 < a < b < 1$):

$$(3.1) \quad P\{\sup_{a\le t\le b} \xi(t) < \lambda\} = P\{\sup_{\frac{a}{1-a}\le t\le \frac{b}{1-b}} [X(t)/(t+1)] < \lambda\}$$

for the $\xi(t)$ and $X(t)$ processes of section 2 is not an easy task, even though we have (2.3) of section 2 now. However, using the argument of section 2, it is seen that the statement of (3.1) is equivalent to

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$$\lim_{n\to\infty} P\{ \sup_{a \leq t \leq b} \xi_n(t) < \lambda\}, \quad 0 < a < b < 1,$$

and this statement has been evaluated by Manija in [4]. His theorem is also given in section 4 of [1].

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