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COMPACT MULTIPOLAR SETS

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ABSTRACT. It is proved that a compact subset of a finite product of Brelot harmonic spaces is multipolar if it is a locally multipolar set.

RÉSUMÉ. On demontre que un ensemble compact dans un produit fini des espaces harmoniques de Brelot est multipolar si l'ensemble est localement multipolar.

Let Ω_j be a Brelot harmonic space with a bounded potential > 0 for each $j = 1, \ldots, n$. We shall consider here the product $X = \Omega_1 \times \cdots \times \Omega_n$ of the above spaces and n-superharmonic functions on open subsets of X. We recall that a subset A of X is said to be n-polar if there is a positive n-superharmonic function on X taking the value $+\infty$ on A. The object of this note is to prove the following result.

THEOREM. *Let K be a compact subset of X which is locally n-polar. Then K is n-polar in X.*

The result in the case $n = 1$ is well known and can be proved in a variety of ways mostly dependent on the characterization of potentials. Such a method is unavailable in the general case. The method of our proof makes use of the characterisation of a compact n-polar set in terms of the possibility of extending an arbitrary continuous function defined on this set to the difference of continuous n-potentials on X. [J]. We also make use of the following result of D. Singman [S].

THEOREM. *Let U_1, U_2, U_3 be relatively compact domains of X with $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U_3 \subset \overline{U}_3 \subset X$. Let $v > 0$ be a n-superharmonic function on U_3 such that v is bounded on ∂U_2. Then there exist n-superharmonic functions $p_1, p_2 > 0$ on X such that p_2 is a continuous n-potential and on U_1, $p_1 = p_2 + v$.*

PROOF. We give below an outline of the proof. Choose an open connected set U_2' such that $\overline{U}_2 \subset U_2' \subset \overline{U_2'} \subset U_3$. Let M be an upper bound for v on ∂U_2 and m a positive lower bound of v on $\partial U_2'$. Choose an $\varepsilon > 0$ with $\varepsilon < m$. Define the continuous function g on the compact set $K = \partial U_2 \cup \partial U_2'$ by setting $g(x) \equiv M$ on ∂U_2 and $g(x) \equiv m$ on $\partial U_2'$. Since the differences of positive continuous n-potentials are "positively" dense in the class of continuous functions with compact support, we may find n-potentials Q and p' such that $|Q - p' - g| < \varepsilon$. We deduce easily that $Q - p' \leq v$ on $\partial U_2'$ and $Q - p' \geq v$ on ∂U_2. It is a fairly routine matter to check that the function p defined by setting $p(x) =$

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$p'(x) + v(x)$ for $x \in U_2$, $p(x) = Q(x)$ for $x \notin U_2'$ and $p(x) = \min\big(p'(x) + v(x), Q(x)\big)$ for $x \in U_2' - U_2$, is indeed a n-superharmonic function. The required conclusion of the theorem is deduced easily.

Let us now prove the following lemma.

LEMMA. *Let K be a compact subset of X which is contained in the product $\boldsymbol{\delta} = \delta_1 \times \cdots \times \delta_n$ of regular domains of the component spaces. Suppose K is a n-polar subset of $\boldsymbol{\delta}$ such that every i-section of K through any point of K is $(n-i)$-polar in the corresponding i-section of $\boldsymbol{\delta}$. Then K is n-polar in X.*

[Note that an i-section is obtained by fixing a set of i-coordinates.]

PROOF. Let U_1, U_2, U_3 be relatively compact open sets such that $K \subset U_1$, $\overline{U}_j \subset U_{j+1}$ and $\overline{U}_3 \subset \boldsymbol{\delta}$. For every positive integer m, let f_m be a non-negative continuous function on $\boldsymbol{\delta}$ such that

(i) $f_m \equiv 1$ on K,
(ii) $f_m(x) = 1/2^m \ \forall x \in \overline{U}_2 \setminus U_1$ and
(iii) $f_m(x) > 0 \ \forall x \in \overline{U}_2$.

We can consider $\boldsymbol{\delta}$ as a product of Brelot harmonic spaces of the type considered here and given $\varepsilon > 0$, we conclude that for each m there is a n-potential on $\boldsymbol{\delta}$ such that $p_m < f_m$ on $\overline{U}_2$ and $p_m > 1 - \varepsilon/2^m$ on K. [J; Proposition 11].

Set $p = \sum_1^\infty p_m$ and note that p is a n-superharmonic function on $\boldsymbol{\delta}$ and $p(x) \leq 1$ for all x in $\overline{U}_2 \setminus U_1$; in particular, p is bounded on the boundary of U_2. By the extension theorem of Singman, we have n-superharmonic > 0 functions q_1, q_2 on X such that q_2 is a continuous potential and $q_1 = p + q_2$ on U_1. Thus $q_1 \equiv +\infty$ on K concluding the proof of the lemma.

PROOF OF THE MAIN RESULT. The result is true for $n = 1$. Let us assume the validity of the result for all integers $n \leq k$. We prove that the result is valid for $n = k + 1$. We present here the proof in the case $n = 3$. The proof in general is very similar except for notational and minor computational differences.

Let us first consider a compact 3-polar subset K of a product of regular domains $\delta_1 \times \delta_2 \times \delta_3$, each $\delta_j \subset \Omega_j$. There exists a 3-superharmonic function $v > 0$ on $\delta_1 \times \delta_2 \times \delta_3$ such that $v \equiv +\infty$ on K. For definiteness, let us take a point (x_1^0, x_2^0, x_3^0) in $\delta_1 \times \delta_2 \times \delta_3$ such that $v(x_1^0, x_2^0, x_3^0) < +\infty$. Let $P_1 = \{x_1 \in \delta_1 : v(x_1, x_2^0, x_3^0) = +\infty\}$ and define P_2, P_3 similarly. Let again, $Q_{12} = \{(x_1, x_2) : v(x_1, x_2, x_3^0) = +\infty\}$ and define Q_{23}, Q_{13} similarly. Clearly, P_j is a polar subset of δ_j, $j = 1, 2, 3$. Further, $Q_{jk} \subset \delta_j \times \delta_k$ is 2-polar in that set and hence as a result of induction assumption, this set is 2-polar in $\Omega_j \times \Omega_k$.

Now, define for each natural number ℓ,

$$A_{1,\ell} = \{x_1 \in \delta_1 : v(x_1, x_2^0, x_3^0) > \ell\}$$

and

$$B_{12,\ell} = \{(x_1, x_2) \in \delta_1 \times \delta_2 : v(x_1, x_2, x_3^0) > \ell\}.$$

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Define similarly the sets $A_{2,\ell}$, $A_{3,\ell}$, $B_{23,\ell}$, and $B_{13,\ell}$. All these sets are open in the respective spaces $\delta_1, \delta_1 \times \delta_2$ etc. Let

$$K_\ell = \big\{(x_1, x_2, x_3) \in \Omega_1 \times \Omega_2 \times \Omega_3 : x_j \in A_{j,\ell} \text{ for some } j \\ \text{or } (x_j, x_k) \in B_{jk,\ell} \text{ for some } (j,k)\big\}.$$

It is easily verified that K_ℓ is an open subset of $\Omega_1 \times \Omega_2 \times \Omega_3$. Let $C_\ell = K \setminus K_\ell$. It is clear that C_ℓ is a compact 3-polar subset of $\delta_1 \times \delta_2 \times \delta_3$. By the construction we have chosen C_ℓ so that every i-section of C_ℓ through any point of this set is $(3 - i)$-polar. From the lemma we derive that C_ℓ is 3-polar in $\Omega_1 \times \Omega_2 \times \Omega_3$. This is true for every ℓ and hence the (increasing) countable union of the sets C_ℓ is again 3-polar in $\Omega_1 \times \Omega_2 \times \Omega_3$.

However, this union is precisely the set $K \setminus A$ where

$$A = \{(x_1, x_2, x_3) : x_j \in P_j \text{ or } (x_j, x_k) \in Q_{jk}\}.$$

It is easily seen from the remarks made earlier that A is a 3-polar set of $\Omega_1 \times \Omega_2 \times \Omega_3$. It follows that $K = (K \setminus A) \cup A$ is a 3-polar set establishing the result in this case.

Suppose now K is an arbitrary compact subset of X such that K is locally n-polar. We can find open sets $U_1, \ldots, U_p$ and $V_1, \ldots, V_p$, each of which is a product of regular domains such that $\overline{V}_j \subset U_j$ for each $j = 1$ to p, $\bigcup_{j=1}^p V_j \supset K$ and $K \cap \overline{V}_j$ is n-polar in U_j. The above proof shows that if $j = 1$ to p, $K \cap \overline{V}_j$ is n-polar in X. Since K is the (finite) union of the sets $K \cap \overline{V}_j$ we conclude that K is n-polar in X completing the proof.

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