COMPOSITIO MATHEMATICA

Analytic representation theory of Lie groups: general theory and analytic globalizations of Harish-Chandra modules

Heiko Gimperlein, Bernhard Krötz and Henrik Schlichtkrull

Compositio Math. **147** (2011), 1581–1607.

doi:10.1112/S0010437X11005392

FOUNDATION
COMPOSITIO
MATHEMATICA

The London Mathematical Society

Compositio Math. 147 (2011) 1581–1607
doi:10.1112/S0010437X11005392

Analytic representation theory of Lie groups: general theory and analytic globalizations of Harish-Chandra modules

Heiko Gimperlein, Bernhard Krötz and Henrik Schlichtkrull

Abstract

In this article a general framework for studying analytic representations of a real Lie group G is introduced. Fundamental topological properties of the representations are analyzed. A notion of temperedness for analytic representations is introduced, which indicates the existence of an action of a certain natural algebra $\mathcal{A}(G)$ of analytic functions of rapid decay. For reductive groups every Harish-Chandra module V is shown to admit a unique tempered analytic globalization, which is generated by V and $\mathcal{A}(G)$ and which embeds as the space of analytic vectors in all Banach globalizations of V.

1. Introduction

While analytic vectors are basic objects in the representation theory of real Lie groups, a coherent framework to study general analytic representations has been lacking so far. It is the aim of this article to introduce categories of tempered and non-tempered such representations and to analyze their fundamental properties. For a representation (π, E) of a Lie group G on a locally convex space E to be *analytic*, we are going to require that every vector in E be analytic and that the topology on the space of analytic vectors coincides with the topology of E. No completeness assumptions on E are imposed, so that the quotient of an analytic representation by a closed invariant subspace is again analytic.

Recall that a vector $v \in E$ is called analytic provided that the orbit map $\gamma_v : x \mapsto \pi(x)v$ extends to a holomorphic E-valued function in a neighborhood of G within the complexification $G_{\mathbb{C}}$. The space E^ω of analytic vectors carries a natural inductive limit topology $E^\omega = \lim_{n\to\infty} E_n$,

$$E_n = \{v \in E \mid \gamma_v \text{ extends to a holomorphic map } GV_n \to E\},$$

indexed by a neighborhood basis $\{V_n\}_{n\in\mathbb{N}}$ of the identity in $G_{\mathbb{C}}$. The induced representation (π, E^ω) turns out to be continuous and indeed satisfies $E^\omega = (E^\omega)^\omega$ in the sense of topological vector spaces. Every analytic representation is obtained in this way. Due to the inductive limit structure of E^ω, interesting examples tend to involve complicated and possibly incomplete topologies. For instance, infinite-dimensional Fréchet spaces do not carry any irreducible analytic representations of a reductive group. Still, in spite of examples by Grothendieck and others which show how incomplete spaces may naturally occur, important special cases are better behaved, like for instance the analytic vectors associated to a Banach representation, the algebra $\mathcal{A}(G)$ below, or the analytic globalization of a Harish-Chandra module.

Received 21 July 2010, accepted in final form 13 December 2010, published online 1 June 2011.
2010 Mathematics Subject Classification 22E45, 22E46 (primary).
Keywords: real Lie group, analytic representation, Harish-Chandra module, globalization.

Moderately growing analytic representations allow for an additional action by an algebra of superexponentially decaying functions. To be specific, consider a Banach representation (π, E). Fix a left-invariant Riemannian metric on G and let $\mathbf{d}$ be the associated distance function. The continuous functions on G decaying faster than $e^{-n\mathbf{d}(\cdot,\mathbf{1})}$ for all $n \in \mathbb{N}$ form a convolution algebra $\mathcal{R}(G)$, which is a G-module under the left regular representation. If we denote the space of analytic vectors of $\mathcal{R}(G)$ by $\mathcal{A}(G)$, the map

$$\Pi : \mathcal{A}(G) \to \mathrm{End}(E^\omega), \quad \Pi(f)v = \int_G f(x)\pi(x)v\, dx, \tag{1.1}$$

gives rise to a continuous algebra action on E^ω. More general representations will be called $\mathcal{A}(G)$-*tempered*, or of *moderate growth*, provided that the integral in (1.1) converges and defines a continuous action of $\mathcal{A}(G)$.

Let us now specify to the case where G is a real reductive group, and let us recall that to each admissible G-representation E of finite length one can associate the Harish-Chandra module E_K of its K-finite vectors. Conversely, a *globalization* of a given Harish-Chandra module V is an admissible representation of G with $V = E_K$. The main result for this case is now as follows.

Theorem 1.1. *Let G be a real reductive group. Then every Harish-Chandra module V for G admits a unique $\mathcal{A}(G)$-tempered analytic globalization $V^{\min}$. Moreover, $V^{\min}$ has the property $V^{\min} = \Pi(\mathcal{A}(G))V$.*

It follows that $E^\omega \simeq V^{\min}$ for every $\mathcal{A}(G)$-tempered globalization E of V (in particular, for every Banach globalization). Let us mention the relationship to the results of [Kas08, KS94] (announced in [Sch85]), which assert in particular that every Harish-Chandra module admits a unique *minimal globalization*, which is equivalent to E^ω for all Banach globalizations E. Our approach is independent of this theory and relies on recent lower bounds for matrix coefficients, see [BK].

The theorem features a worthwhile corollary, as follows.

Corollary 1.2. *For an irreducible admissible Banach representation (π, E) of a real reductive group G, the space of analytic vectors E^ω is an algebraically simple $\mathcal{A}(G)$-module.*

This corollary suggests a notion of irreducible analytic tempered representations for a general Lie group.

2. Banach representations and *F*-representations

All topological vector spaces E considered in this paper are assumed to be Hausdorff and locally convex. If E is a topological vector space, then we denote by $\mathrm{GL}(E)$ the group of isomorphisms of E.

Let G be a connected Lie group. By a *representation* of G we shall understand a continuous action

$$G \times E \to E, \quad (g, v) \mapsto g \cdot v,$$

on some topological vector space E. Each representation gives rise to a group homomorphism

$$\pi : g \to E, \quad g \mapsto \pi(g), \quad \pi(g)v := g \cdot v \quad (v \in E),$$

and it is customary to denote the representation by the symbol (π, E).

A representation (π, E) is called a *Banach representation* if E is a Banach space. We say that (π, E) is an F-representation if E is a Fréchet space for which there exists a defining family of

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

seminorms $(p_n)_{n\in\mathbb{N}}$ such that for all $n\in\mathbb{N}$ the action

$$G\times(E,p_n)\to(E,p_n)$$

is continuous. Here, (E,p_n) refers to the vector space E endowed with the locally convex structure induced by p_n.

Remark 2.1. (a) Every Banach representation is an F-representation.

(b) Let (π,E) be an F-representation. For each $n\in\mathbb{N}$, let us denote by $\hat{E}_n$ the Banach completion of (E,p_n), i.e. the completion of the normed space $E/\{p_n=0\}$. The action of G on (E,p_n) factors to a continuous action on the normed space $E/\{p_n=0\}$ and thus induces a Banach representation of G on $\hat{E}_n$.

(c) The left regular action of G on the Fréchet space $C(G)$ defines a representation, but in general not an F-representation.

Let E^∞ denote the space of smooth vectors in E, that is, the vectors $v\in E$ for which the orbit map $g\mapsto\pi(g)v$ is smooth into E. Then $E^\infty\subset E$ is an invariant subspace, and it is dense if E is complete. The orbit map provides an injection of E^∞ into $C^\infty(G,E)$, from which E^∞ inherits a topological vector space structure. Then (π,E^∞) is a representation. Furthermore, E^∞ is a Fréchet space if E is a Fréchet space, and (π,E^∞) is an F-representation if (π,E) is an F-representation. By definition, a *smooth representation* is a representation for which $E^\infty=E$ as topological vector spaces.

2.1 Growth of representations

We call a function $w:g\to\mathbb{R}^+$ a *weight* if:

- w is locally bounded;
- w is sub-multiplicative, i.e. $w(gh)\leqslant w(g)w(h)$ for all $g,h\in G$.

To every Banach representation (π,E) we associate the function

$$w_\pi(g):=\|\pi(g)\|\quad(g\in G),$$

where $\|\cdot\|$ denotes the standard operator norm. It follows from the uniform boundedness principle that w_π is locally bounded. Hence, w_π is a weight.

Sub-multiplicative functions can be dominated in a geometric way. For that, let us fix a left-invariant Riemannian metric $\mathbf{g}$ on G. Associated to $\mathbf{g}$ we obtain the Riemannian distance function $\mathbf{d}:G\times G\to\mathbb{R}_{\geqslant 0}$. The distance function is left G-invariant and hence is recovered as $\mathbf{d}(g,h)=d(g^{-1}h)$ from the function

$$d(g):=\mathbf{d}(g,\mathbf{1})\quad(g\in G),$$

where $\mathbf{1}\in G$ is the neutral element. Notice that it follows from the elementary properties of the metric that d is compatible with the group structure in the sense that

$$d(g^{-1})=d(g)\quad\text{and}\quad d(gh)\leqslant d(g)+d(h)\tag{2.1}$$

for all $g,h\in G$. In particular, $g\mapsto e^{d(g)}$ is a weight. Note also that the metric balls $\{g\in G\mid d(g)\leqslant R\}$ in G are compact [Gar60, p. 74].

If w is an arbitrary weight on G, then there exist constants $c,C>0$ (depending on w) such that [Gar60, p. 75, Lemme 3]

$$w(g)\leqslant Ce^{cd(g)}\quad(g\in G).\tag{2.2}$$

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

In particular, it follows that a Banach representation has at most exponential growth

$$\|\pi(g)\| \leqslant Ce^{cd(g)}.$$

By applying Remark 2.1(b), we obtain for an F-representation (π, E) with defining seminorms $(p_n)_{n\in\mathbb{N}}$ that for each n there exist constants c_n, C_n such that

$$p_n(\pi(g)v) \leqslant C_n e^{c_n d(g)} p_n(v) \quad (g \in G, v \in E). \tag{2.3}$$

Finally, notice that it follows from (2.2) that if $d_1(g) = d_{\mathbf{g}_1}(g, \mathbf{1})$ is the function associated to a different choice of a G-invariant metric, then d_1 is compatible with d, in the sense that there exist constants $c, C > 0$ such that

$$d_1(g) \leqslant cd(g) + C \quad (g \in G)$$

(and vice versa with d, d_1 interchanged).

Remark 2.2. Suppose that G is a real reductive group and $\|\cdot\|$ is a norm of G (see [Wal88, § 2.A.2]). Then $\|\cdot\|$ is a weight and hence there exist constants $c_1, C_1 > 0$ such that

$$\log \|g\| \leqslant c_1 d(g) + C_1 \quad (g \in G).$$

Conversely, by following the proof of [Wal88, Lemma 2.A.2.2], one finds constants $c_2, C_2 > 0$ such that

$$d(g) \leqslant c_2 \log \|g\| + C_2 \quad (g \in G).$$

3. Analytic representations

Let us start by setting up some notation in order to discuss the issue of analyticity in a convenient way.

Let us denote by $\mathfrak{g}$ the Lie algebra of G. To simplify the exposition, we will assume that $G \subset G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a complex group with Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} =: \mathfrak{g}_{\mathbb{C}}$. We stress, however, that this assumption is not necessary, since the use of $G_{\mathbb{C}}$ essentially only takes place locally in neighborhoods G.

We extend the left-invariant metric $\mathbf{g}$ to a left $G_{\mathbb{C}}$-invariant metric on $G_{\mathbb{C}}$ and denote the associated distance function as before by d. For every $n \in \mathbb{N}$, we set

$$V_n := \left\{ g \in G_{\mathbb{C}} \mid d(g) < \frac{1}{n} \right\} \quad \text{and} \quad U_n := V_n \cap G.$$

It is clear that the V_n, respectively U_n, form a base of the neighborhood filter of $\mathbf{1}$ in $G_{\mathbb{C}}$, respectively G. Note that V_n is symmetric, and that $xy \in V_n$ for all $x, y \in V_{2n}$.

3.1 The space of analytic vectors

Let (π, E) be a representation of G. For each $v \in E$, we denote by

$$\gamma_v : g \to E, \quad x \mapsto \pi(x)v,$$

the associated continuous orbit map. We call v an *analytic* vector if γ_v extends to a holomorphic E-valued function (see Appendix A) on some open neighborhood of G in $G_{\mathbb{C}}$.

If v is analytic, then γ_v is a real analytic map $G \to E$. The converse statement, that real analyticity of the orbit map implies the analyticity of v, holds under the assumption that E is

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

sequentially complete. Hence, our definition agrees with the standard notion of analytic vectors for Banach representations, see for example [Gar60, Goo69, Nel59].

Remark 3.1. If E is a Banach space or more generally a complete *DF*-space (see [MV97, ch. 25]), then it follows from [Net64, Theorem 1] that v is an analytic vector already if the orbit map is *weakly analytic*, that is, $\lambda \circ \gamma_v : g \to \mathbb{C}$ is real analytic for all $\lambda \in E'$. Here, E' denotes the dual space of continuous linear forms.

The space of analytic vectors is denoted by E^ω. A theorem of Nelson [Nel59, p. 599] asserts that E^ω is dense in E if E is a Banach space. More precisely, Nelson's theorem asserts the following. Let $h_t \in C^\infty(G)$ denote the *heat kernel* on G, where $t > 0$; then $\Pi(h_t)v \in E^\omega$ and $\Pi(h_t)v \to v$ for $t \to 0$ for all $v \in E$. In fact, the proof of Nelson's theorem is valid more generally if E is sequentially complete and with suitably restricted growth of π. In particular, this is the case for *F*-representations, see (2.3). The density is false in general, as easy examples such as the left regular representation of $\mathbb{R}$ on $C_c(\mathbb{R})$ show.

We wish to emphasize that E^ω is a G-invariant vector subspace of E. This follows immediately from the identity $\gamma_{\pi(g)v}(x) = \gamma_v(xg)$. We also note that E^ω is a $\mathfrak{g}$-invariant subset of the space E^∞ of smooth vectors.

It is convenient to introduce the following notation. For every $n \in \mathbb{N}$, we define the subspace of E^ω,

$$E_n = \{v \in E \mid \gamma_v \text{ extends to a holomorphic map } GV_n \to E\}.$$

Since G is totally real in $G_\mathbb{C}$ and GV_n is connected, the holomorphic extension of γ_v is unique if it exists. Let us denote the extension by $\gamma_{v,n} \in \mathcal{O}(GV_n, E)$. For each $z \in GV_n$, the operator

$$\pi_n(z) : E_n \to E, \quad \pi_n(z)v := \gamma_{v,n}(z),$$

is linear. In particular, uniqueness implies that

$$\pi_n(gz) = \pi(g)\pi_n(z)$$

for all $g \in G$, $z \in GV_n$. It is easily seen that if $m < n$, then $E_m \subset E_n$ and $\pi_m(z)v = \pi_n(z)v$ for $z \in GV_n$, $v \in E_m$. We shall omit the subscript n from the operator $\pi_n(z)$ if no confusion is possible.

A closely related space is

$$\tilde{E}_n = \{v \in E \mid \gamma_v|_{U_n} \text{ extends holomorphically to } V_n\}.$$

Lemma 3.2. *The space of analytic vectors is given by the increasing unions*

$$E^\omega = \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} \tilde{E}_n.$$

Furthermore,

$$E_n \subset \tilde{E}_n \subset E_{4n} \tag{3.1}$$

for all $n \in \mathbb{N}$.

Proof. The inclusions

$$\bigcup_{n\in\mathbb{N}} E_n \subset E^\omega \subset \bigcup_{n\in\mathbb{N}} \tilde{E}_n$$

as well as the first inclusion in (3.1) are clear. Hence, it suffices to prove the second inclusion in (3.1). Let $v \in V_n$ and let us denote the extension of γ_v by $f : V_n \to E$. For $g \in G$ and $z \in V_{4n}$,

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

we define

$$F(gz) := \pi(g)f(z) \in E.$$

We need to show that the expression is well defined. Assume that $gz = g'z'$ with $g, g' \in G$ and $z, z' \in V_{4n}$. Then $g^{-1}g' = zz'^{-1} \in V_{2n}$ and hence $g^{-1}g'x \in V_n$ for all $x \in V_{2n}$. Since $\pi(g)\pi(g^{-1}g'x)v = \pi(g')\pi(x)v$ for $x \in G$, analytic continuation from U_{2n} implies that $\pi(g)f(g^{-1}g'x) = \pi(g')f(x)$ for $x \in V_{2n}$. In particular, with $x = z'$ we obtain $\pi(g)f(z) = \pi(g')f(z')$, showing that F is well defined. As F is clearly holomorphic, we conclude that $v \in E_{4n}$. □

Next we want to topologize E^ω. For that, we notice that the holomorphic extensions provide injections of E_n and $\tilde{E}_n$ into $\mathcal{O}(GV_n, E)$ and $\mathcal{O}(V_n, E)$, respectively. We topologize E_n and $\tilde{E}_n$ by means of these maps and the standard compact open topologies. It is easily seen that the inclusion maps $E_n \to E_{n+1} \to E$ and $\tilde{E}_n \to \tilde{E}_{n+1} \to E$ are all continuous. Furthermore, we have the following lemma.

Lemma 3.3. *The inclusion maps in (3.1) are continuous for all $n \in \mathbb{N}$.*

Proof. Identifying E_n and $\tilde{E}_n$ with the corresponding spaces of holomorphic functions, we obtain the following neighborhood bases of 0. In E_n, the members are all sets

$$W_{K,Z} := \{f \in E_n \mid f(K) \subset Z\},$$

where $K \subset GV_n$ is compact and $Z \subset E$ is a zero neighborhood. Similarly, in $\tilde{E}_n$ the members are

$$\tilde{W}_{K,Z} := \{f \in \tilde{E}_n \mid f(K) \subset Z\},$$

where $K \subset V_n$ is compact and $Z \subset E$ is a zero neighborhood. The continuity of the first inclusion is then obvious.

With the mentioned identifications, the second inclusion is given by the map $f \to F$ described in the previous proof. Let a neighborhood $W = W_{K,O} \subset E_{4n}$ be given. Let $K' \subset V_{4n}$ be an arbitrary compact neighborhood of 0. By compactness of $K \subset GV_{4n}$, we obtain a finite union $K \subset \bigcup g_i K' \subset GV_{4n}$. Let $O' = \bigcap \pi(g_i)^{-1}(O)$; then $\tilde{W} = \tilde{W}_{K',O'}$ is an open neighborhood of 0 in $\tilde{E}_n$, and $f \in \tilde{W} \Rightarrow F \in W$. □

We endow E^ω with the inductive limit topology of the ascending unions in Lemma 3.2. The Hausdorff property follows, since E is assumed to be Hausdorff. It follows from Lemma 3.3 that the two unions give rise to the same topology. In symbols:

$$E^\omega = \lim_{n\to\infty} E_n = \lim_{n\to\infty} \tilde{E}_n \subset E, \tag{3.2}$$

with continuous inclusion into E. Since the restriction $\mathcal{O}(GV_n, E) \to C^\infty(G, E)$ is continuous for all $n \in \mathbb{N}$, we have $E^\omega \subset E^\infty$ with continuous inclusion.

Observe that an intertwining operator $T : E \to F$ between two representations (π, E), (ρ, F) carries E^ω continuously into F^ω. In fact, if $v \in E_n$ with the holomorphically extended orbit map $z \mapsto \pi(z)v$, then $Tv \in F_n$, since $z \mapsto T\pi(z)v$ is a holomorphic extension of the orbit map $g \mapsto \rho(g)Tv = T\pi(g)v$. It follows that T maps E_n continuously into F_n for each n.

Notice that if we define a continuous action of G on $\mathcal{O}(GV_n, E)$ by

$$(g \cdot f)(z) := \pi(g)f(g^{-1}z) \quad (g \in G, z \in GV_n),$$

then the image of $v \mapsto \pi_n(\cdot)v$ is the subspace $\mathcal{O}(GV_n, E)^G$ of G-invariant functions, with inverse map given by evaluation at $\mathbf{1}$. Thus, E_n is identified with a *closed* subspace of $\mathcal{O}(GV_n, E)$. In particular, it follows (see [Jar81, p. 365]) that E_n is complete/Fréchet if E has this property.

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

Let us briefly recall the structure of the open neighborhoods of zero in the limit E^ω. If A is a subset of some vector space, then we write $\Gamma(A)$ for the convex hull of A. Now, given for each n an open 0-neighborhood W_n in E_n (or $\tilde{E}_n$), the set

$$W := \Gamma\Big(\bigcup_{n\in\mathbb{N}} W_n\Big) \tag{3.3}$$

is an open convex neighborhood of 0 in E^ω. The set of neighborhoods W thus obtained forms a filter base of the 0-neighborhoods in E^ω.

Proposition 3.4. *Let (π, E) be a representation of a Lie group on a topological vector space E. Then the following assertions hold:*

(i) *the action $G \times E^\omega \to E^\omega$ is continuous and hence defines a representation (π, E^ω) of G;*

(ii) *each $v \in E^\omega$ is an analytic vector for (π, E^ω) and*

$$(E^\omega)^\omega = E^\omega$$

as topological vector spaces.

Proof. In (i) it suffices to prove continuity at $(\mathbf{1}, v)$ for each $v \in E^\omega$. We first prove the separate continuity of $g \mapsto \pi(g)v \in E^\omega$. Let $v \in E_n$ and consider the E-valued holomorphic extension of $g \mapsto \pi(g)v$. Since multiplication in $G_\mathbb{C}$ is holomorphic and $V_{2n} \cdot V_{2n} \subset V_n$, it follows that for each $z_1 \in V_{2n}$, the element $\pi_{2n}(z_1)v$ belongs to E_{2n}, with the holomorphic extension

$$z_2 \mapsto \pi_{2n}(z_2)\pi_{2n}(z_1)v := \pi_n(z_2 z_1)v \quad (z_1, z_2 \in GV_{2n}, v \in E_n) \tag{3.4}$$

of the orbit map. In particular, (3.4) holds for $z_1 = g \in U_{2n}$. The element $\pi(z_2 g)v \in E$ depends continuously on g, locally uniformly with respect to z_2. It follows that $g \mapsto \pi(g)v$ is continuous $U_{2n} \to E_{2n}$ and hence into E^ω.

In order to conclude the full continuity of (i), it now suffices to establish the following.

$(*)$ *For all compact subsets $B \subset G$, the operators $\{\pi(g) \mid g \in B\}$ form an equicontinuous subset of* $\mathrm{End}(E^\omega)$.

Before proving this, we note that for every compact subset $B \subset G$ and every $m \in \mathbb{N}$, there exists $n > m$ such that

$$b^{-1} V_n b \subset V_m \quad (b \in B).$$

This follows from the continuity of the adjoint action. Then $zb \in GV_m$ for all $z \in GV_n$ and hence $\pi(b)v \in E_n$ for all $b \in B$, $v \in E_m$ with

$$\pi_n(z)\pi(b)v = \pi_m(zb)v. \tag{3.5}$$

In order to prove $(*)$, we fix a compact set $B \subset G$. Given $m \in N$, we choose $n > m$ as above. We are going to prove equicontinuity $B \times E_m \to E_n$. An open neighborhood of 0 in E_n can be assumed of the form

$$(K, Z) := \{f \in E_n \mid f(K) \subset Z\},$$

where $K \subset GV_n$ is compact and $Z \subset E$ is a zero neighborhood. Then, with $K' = \bigcup_{b\in B} b^{-1} K b$ and $Z' = \bigcap_{b\in B} \pi(b)^{-1}(Z)$, we obtain

$$f(K') \subset Z' \Rightarrow \pi(b) f(b^{-1} K b) \subset Z$$

for all $b \in B$ and all functions $f : GV_m \to E$. If in addition f is G-invariant, then the conclusion is $f(Kb) \subset Z$ and we have shown that the right translation by b maps the zero neighborhood (K', Z') in E_m into the zero neighborhood (K, Z) in E_n.

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

The equicontinuity $B \times E^\omega \to E^\omega$ is an easy consequence given the description (3.3) of the neighborhoods in the inductive limit. This completes the proof of (i).

For the proof of (ii), let $v \in E_n$. In the first part of the proof we saw that $\pi(z_1)v \in E_{2n}$ for each $z_1 \in V_{2n}$, with the holomorphically extended orbit map given by (3.4). It then follows from Lemma A.1, applied to $V_{2n} \times GV_{2n}$ and the map $(z_1, z_2) \mapsto \pi(z_2 z_1)v$, that $z_1 \mapsto \pi(\cdot)\pi(z_1)v$ is holomorphic $V_{2n} \to \mathcal{O}(GV_{2n}, E)$. Hence, $z_1 \mapsto \pi(z_1)v$ is holomorphic into E_{2n} and hence also into E^ω. Thus, $g \mapsto \pi(g)v$ extends to a holomorphic E^ω-valued map on V_{2n} and hence $v \in (E^\omega)^\omega$ by the second description in (3.2).

For the topological statement in (ii), we need to show that the identity map is continuous $E^\omega \to (E^\omega)^\omega$. We just saw that the identity map takes

$$E_n \to \widetilde{(E^\omega)}_{2n};$$

hence, it suffices to show continuity of this map for each n. The proof given above reduces to the statement that the map mentioned below (A.1) is continuous. □

Corollary 3.5. *$(E^\infty)^\omega = (E^\omega)^\infty = E^\omega$ as topological vector spaces.*

Proof. The continuous inclusions $E^\omega \subset E^\infty \subset E$ induce continuous inclusions $E^\omega = (E^\omega)^\omega \subset (E^\infty)^\omega \subset E^\omega$. With E replaced by E^ω, the same inclusions also imply that $(E^\omega)^\omega \subset (E^\omega)^\infty \subset E^\omega$. □

We are interested in the functorial properties of the construction.

Lemma 3.6. *Let (π, E) be a representation and let $F \subset E$ be a closed invariant subspace. Then:*

(i) *$F^\omega = E^\omega \cap F$ as a topological space;*

(ii) *$E^\omega/F^\omega \subset (E/F)^\omega$ continuously.*

Proof. (i) Obviously, $F_n \subset E_n$ for all n. Conversely, if $v \in E_n \cap F$ with holomorphically extended orbit map $z \mapsto \pi(z)v \in E$, then $\pi(g)v \in F$ for all $g \in G$ implies that $\pi(z)v \in F$ for all $z \in GV_n$. Hence, $v \in F_n$. The topological statement follows easily.

(ii) The quotient map induces a continuous map $E^\omega \to (E/F)^\omega$, which in view of (i) induces the mentioned continuous inclusion. □

Notice also that if E_1, E_2 are representations, then the product representations satisfy $E_1^\omega \times E_2^\omega \simeq (E_1 \times E_2)^\omega$.

3.2 Completeness

In general, completeness of E does not ensure that E^ω is complete. For Banach representations this is the case, as the following result shows.

Proposition 3.7. *Let (π, E) be a representation of G on a complete DF-space. Then E^ω is complete.*

Proof. Let (v_i) be a Cauchy net in E^ω. It is Cauchy in E and hence converges to some element $v \in E$. Moreover, the net of orbit maps (γ_{v_i}) converges pointwise on G to γ_v. We need to show that γ_v is real analytic and, using our assumptions on E, it suffices to prove weak analyticity, see Remark 3.1.

Let $K \subset G$ be any compact set. We consider the space $A(K)$ of real analytic functions on K. These are germs of holomorphic functions defined on open neighborhoods V of K in $G_{\mathbb{C}}$,

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

and $A(K)$ is equipped with the inductive topology. Since each $\mathcal{O}(V)$ has the Montel property, the limit is compact, so that $A(K)$ inherits completeness from $\mathcal{O}(V)$.

For every $\lambda \in E'$, we consider the mapping

$$E^\omega \to A(K), \quad E_n \ni v \mapsto \text{germ of } \lambda \circ \gamma_v.$$

It is clear that this is a continuous map. It follows that $\lambda \circ \gamma_{v_i}|_K$ converges in $A(K)$, so that $\lambda \circ \gamma_v$ is real analytic on K. □

Remark 3.8. Combining the proof above with [BD01, Theorem 3] leads to a more general result for representations on Fréchet spaces. In this case, E^ω is complete whenever there is a fundamental system of seminorms $\{p_n\}_{n\in\mathbb{N}}$ for the topology of E such that

$$\exists n\ \forall m \geqslant n\ \exists k \geqslant m\ \exists C > 0\ \forall v \in E : p_m(v)^2 \leqslant C p_k(v) p_n(v).$$

Remark 3.9. An example by Grothendieck [Gro53b, p. 95] may be adapted to give an example of an incomplete space of analytic vectors. Consider the regular representation of $G = S^1$ on the (complete) space $E = C(S^1, \mathbb{C}^{\mathbb{N}})$, where $\mathbb{C}^{\mathbb{N}}$ is endowed with the product topology. The analytic vectors for this action are sequences of functions, which extend holomorphically to a common annulus $\{z \in \mathbb{C} \mid 1 - \varepsilon < |z| < 1 + \varepsilon\}$ for some $\varepsilon > 0$. Being a dense subspace of $(C(S^1)^\omega)^{\mathbb{N}}$, E^ω fails to be complete as well as sequentially complete.

3.3 Definition of analytic representation

Motivated by Proposition 3.4, we shall give the following definition.

Definition 3.10. A representation (π, E) is called *analytic* if $E = E^\omega$ holds as topological vector spaces.

Given a representation (π, E), Proposition 3.4 implies that (π, E^ω) is an analytic representation.

Lemma 3.11. *Let (π, E) be an analytic representation and let $F \subset E$ be a closed invariant subspace. Then π induces analytic representations on both F and E/F.*

Proof. This follows from Lemma 3.6. From (i) in that lemma we infer immediately that $F^\omega = F$, and from (ii) we then conclude that $E/F = E^\omega/F^\omega \to (E/F)^\omega$ is continuous. The opposite inclusion is trivially valid and continuous. □

Example 3.12. We consider the Fréchet space $E := \mathcal{O}(G_{\mathbb{C}})$ with the right regular action of G,

$$\pi(g)f(z) = f(zg) \quad (g \in G, z \in G_{\mathbb{C}}, f \in \mathcal{O}(G_{\mathbb{C}})).$$

It is easy to see that (π, E) defines a representation. Given $v \in E$, it follows from (A.1) that the orbit map $\gamma_v : g \to E$ extends to a holomorphic mapping from $G_{\mathbb{C}}$ to E. The same equation implies easily that $E = E^\omega$ as topological spaces. Thus, (π, E) is analytic.

3.4 Irreducible analytic representations

It is a natural question on which type of topological vector spaces E one can model irreducible analytic representations. The next result shows that this class is rather restrictive.

Theorem 3.13. *Let (π, E) be an irreducible representation of a reductive group on a Fréchet space E. If $E = E^\omega$ as vector spaces, then E is finite dimensional.*

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

Proof. By passing to a covering group if necessary, we may assume that $G_{\mathbb{C}}$ is simply connected. By assumption, $E^\omega = \lim E_n$ identifies with E as vector spaces. The Grothendieck factorization theorem implies that $E = E_n$ for some n (see [Gro73, ch. 4, § 5, Theorem 1]). Hence, the operator $\pi(x) := \pi_n(x)$ is defined on E for all $x \in V_n$. We shall holomorphically extend to all $x \in G_{\mathbb{C}}$.

Let $v \in E$. By the monodromy theorem, it suffices to extend $\pi(x)v$ along all simple smooth curves starting at $\mathbf{1}$. Let $\gamma : [0,1] \to G_{\mathbb{C}}$ be such a curve with $\gamma(0) = \mathbf{1}$. We select finitely many open sets $U_1, \dots, U_k \subset G_{\mathbb{C}}$ which cover the curve $\gamma([0,1])$ and points

$$x_i = \gamma(t_i), \quad 0 = t_1 < \dots < t_k < 1,$$

such that $\mathbf{1} = x_1 \in U_1$ and $x_i \in U_i \cap U_{i-1}$ for $i > 1$. By choosing the sets U_i sufficiently small (and sufficiently many), we may assume that $U_i \subset V_{2n} x_i$ for each i and also that the only non-empty overlaps are among neighboring sets U_i and U_{i-1} (to attain these properties, it may be useful from the outset to select the sets inside a tubular neighborhood around the curve).

In particular, $\pi(x)v$ is already defined for $x \in U_1 \subset V_{2n}$. On $U_2, \dots, U_k$, we recursively define

$$\pi(x)v = \pi(z)\pi(x_i)v, \quad x = zx_i \in U_i \subset V_{2n}x_i,$$

where $\pi(x_i)v$ is defined in the preceding step. Clearly, this depends holomorphically on x. However, in order to obtain a proper extension of $x \mapsto \pi(x)v$, we need to verify that $\pi(x)v$ is well defined on overlaps between the U_i. What we need to show is that

$$\pi(z)\pi(x_i)v = \pi(zx_i)v, \quad zx_i \in U_i \cap U_{i-1}.$$

Let $x_i = yx_{i-1}$, where $y \in V_{2n}$. By the recursive definition, we have $\pi(x_i)v = \pi(y)\pi(x_{i-1})v$ and $\pi(zx_i)v = \pi(zy)\pi(x_{i-1})v$. Then the desired identity follows, since $\pi(z)\pi(y) = \pi(zy)$ by (3.4).

Thus, the representation extends to an irreducible holomorphic representation of $G_{\mathbb{C}}$ (also denoted by π). If $U < G_C$ is a compact real form, then the Peter–Weyl theorem implies that $\pi|_U$ is irreducible and finite dimensional. □

Remark 3.14. Non-reductive groups, on the other hand, may have irreducible analytic actions on a Fréchet space. As an example, consider the Schrödinger representation of the Heisenberg group $\mathbb{H}^n$ on the Fréchet space

$$E = \left\{ f \in \mathcal{O}(\mathbb{C}^n) \mid \forall N, M \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} \sup_{y \in (-N,N)^n} |f(x+iy)| e^{M|x|} < \infty \right\}.$$

It is irreducible as a restriction of the Schrödinger representation on $L^2(\mathbb{R}^n)$, and one readily verifies that $E = E^\omega$.

4. The algebra of analytic superdecaying functions

We define a convolution algebra of analytic functions with fast decay. The purpose is to obtain an algebra which acts on representations of restricted growth, such as F-representations.

4.1 Superdecaying functions

Let us denote by dg the Riemannian measure on G associated to the metric $\mathbf{g}$ and note that dg is a left Haar measure. It is of some relevance below that there is a constant $c > 0$ such that

$$\int_G e^{-cd(g)}\, dg < \infty \tag{4.1}$$

(see [Gar60, p. 75, Lemme 2]).

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

We define the space of superdecaying continuous function on G by

$$\mathcal{R}(G) := \left\{ f \in C(G) \mid \forall N \in \mathbb{N} : \sup_{g \in G} |f(g)| e^{Nd(g)} < \infty \right\}$$

and equip it with the corresponding family of seminorms. Note that $\mathcal{R}(G)$ is independent of the choice of the left-invariant metric, and that it has the following properties.

Proposition 4.1. (i) *$\mathcal{R}(G)$ is a Fréchet space and the natural action of $G \times G$ by left–right displacements defines an F-representation.*

(ii) *$\mathcal{R}(G)$ becomes a Fréchet algebra under convolution:*

$$f * h(x) = \int_G f(y) h(y^{-1}x)\, dy$$

for $f, h \in \mathcal{R}(G)$ and $x \in G$.

(iii) *Every F-representation (π, E) of G gives rise to a continuous algebra representation of $\mathcal{R}(G)$,*

$$\mathcal{R}(G) \times E \to E, \quad (f, v) \mapsto \Pi(f)v,$$

where

$$\Pi(f)v := \int_G f(g)\pi(g)v\, dg \quad (f \in \mathcal{R}(G), v \in E)$$

as an E-valued integral.

Proof. Easy. Use (2.1), (2.3), and (4.1). □

4.2 Analytic superdecaying functions

We shall start with a discussion of the analytic vectors in $\mathcal{R}(G)$. Henceforth, we shall view $\mathcal{R}(G)$ as a G-module for the left regular representation of G. We set $\mathcal{A}(G) := \mathcal{R}(G)^\omega$ and equip $\mathcal{A}(G)$ with the corresponding vector topology. With the notation from the preceding section, we put $\mathcal{A}_n(G) := \mathcal{R}(G)_n$ for each $n \in \mathbb{N}$. Notice that $\mathcal{A}_n(G)$ is a Fréchet space for each n, since $\mathcal{R}(G)$ is Fréchet. Hence, $\mathcal{A}(G)$ is an *LF*-space (inductive limit of Fréchet spaces). In the appendix, we show that $\mathcal{A}(G)$ is complete and reflexive.

Proposition 4.2. (i) *$\mathcal{A}(G)$ carries representations of G by left and right actions.*

(ii) *$\mathcal{A}(G)$ is a subalgebra of $\mathcal{R}(G)$ and convolution is continuous*

$$\mathcal{A}(G) \times \mathcal{A}(G) \to \mathcal{A}(G).$$

Proof. (i) The statement about the left action is immediate from Proposition 3.4(i). It is clear that $\mathcal{A}(G)$ is right invariant, since every right displacement $f \mapsto R_g f$ is an intertwining operator for the left regular representation. The continuity of the right action follows from Lemma 4.3 below, see Remark 4.4.

(ii) This follows from Proposition 4.6 (to be proved below) by taking $E = \mathcal{R}(G)$. □

The next lemma gives us a concrete realization of $\mathcal{A}_n(G)$.

Lemma 4.3. *For all $n \in \mathbb{N}$, restriction to G provides a topological isomorphism of*

$$\left\{ f \in \mathcal{O}(V_n G) \mid \forall N > 0, \forall \Omega \subset V_n \text{ compact} : \sup_{g \in G, z \in \Omega} |f(zg)| e^{Nd(g)} < \infty \right\}$$

onto $\mathcal{A}_n(G)$. Here, the space above is topologized by the seminorms mentioned in its definition.

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

Proof. Let $f \in \mathcal{A}_n(G)$. Then $\gamma_f : g \to \mathcal{R}(G), g \mapsto f(g^{-1}\cdot)$ extends to a holomorphic map $\gamma_{f,n} : gV_n \to \mathcal{R}(G)$. As point evaluations $\mathcal{R}(G) \to \mathbb{C}$ are continuous, it follows that $F(z) := \gamma_{f,n}(z^{-1})(\mathbf{1})$ defines a holomorphic extension of f to V_nG. Moreover, $F(zg) = \gamma_{f,n}(z^{-1})(g)$ for $z \in V_n, g \in G$. Let $N > 0$ and a compact set $\Omega \subset V_n$ be given; then

$$\sup_{g\in G, z\in\Omega} |F(zg)|e^{Nd(g)} = \sup_{z\in\Omega} p_N(\gamma_{f,n}(z^{-1})) < \infty,$$

where $p_N(h) = \sup_{g\in G} |h(g)|e^{Nd(g)}$ is a defining seminorm of $\mathcal{R}(G)$. Hence, F belongs to the space above. Moreover, we see that $f \mapsto F$ is an isomorphism onto its image.

Conversely, let F belong to the space above and put $f := F|_G$. Then it is clear that $f \in \mathcal{R}(G)$ (take $\Omega = \{\mathbf{1}\}$). We need to show that $f \in \mathcal{A}_n(G)$, i.e. that $\gamma_f : g \to \mathcal{R}(G)$ extends to a holomorphic map $GV_n \to \mathcal{R}(G)$. The extension is $z \mapsto F(z^{-1}\cdot)$, and we need to show that it is holomorphic.

We first show that $z \mapsto F(z^{-1}\cdot)$ is continuous into $\mathcal{R}(G)$. To see this, let $z_0 \in GV_n$ and $\epsilon, N > 0$ be given. We wish to find a neighborhood D of z_0 such that

$$p_N(F(z^{-1}\cdot) - F(z_0^{-1}\cdot)) < \epsilon \tag{4.2}$$

for all $z \in D$.

Let us fix a compact neighborhood D_0 of z_0 in GV_n. As

$$\sup_{g\in G, z\in D_0} |F(z^{-1}g)|e^{md(g)} < \infty$$

for all $m > N$, we find a compact subset $K \subset G$ such that

$$\sup_{g\in G\setminus K, z\in D_0} |F(z^{-1}g)|e^{Nd(g)} < \epsilon/2.$$

Shrinking D_0 to some possibly smaller neighborhood D, we may request that

$$\sup_{g\in K, z\in D} |F(z^{-1}g) - F(z_0^{-1}g)|e^{Nd(g)} < \epsilon.$$

The required estimate (4.2) follows.

As continuity has been verified, holomorphicity follows provided that $z \mapsto \lambda(F(z^{-1}\cdot))$ is holomorphic for λ ranging in a subset whose linear span is weakly dense in $\mathcal{R}(G)'$ (see [Gro53a, p. 39, Remarque 1]). A convenient such subset is $\{\delta_g \mid g \in G\}$, and the proof is complete. □

Remark 4.4. Let $q(f) := \sup_{g\in G, z\in\Omega} |f(zg)|e^{Nd(g)}$ be a seminorm on $\mathcal{A}_n(G)$ as above. Then (2.1) implies that

$$q(R_x f) \leqslant e^{Nd(x)} q(f) \quad (f \in \mathcal{A}_n(G)),$$

for $x \in G$, so $\mathcal{A}_n(G)$ is an F-representation for the right action.

4.3 Analytic vectors of F-representations

Let (π, E) be an F-representation of G and let $v \in E$. The map $f \mapsto \Pi(f)v$ is intertwining from $\mathcal{R}(G)$ (with left action) to E. Hence, $\Pi(f)v \in E_n$ for $f \in \mathcal{A}_n(G)$ and $\Pi(f)v \in E^\omega$ for $f \in \mathcal{A}(G)$. With the preceding characterization of $\mathcal{A}_n(G)$, we have

$$\pi(z)\Pi(f)v = \int_G f(z^{-1}g)\pi(g)v \, dg \tag{4.3}$$

for $f \in \mathcal{A}_n(G)$, $z \in GV_n$.

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

Remark 4.5. In particular,

$$\Pi(\mathcal{A}(G))E^\omega \subset E^\omega$$

for F-representations. In fact, one can show (see [GKL]) that

$$\Pi(\mathcal{A}(G))E^\omega = E^\omega.$$

It is easily seen that the action of $\mathcal{A}(G)$ on E^ω is an algebra action. We shall now see that it is continuous.

Proposition 4.6. *Let (π, E) be an F-representation. The bilinear map $(f, v) \mapsto \Pi(f)v$ is continuous*

$$\mathcal{A}_n(G) \times E \to E_n$$

for every $n \in \mathbb{N}$. Likewise, it is continuous

$$\mathcal{A}(G) \times E \to E^\omega.$$

Notice that since E^ω injects continuously in E, the last statement implies continuity of both

$$\mathcal{A}(G) \times E \to E \quad \text{and} \quad \mathcal{A}(G) \times E^\omega \to E^\omega.$$

Proof. Let $n \in \mathbb{N}$ be fixed and let $W \subset E_n$ be an open neighborhood of 0. We may assume that

$$W = W_{K,p} := \{v \in E_n \mid p(\pi(K)v) < 1\},$$

with $K \subset GV_n$ compact and p a continuous seminorm on E such that

$$p(\pi(g)v) \leqslant Ce^{cd(g)}p(v) \quad (g \in G, v \in E)$$

for some constants c, C (see (2.3)).

Choose $N > 0$ so that (cf. (4.1))

$$C_1 := \int_G e^{(c-N)d(g)}\, dg < \infty$$

and let

$$O := \left\{ f \in \mathcal{O}(V_nG) \;\middle|\; \sup_{z\in K, g\in G} |f(z^{-1}g)|e^{Nd(g)} < \epsilon \right\} \subset \mathcal{A}_n(G)$$

(with ϵ to be specified below). According to Lemma 4.3, O is open.

For $f \in O$ and $z \in K$, we obtain by (4.3)

$$p(\pi(z)\Pi(f)v) \leqslant \int_G |f(z^{-1}g)|p(\pi(g)v)\, dg \leqslant \epsilon CC_1p(v).$$

With $\epsilon < 1/(CC_1)$, we conclude that $\Pi(f)v \in W$ if $f \in O$ and $p(v) < 1$.

This proves the first statement. By taking inductive limits, we infer continuity of $\lim(\mathcal{A}_n(G) \times E) \to E^\omega$. For the continuity of $\mathcal{A}(G) \times E \to E^\omega$, it now suffices to verify that $\lim(\mathcal{A}_n(G) \times E)$ and $\mathcal{A}(G) \times E = (\lim \mathcal{A}_n(G)) \times E$ are isomorphic. The map

$$\lim(\mathcal{A}_n(G) \times E) \to (\lim \mathcal{A}_n(G)) \times E$$

is clearly bijective and continuous. The left-hand side is LF, and the right-hand side is a product of ultra-bornological spaces and hence itself ultra-bornological. It follows that the open mapping theorem can be applied (see [MV97, Theorem 24.30 and Remarks 24.15 and 24.36]). □

For later use, we note that $\mathcal{A}(G)$ contains a Dirac sequence.

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

Lemma 4.7. *The heat kernel h_t belongs to $\mathcal{A}(G)$ for each $t>0$. Let E be an F-representation. Then $\Pi(h_t)v \to v$ in E for all $v \in E$.*

Proof. The convergence in E is Nelson's theorem (see § 3). The heat kernel belongs to $\mathcal{A}(G)$ for all $t>0$ by [GKL, Theorem 4.2].

Remark 4.8. It follows from the proof of [GKL, Theorem 4.2] that there exists a common m such that $h_t \in \mathcal{A}_m(G)$ for all $t>0$.

4.4 $\mathcal{A}(G)$-tempered representations

As we have seen that there is a continuous algebra action of $\mathcal{A}(G)$ on the analytic vectors of F-representations, we shall make this property part of a definition.

Definition 4.9. A representation (π, E) is called $\mathcal{A}(G)$-tempered if for all $f \in \mathcal{A}(G)$ and $v \in E$ the vector-valued integral

$$\Pi(f)v = \int_G f(g)\pi(g)v\, dg$$

converges absolutely in E, and $(f, v) \mapsto \Pi(f)v$ defines a continuous algebra action

$$\mathcal{A}(G) \times E \to E.$$

The absolute convergence of the vector-valued integral is assumed with respect to all continuous seminorms on E.

Example 4.10. (a) For every F-representation (π, E), both (π, E) itself and (π, E^ω) are $\mathcal{A}(G)$-tempered according to Proposition 4.6. In particular, this holds for all Banach representations and also for $E = \mathcal{R}(G)$ with the left action (so that $E^\omega = \mathcal{A}(G)$).

(b) If (π, E) is an $\mathcal{A}(G)$-tempered representation and $F \subset E$ is a closed G-invariant subspace, then the induced representations on F and E/F are $\mathcal{A}(G)$-tempered.

5. Analytic globalizations of Harish-Chandra modules

In this section we will assume that G is a real reductive group. Let us fix a maximal compact subgroup $K < G$. We say that a complex vector space V is a $(\mathfrak{g}, K)$-module if V is endowed with a Lie algebra action of $\mathfrak{g}$ and a locally finite group action of K which are compatible in the sense that the derived and restricted actions of $\mathfrak{k}$ agree and, in addition,

$$k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v) \quad (k \in K, X \in \mathfrak{g}, v \in V).$$

We call a $(\mathfrak{g}, K)$-module *admissible* if for any irreducible representation (σ, W) of K the multiplicity space $\mathrm{Hom}_K(W, V)$ is finite dimensional. Finally, an admissible $(\mathfrak{g}, K)$-module is called a *Harish-Chandra module* if V is finitely generated as a $\mathcal{U}(\mathfrak{g})$-module. Here, as usual, $\mathcal{U}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$.

By a *globalization* of a Harish-Chandra module V we understand a representation (π, E) of G such that the space of K-finite vectors

$$E_K := \{v \in E \mid \dim \mathrm{span}_{\mathbb{C}}\{\pi(K)v\} < \infty\}$$

is $(\mathfrak{g}, K)$-isomorphic to V and dense in E. Density of E_K is automatic whenever E is quasi-complete, see [Har66, Lemma 4]. Each element $v \in E$ allows an expansion in K-types $v = \sum_{\tau \in \hat{K}} v_\tau$, where $v_\tau = \dim \tau \pi(\chi_\tau)v \in E_K$. Here, the integral over K that defines $\pi(\chi_\tau)v$ may

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

take place in the completion of E, but v_τ belongs to E_K by density and finite dimensionality of K-type spaces.

A *Banach (F-, analytic, $\mathcal{A}(G)$-tempered) globalization* is a globalization by a Banach (F-, analytic, $\mathcal{A}(G)$-tempered) representation. Note that according to Harish-Chandra [Har53], $E_K \subset E^\omega$ if E is a Banach globalization. In general, the orbit map of a vector $v \in E_K$ is weakly analytic (see Remark 3.1).

According to the subrepresentation theorem of Casselman (see [Wal88, Theorem 3.8.3]), V admits a Banach globalization E. The space E^ω is then an analytic $\mathcal{A}(G)$-tempered globalization.

If V is a Harish-Chandra module, we denote by $\tilde{V}$ the Harish-Chandra module dual to V, i.e. the space of K-finite linear forms on V (see [Wal88, p. 115]). We note that if E is a globalization of V, then $\tilde{V}$ embeds into E' and identifies with the subspace of K-finite continuous linear forms (see [Cas89, Proposition 2.2]). Furthermore, $\tilde{V}$ separates on E. Since the matrix coefficients $x \mapsto \xi(\pi(x)v)$ for $v \in V, \xi \in \tilde{V}$ are real analytic functions on G, they are determined by their germs at $\mathbf{1}$. It follows that these functions on G are independent of the globalization (see [Cas89, p. 396]).

5.1 Minimal analytic globalizations

Let V be a Harish-Chandra module and $\mathbf{v} = \{v_1, \dots, v_k\}$ be a set of $\mathcal{U}(\mathfrak{g})$-generators. We shall fix an arbitrary $\mathcal{A}$-tempered globalization (π, E) and regard V as a subspace in E.

On the product space $\mathcal{A}(G)^k = \mathcal{A}(G) \times \cdots \times \mathcal{A}(G)$ with diagonal G-action, we consider the G-equivariant map

$$\Phi_{\mathbf{v}} : \mathcal{A}(G)^k \to E, \quad \mathbf{f} = (f_1, \dots, f_k) \mapsto \sum_{j=1}^{k} \Pi(f_j) v_j,$$

and write $I_{\mathbf{v}}$ for its kernel. This map is evidently continuous and thus $I_{\mathbf{v}}$ is a closed G-invariant subspace of $\mathcal{A}(G)^k$. We note that $\mathbf{f} \in I_{\mathbf{v}}$ if and only if $\sum_j \int f_j(g)\xi(\pi(g)v_j)\, dg = 0$ for all $\xi \in \tilde{V}$. It follows that $I_{\mathbf{v}}$ is independent of the choice of globalization. Furthermore, the dependence on generators is easily described: if $\mathbf{v}'$ is another set of generators, say k' in number, then there exists a $k \times k'$ matrix u of elements from $\mathcal{U}(\mathfrak{g})$ such that $\mathbf{f} \in I_{\mathbf{v}}$ if and only if $R_u \mathbf{f} \in I_{\mathbf{v}'}$.

Since $I_{\mathbf{v}}$ is closed and G-invariant, the quotient

$$V^{\min} := \mathcal{A}(G)^k / I_{\mathbf{v}}$$

carries a representation of G, which we denote by $(\pi, V^{\min})$. It is independent of the choice of the globalization (π, E) and (up to equivalence) of the set $\mathbf{v}$ of generators.

Lemma 5.1. *Let V be a Harish-Chandra module. Then the following assertions hold.*

(i) *$V^{\min}$ is an analytic $\mathcal{A}(G)$-tempered globalization of V.*

(ii) *$V^{\min} = \Pi(\mathcal{A}(G))V$, that is, $V^{\min}$ is spanned by the vectors of the form $\Pi(f)v$.*

(iii) *If (λ, F) is any $\mathcal{A}$-tempered globalization of V, then the identity mapping $V \to F$ lifts to a G-equivariant continuous injection $V^{\min} \to F^\omega$.*

Proof. (i) It follows from the definition that $V^{\min}$ is analytic (see Lemma 3.6) and $\mathcal{A}(G)$-tempered (see Example 4.10(b)). It remains to be seen that $(V^{\min})_K$ is $(\mathfrak{g}, K)$-isomorphic to V. By definition, $\Phi_{\mathbf{v}}$ induces a continuous G-equivariant injection $V^{\min} \to E$. In particular, $(V^{\min})_K$ is isomorphic to a $(\mathfrak{g}, K)$-submodule of $V = E_K$. Moreover, as $\mathcal{A}(G)$ contains a Dirac sequence by Lemma 4.7, and as we may assume E to be a Banach space, each generator v_j belongs

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

to the E-closure of the image of $V^{\min}$. By admissibility and finite dimensionality of K-types, v_j belongs to $(V^{\min})_K$ for each j. Thus, $(V^{\min})_K \simeq V$ and (i) follows. Assertions (ii) and (iii) are clear. □

Because of property (iii), we shall refer to $V^{\min}$ as the *minimal $\mathcal{A}(G)$-tempered globalization* of V. We record the following functorial properties of the construction.

Lemma 5.2. *Let V, W be Harish-Chandra modules.*

(i) *Every $(\mathfrak{g}, K)$-homomorphism $T : W \to V$ lifts to a unique intertwining operator $T^{\min} : W^{\min} \to V^{\min}$ with restriction T on $W = (W^{\min})_K$ and with closed image.*

(ii) *Assume that $W \subset V$ is a submodule. Then:*

(a) *$W^{\min}$ is equivalent with a subrepresentation of $V^{\min}$ on a closed invariant subspace;*
(b) *$(V/W)^{\min}$ is equivalent with the quotient representation $V^{\min}/W^{\min}$.*

Proof. (i) Let $\tilde{T} : \tilde{V} \to \tilde{W}$ denote the dual map of T and observe that

$$\tilde{T}\xi(\pi(g)w) = \xi(\pi(g)Tw)$$

for all $w \in W, \xi \in \tilde{V}$ and $g \in G$. Indeed, these are analytic functions of g whose power series at $\mathbf{1}$ agree because T is a $\mathfrak{g}$-homomorphism. It follows that if we choose generators $w_1, \dots, w_l$ for W and $v_1, \dots, v_k$ for V such that $v_j = Tw_j$ for $j = 1, \dots, l$, then the inclusion map $\mathbf{f} \mapsto (\mathbf{f}, \mathbf{0})$ of $\mathcal{A}(G)^l$ into $\mathcal{A}(G)^k$ takes $I_{\mathbf{w}}$ into $I_{\mathbf{v}}$. Hence, this inclusion map induces a map

$$T^{\min} : \mathcal{A}(G)^l/I_{\mathbf{w}} \to \mathcal{A}(G)^k/I_{\mathbf{v}}$$

which is continuous, intertwining, and has closed image. Moreover, this map restricts to T on W, since it maps each generator w_j to $v_j = Tw_j$.

(ii) is obtained from (i) with T equal to (a) the inclusion map $W \to V$ or (b) the quotient map $V \to V/W$. □

Our next concern will be to realize the analytic globalizations inside Banach modules.

Proposition 5.3. *Let (π, E) be an analytic $\mathcal{A}(G)$-tempered globalization of a Harish-Chandra module V. Then there exist a Banach representation (σ, F) of G and a continuous G-equivariant injection $(\pi, E) \to (\sigma, F)$.*

Proof. We fix generators $\xi = \{\xi_1, \dots, \xi_l\}$ of the dual Harish-Chandra module $\tilde{V} \subset E'$ and put $U := \{v \in E \mid \max_{1 \leqslant j \leqslant l} |\xi_j(v)| < 1\}$. Then U is an open neighborhood of 0 in E.

Fix $m \in \mathbb{N}$ such that $\mathcal{A}_m(G)$ contains a Dirac sequence (see Remark 4.8). As $\mathcal{A}_m(G) \times E \to E$ is continuous, we find an open neighborhood O of 0 in $\mathcal{A}_m(G)$ and an open neighborhood W of 0 in E such that $\Pi(O)W \subset U$. We may assume that O is of the type $O = \{f \in \mathcal{A}_m(G) \mid q(f) < 1\}$, where

$$q(f) = \sup_{\substack{g \in G \\ z \in \Omega}} |f(zg)| e^{Nd(g)}$$

for some $N \in \mathbb{N}$ and $\Omega \subset V_m$ compact. Define the normed space $X := (\mathcal{A}_m(G), q)$. It follows from Remark 4.4 that the right regular action of G is a representation by bounded operators on X. Let $F := (X^*)^l$ be the topological dual of X^l and σ the corresponding dual diagonal action of G. Note that F is a Banach space, being the dual of a normed space, so that σ is a Banach representation.

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

We claim that the map

$$\phi : e \to F, \quad v \mapsto \left(\mathbf{f} = (f_1, \dots, f_l) \mapsto \sum_{j=1}^{l} \xi_j(\Pi(f_j)v) \right)$$

is G-equivariant, continuous, and injective. Equivariance is clear, and in order to establish continuity we fix a closed convex neighborhood $\tilde{O}$ of 0 in F. We may assume that $\tilde{O}$ is a polar of the form $\tilde{O} = [B^l]^o$, where B is a bounded set $B \subset X$. Because B is bounded, there exists $\lambda > 0$ such that $B \subset \lambda O$. Choosing $\tilde{W} := (1/\lambda)W$, we have $\phi(\tilde{W}) \subset \tilde{O}$, as

$$\phi(\tilde{W})(B^l) \subset \frac{1}{l}\phi(W)(O^l) \subset \frac{1}{l}\sum_{j=1}^{l} \xi_j(\Pi(O)W)$$

$$\subset \frac{1}{l}\sum_{j=1}^{l} \xi_j(U) \subset \{z \in \mathbb{C} \mid |z| \leqslant 1\}.$$

It remains to be shown that ϕ is injective. Suppose that $\phi(v) = 0$. Then $\phi(v_\tau) = 0$ for each element v_τ in the K-finite expansion of v, so that we may assume that v is K-finite. Then for all $f \in \mathcal{A}_m(G)$ and $\eta \in \tilde{V}$ one would have $\eta(\Pi(f)v) = 0$. Since K-finite matrix coefficients are independent of globalizations, we conclude by Lemma 4.7 that $\eta(v) = 0$ and hence $v = 0$. □

5.2 The minimal analytic globalization of a spherical principal series representation

Let $G = KAN$ be an Iwasawa decomposition of G and denote by M the centralizer of A in K, i.e. $M = Z_K(A)$. Then $P = MAN$ is a minimal parabolic subgroup. Let us denote by $\mathfrak{a}, \mathfrak{n}$ the Lie algebras of A and N and define $\rho \in \mathfrak{a}^*$ by $\rho(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}\, X|_{\mathfrak{n}})$, $X \in \mathfrak{a}$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $a \in A$, we set $a^\lambda := e^{\lambda(\log a)}$.

The *smooth spherical principal series with parameter* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is defined by

$$V_\lambda^\infty := \{f \in C^\infty(G) \mid \forall man \in P\ \forall g \in G : f(man\, g) = a^{\rho+\lambda} f(g)\}.$$

The action of G on V_λ^∞ is by right displacements in the arguments, and in this way we obtain a smooth F-representation $(\pi_\lambda, V_\lambda^\infty)$ of G. We denote the Harish-Chandra module of V_λ^∞ by V_λ.

It is useful to observe that the restriction mapping to K,

$$\mathrm{Res}_K : v_\lambda^\infty \to C^\infty(M\backslash K),$$

is an K-equivariant isomorphism of Fréchet spaces, and henceforth we will identify V_λ^∞ with $C^\infty(M\backslash K)$. The space V_λ of K-finite vectors in V_λ^∞ is then identified as a K-module with the space $C(M\backslash K)_K$ of K-finite functions on $M\backslash K$.

Likewise, the Hilbert space $\mathcal{H}_\lambda := L^2(M\backslash K)$ is provided with the representation π_λ. The space of smooth vectors for this representation is $\mathcal{H}_\lambda^\infty = V_\lambda^\infty = C^\infty(M\backslash K)$, and the space of analytic vectors is the space $\mathcal{H}_\lambda^\omega = V_\lambda^\omega := C^\omega(M\backslash K)$ of analytic functions on $M\backslash K$ with its usual topology.

Theorem 5.4. *For every* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, *one has*

$$\Pi_\lambda(\mathcal{A}(G))V_\lambda = C^\omega(M\backslash K).$$

In particular, $V_\lambda^{\min} \simeq V_\lambda^\omega = C^\omega(M\backslash K)$ *as analytic representations.*

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

The proof of this theorem is similar to the corresponding result in the smooth case (see [BK, § 4]). Note that from Lemma 5.1, we have

$$\Pi_\lambda(\mathcal{A}(G))V_\lambda = V_\lambda^{\min} \subset V_\lambda^\omega$$

with continuous inclusion. As the space $V_\lambda^{\min}$ admits a web (see [MV97, 24.8 and 24.28]) and $C^\omega(M\backslash K)$ is ultra-bornological (see [MV97, 24.16]), we can apply the open mapping theorem [MV97, 24.30] to obtain an identity of topological spaces from the set-theoretical identity. It thus suffices to prove that for each $v \in V_\lambda^\omega$, there exist $\xi \in V_\lambda$ and $F \in \mathcal{A}(G)$ such that $\Pi(F)\xi = v$.

We need some technical preparations. Let us denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$, and write θ for the corresponding Cartan involution. Let $(\cdot,\cdot)$ be a non-degenerate invariant bilinear form on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Then $\langle\cdot,\cdot\rangle = -(\theta\cdot,\cdot)$ defines an inner product on $\mathfrak{g}$, which we use to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. We write $|\cdot|$ for the norms induced on $\mathfrak{g}$ and $\mathfrak{g}^*$.

Let $X_1,\dots,X_s$ be an orthonormal basis of $\mathfrak{k}$ and $Y_1,\dots,Y_l$ be an orthonormal basis of $\mathfrak{p}$. We define elements in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ by

$$\Delta = \sum_{j=1}^{s} X_j^2 + \sum_{i=1}^{l} Y_i^2, \quad \Delta_K = \sum_{j=1}^{s} X_j^2, \quad \text{and} \quad \mathrm{C} := \Delta - 2\Delta_K.$$

Note that C is a Casimir element. In particular, it belongs to the center of $\mathcal{U}(\mathfrak{g})$.

Let $\mathfrak{t} \subset \mathfrak{k}$ be a maximal torus. We fix a positive system of the root system $\Sigma(\mathfrak{t}_\mathbb{C}, \mathfrak{k}_\mathbb{C})$ and identify the unitary dual $\hat{K}$ via their highest weights with a subset of $i\mathfrak{t}^*$. If (τ, W_τ) is an irreducible representation of K, then Δ_K acts as the scalar multiple $|\tau+\rho_\mathfrak{k}|^2 - |\rho_\mathfrak{k}|^2$. For every $\tau \in \hat{K}$, we denote by $\chi_\tau \in C(K)$ the normalized character $\chi_\tau(k) = (\dim W_\tau)^{-1}\mathrm{tr}\,\tau(k)$. Note that $C(K)$ acts on $\mathcal{A}(G)$ by left convolution.

We denote the left regular representation of G on $\mathcal{A}(G)$ by L. The following proposition will be crucial in the proof of Theorem 5.4.

Proposition 5.5. *Let $(c_\tau)_{\tau\in\hat{K}}$ be a sequence of complex numbers and $(a_\tau)_{\tau\in\hat{K}}$ a sequence of elements in G. Assume that*

$$|c_\tau| \leqslant Ce^{-\epsilon|\tau|}, \quad d(a_\tau) \leqslant c_1 \log(1+|\tau|) + c_2$$

for some $C, \epsilon, c_1, c_2 > 0$. Let $f \in \mathcal{A}(G)$. Then

$$F := \sum_{\tau\in\hat{K}} c_\tau \chi_\tau * L(a_\tau) f \in \mathcal{A}(G).$$

Proof. As $(L, \mathcal{R}(G))$ is an F-representation, it follows from [GKL] that $h \in \mathcal{R}(G)$ belongs to $\mathcal{A}(G)$ if and only if there exists an $M > 0$ such that for all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ with

$$\sup_{g\in G} e^{Nd(g)}|\Delta^k h(g)| \leqslant C_N M^{2k}(2k)! \tag{5.1}$$

for all $k \in \mathbb{N}$.

Observe that $\Delta = \mathrm{C} + 2\Delta_K$. For every $h \in \mathcal{R}(G)$, one has

$$\Delta_K(\chi_\tau * h) = (|\tau+\rho_\mathfrak{k}|^2 - |\rho_\mathfrak{k}|^2)\chi_\tau * h. \tag{5.2}$$

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

Moreover, as C is central, we obtain for every $g \in G$ and $h \in \mathcal{A}(G)$ that

$$\mathrm{C}(\chi_\tau * L(g)h) = \chi_\tau * L(g)(\mathrm{C}h). \tag{5.3}$$

Let now $f \in \mathcal{A}(G)$. As f is an analytic vector for $\mathcal{R}(G)$ and hence also for $L^2(G)$, we find (see [War72, Corollary 4.4.6.4]) a constant $M_1 > 0$ such that for all $N > 0$ there exists a constant $C_N > 0$ such that

$$\sup_{g\in G} e^{Nd(g)}|\mathrm{C}^k f(g)| \leqslant C_N M_1^{2k}(2k)!. \tag{5.4}$$

We first estimate $\Delta^k(\chi_\tau * L(a_\tau)f)$. For that, we employ (5.2) and (5.3) in order to obtain that

$$\begin{aligned}\Delta^k(\chi_\tau * L(a_\tau)f) &= \sum_{j=0}^{k} \binom{k}{j} \mathrm{C}^j (2\Delta_K)^{k-j}(\chi_\tau * L(a_\tau)f)\\ &= \sum_{j=0}^{k} 2^{k-j}\binom{k}{j}(|\tau+\rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2)^{k-j}(\chi_\tau * L(a_\tau)\mathrm{C}^j f).\end{aligned}$$

For $N > 0$, we thus obtain using (5.4) that

$$\begin{aligned}\sup_{g\in G} e^{Nd(g)}|\Delta^k(\chi_\tau * L(a_\tau)f)(g)| &\leqslant C_N 2^{2k}\sum_{j=0}^{k}(1+|\tau|)^{2(k-j)} \cdot \sup_{g\in G} e^{Nd(g)}|L(a_\tau)\mathrm{C}^j f(g)|\\ &\leqslant C'_N 2^{2k} e^{Nd(a_\tau)}\sum_{j=0}^{k} M_1^{2j}(1+|\tau|)^{2(k-j)}(2j)!\\ &\leqslant C''_N M_2^{2k}\sum_{j=0}^{k}(1+|\tau|)^{2(k-j)+Nc_1}(2j)!\end{aligned}$$

for some $C_N, M_2 > 0$ independent of τ. Using these inequalities for F, we arrive at

$$\sup_{g\in G} e^{Nd(g)}|\Delta^k F(g)| \leqslant C''_N M_2^{2k}\sum_{\tau\in\hat{K}}\sum_{j=0}^{k}|c_\tau|(1+|\tau|)^{2(k-j)+c_1}(2j)!.$$

From the lemma below, we obtain that

$$\sum_{\tau\in\hat{K}}|c_\tau|(1+|\tau|)^{2(k-j)+c_1} \leqslant CM^{2k-2j}(2k-2j)!$$

for some constants $C, M > 0$ independent of k, j. Since

$$\sum_{j=0}^{k}(2k-2j)!(2j)! \leqslant 2^{2k}(2k)!,$$

we conclude that F satisfies the estimates (5.1). □

Lemma 5.6. *Let $\epsilon > 0$. There exist $C, M > 0$ such that*

$$\sum_{\tau\in\hat{K}} e^{-\epsilon|\tau|}(1+|\tau|)^n \leqslant CM^n n!$$

for all $n \in \mathbb{N}$.

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

Proof. We assume for simplicity that K is semisimple. The proof is easily adapted to the general case. The set $\hat{K}$ is parameterized by a semilattice in $i\mathfrak{t}^*$, say

$$\hat{K} = \{m_1\tau_1 + \cdots + m_l\tau_l \mid m_1, \ldots, m_l \in \mathbb{N}\}.$$

We shall perform the summation over $\hat{K}$ by summing over $m \in \mathbb{N}$, and over those elements τ for which the maximal m_j is m. There are lm^{l-1} such elements, and they all satisfy $am \leqslant |\tau| \leqslant bm$ for some $a, b > 0$ independent of m. It follows that the sum above is dominated by

$$\sum_{m\in\mathbb{N}} lm^{l-1}e^{-\epsilon am}(1+bm)^n.$$

The given estimate now follows easily. □

Before we give the proof of Theorem 5.4, we recall some harmonic analysis for the compact homogeneous space $M\backslash K$. We denote by $K_M^\wedge$ the M-spherical part of $\hat{K}$, that is, the equivalence classes of irreducible representations τ for which the space V_τ^M of M-fixed vectors is non-zero. Then

$$L^2(M\backslash K) = \hat{\bigoplus_{\tau\in K_M^\wedge}} \operatorname{Hom}(V_\tau^M, V_\tau) \tag{5.5}$$

by the Peter–Weyl theorem. We write $v = \sum_\tau v_\tau$ for the corresponding decomposition of a function v on $M\backslash K$ and note that with the right action of $k \in K$ on $L^2(M\backslash K)$ we have $[\pi(k)v]_\tau = \tau(k) \circ v_\tau$.

Furthermore,

$$C^\omega(M\backslash K) = \left\{ v = \sum_\tau v_\tau \mid \exists \epsilon, C > 0 \ \forall \tau : \|v_\tau\| \leqslant Ce^{-\epsilon|\tau|} \right\},$$

where $\|v_\tau\|$ denotes the operator norm of v_τ.

Let $\tau \in K_M^\wedge$. The integral $\delta_\tau(k) = \dim(\tau) \int_M \chi_\tau(mk)\, dm$ of the character is bi-invariant under M. The components of δ_τ in the decomposition (5.5) are all 0 except the τ-component, which is the inclusion operator I_τ of V_τ^M into V_τ.

Proof. We can now finally give the proof of Theorem 5.4. Let $v = \sum_\tau v_\tau \in C^\omega(M\backslash K)$ be given, and let $\epsilon > 0$ be as above.

It follows from [BK, § 6] that there exists a K-finite function $\xi \in V_\lambda$ and, for each $\tau \in K_M^\wedge$, elements $a_\tau \in A$ and $c_\tau \in \mathbb{C}$ such that

$$d(a_\tau) \leqslant c_1 \log(1 + |\tau|) + c_2, \quad |c_\tau| \leqslant 2(1 + |\tau|)^{c_3}$$

for some constants $c_1, c_2, c_3 > 0$ independent of τ, and such that

$$R_\tau := \delta_\tau - c_\tau[\pi_\lambda(a_\tau)\xi]_\tau$$

satisfies $\|R_\tau\| \leqslant 1/2$ for all τ. By integration of ξ and R_τ over M, we can arrange that they are both M-bi-invariant.

We now choose a function $f \in \mathcal{A}(G)$ such that $\Pi(f)\xi = \xi$. It exists because $\Pi(\sum_{\tau\in F} \chi_\tau * h_t)\xi$ converges to ξ for $t \to 0$ and some finite set F of K-types by Lemma 4.7, so that ξ belongs to the closure of a finite-dimensional subspace of $\Pi(\mathcal{A}(G))\xi$. According to Proposition 5.5, the function

$$F = \sum_\tau c_\tau e^{-\frac{1}{2}\epsilon|\tau|}\chi_\tau * L(a_\tau)f$$

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

belongs to $\mathcal{A}(G)$. An easy calculation shows that

$$\Pi(F)\xi = \sum_{\tau} e^{-\frac{1}{2}\epsilon|\tau|}(\delta_\tau - R_\tau).$$

Being of type τ and M-bi-invariant, R_τ corresponds in (5.5) to an operator $R_\tau \in \mathrm{End}(V_\tau^M)$. Since $\|R_\tau\| \leqslant \frac{1}{2}$, the operator $I_\tau - R_\tau \in \mathrm{End}(V_\tau^M)$ is invertible with $\|(I_\tau - R_\tau)^{-1}\| \leqslant 2$. Then $v_\tau(I_\tau - R_\tau)^{-1} \in \mathrm{Hom}(V_\tau^M, V_\tau)$ with $\|v_\tau(I_\tau - R_\tau)^{-1}\| \leqslant 2Ce^{-\epsilon|\tau|}$. It follows that the function on $M\backslash K$ with the expansion

$$\sum_{\tau} e^{\frac{1}{2}\epsilon|\tau|} v_\tau(I_\tau - R_\tau)^{-1}$$

belongs to $C^\omega(M\backslash K)$. We denote by $h(k^{-1})$ this function, so that h is a right M-invariant function on K. Another easy calculation now shows that

$$\Pi(h)\Pi(F)\xi = \sum_{\tau} v_\tau = v$$

and hence $h * F \in \mathcal{A}(G)$ is the function we seek. □

5.3 Unique analytic globalization

The goal of this section is to prove the following version of Schmid's minimal globalization theorem [KS94, Theorem 2.13].

Theorem 5.7. *Let V be a Harish-Chandra module. Every analytic $\mathcal{A}(G)$-tempered globalization of V is isomorphic to $V^{\min}$.*

In particular, if (π, E) is an arbitrary F-globalization of V, then

$$E^\omega \simeq V^{\min}.$$

Proof. We first treat the case of an irreducible Harish-Chandra module V.

We first claim that V admits a Hilbert globalization $\mathcal{H}$ such that $\mathcal{H}^\omega = \Pi(\mathcal{A}(G))V$ and hence in particular (see Lemma 5.1(ii))

$$\mathcal{H}^\omega \simeq V^{\min}.$$

In the case $V = V_\lambda$, we can take $\mathcal{H}_\lambda = L^2(M\backslash K)$ and the assertion follows from Theorem 5.4. If the Harish-Chandra module is of the type $V = V_\lambda \otimes W$, where W is a finite-dimensional G-module, then $\mathcal{H} = \mathcal{H}_\lambda \otimes W$ is a Hilbert globalization with $\mathcal{H}^\omega = \mathcal{H}_\lambda^\omega \otimes W$. A straightforward generalization of [BK, Lemma 5.4] yields that

$$(\Pi_\lambda \otimes \Sigma)(\mathcal{A}(G))V = \mathcal{H}^\omega.$$

Finally, every irreducible Harish-Chandra module is a subquotient of some $V_\lambda \otimes W$ (see for example [LW73, Theorem 4.10]), and the claim follows by Lemma 5.2.

Let now (π, E) be an arbitrary analytic $\mathcal{A}(G)$-tempered globalization of V. We aim to prove that $E \simeq V^{\min}$. From Lemma 5.1, we know that $V^{\min}$ injects G-equivariantly and continuously into $E = E^\omega$ and hence it suffices to establish surjectivity of the injection.

We now fix the Hilbert globalization $\mathcal{H}$ of the above. In view of Proposition 5.3, we can embed (π, E) into a Banach globalization F of V. As E is analytic, we obtain a continuous G-equivariant injection $E \to F^\omega$. In order to proceed, we recall the Casselman–Wallach theorem (cf. [Cas89, Wal88], or [BK] for a more recent proof), which implies that F^∞ is equivalent to $\mathcal{H}^\infty$

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

as an F-representation. It follows, see Corollary 3.5, that $F^\omega \simeq \mathcal{H}^\omega$. Collecting the established isomorphisms, we have

$$V^{\min} \to E \subset F^\omega \simeq \mathcal{H}^\omega \simeq V^{\min}.$$

The surjectivity follows from the completeness of $\mathcal{H}^\omega$ (see Proposition 3.7).

Finally, we prove the case of an arbitrary Harish-Chandra module. As Harish-Chandra modules have finite composition series, it suffices to prove the following statement: let $0 \to V_1 \to V \to V_2 \to 0$ be an exact sequence of Harish-Chandra modules and suppose that both V_1 and V_2 have unique analytic $\mathcal{A}(G)$-tempered globalizations. Then so does V.

Let E be an analytic $\mathcal{A}(G)$-tempered globalization of V. Let E_1 be the closure of V_1 in E and $E_2 = E/E_1$. Then E_1 and E_2 are analytic $\mathcal{A}(G)$-tempered globalizations of V_1 and V_2. By assumption, we get $E_1 = V_1^{\min}$ and $E_2 = V_2^{\min}$ and, from Lemma 5.2, we infer that $V_2^{\min} = V^{\min}/V_1^{\min}$. Observe that in an exact sequence of topological vector spaces $0 \to E_1 \to E \to E_2 \to 0$ the topology on E is uniquely determined by the topology of E_1 and E_2 (see [DS79, Lemma 1]). We thus conclude that $E = V^{\min}$. □

We conclude by summarizing the topological properties of $V^{\min}$. Recall that an inductive limit $E = \lim_{n\to\infty} E_n$ of Fréchet spaces is called regular if every bounded set is contained and bounded in one of the steps E_n.

Corollary 5.8. *The minimal globalization $V^{\min}$ is a nuclear, regular, reflexive, and complete inductive limit of Fréchet–Montel spaces.*

Proof. Theorem 5.7 and Proposition 3.7 imply that $V^{\min}$ is complete. Furthermore, it then follows from [Kuc04, Wen96] that $V^{\min}$ is regular and reflexive (see also Appendix B). It is an inductive limit of Fréchet–Montel spaces, because $\mathcal{A}(G)$ is an inductive limit of Fréchet–Schwarz spaces, and Hausdorff quotients of such spaces are Fréchet–Montel. Nuclearity is inherited from $C^\omega(M\backslash K)$, which is the strong dual of a nuclear Fréchet space, and this property is preserved when passing to the quotient of a finite-dimensional tensor product. Finally, a Fréchet space is nuclear if and only if its strong dual is nuclear (see [Jar81, § 21.5]). □

Appendix A. Vector-valued holomorphy

Here, we collect some results about analytic functions with values in a locally convex Hausdorff topological vector space E. Let $\Omega \subset \mathbb{C}^n$ be open.

It is a natural and common assumption that E is sequentially complete. Let us recall that under this assumption an E-valued function f on Ω is said to be holomorphic if it satisfies one of the following conditions, which are equivalent in this case:

(a) f is weakly holomorphic, that is, the scalar function $z \mapsto \zeta(f(z))$ is holomorphic for each continuous linear form $\zeta \in E'$;

(b) f is $\mathbb{C}$-differentiable in each variable at each $z \in \Omega$;

(c) f is infinitely often $\mathbb{C}$-differentiable at each $z \in \Omega$;

(d) f is continuous and is represented by a converging power series expansion with coefficients in E, in a neighborhood of each $z \in \Omega$.

In general, the conditions (c) and (d) are mutually equivalent and they imply (a) and (b). This follows by regarding f as a function into the completion $\bar{E}$ of E (see [Glo02, Proposition 2.4]).

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

We shall call a function $f : \Omega \to E$ *holomorphic* if (c) or (d) is satisfied or, equivalently, if it is holomorphic into $\bar{E}$ with E-valued derivatives up to all orders.

Let M be an n-dimensional complex manifold. An E-valued function on M is called holomorphic if all its coordinate expressions are holomorphic. We denote by $\mathcal{O}(M, E)$ the space of E-valued holomorphic functions on M. Endowed with the compact open topology, it is a Hausdorff topological vector space, which is complete whenever E is complete.

The following isomorphism of topological vector spaces is useful.

Lemma A.1. *Let M and N be complex manifolds; then*

$$\mathcal{O}(M \times N, E) \simeq \mathcal{O}(M, \mathcal{O}(N, E)) \tag{A.1}$$

under the natural map $f \mapsto (x \mapsto f(x, \cdot))$ from left to right.

Proof. Apart from the statement that $x \mapsto f(x, \cdot) \in \mathcal{O}(N, E)$ is holomorphic, this is straightforward from definitions. It is clear that $f(x, \cdot) \in \mathcal{O}(N, E)$. By regarding $\mathcal{O}(N, E)$ as a subspace of $\mathcal{O}(N, \bar{E})$ and noting that it carries the relative topology, we reduce to the case that E is complete, so that condition (b) applies. Assume for simplicity that $M = \mathbb{C}$. What needs to be established is then only that the complex differentiation

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{1}{h}[f(x + h, y) - f(x, y)] \in E$$

is valid locally uniformly with respect to $y \in N$. This follows from uniform continuity on compacta of the derivative. □

Appendix B. Topological properties of $\mathcal{A}(G)$

While the topology of a general inductive limit of Fréchet spaces may be complicated, $\mathcal{A}(G)$ inherits certain properties from the steps $\mathcal{A}(G)_n$.

Theorem B.1. *The algebra $\mathcal{A}(G)$ is regular, complete, and reflexive.*

A regular inductive limit of Fréchet–Montel spaces is known to be reflexive [Kuc04] and complete [Wen96], so that we only have to show regularity. The following criterion from [Wen96, Theorem 3.3], in terms of interpolation inequalities, will be convenient.

Proposition B.2. *An inductive limit $E = \lim_{n\to\infty} E_n$ of Fréchet–Montel spaces is regular if and only if for some fundamental system $\{p_{n,\nu}\}_{\nu\in\mathbb{N}}$ of seminorms on E_n : $\forall n\ \exists m > n\ \exists \nu\ \forall k > m\ \forall \mu\ \exists \kappa\ \exists C\ \forall f \in E_n$*

$$p_{m,\mu}(f) \leqslant C(p_{k,\kappa}(f) + p_{n,\nu}(f)). \tag{B.1}$$

In the case of $\mathcal{A}(G)$, condition (B.1) should be thought of as a weighted geometric relative of Hadamard's three-lines theorem. To verify it, we need to introduce some notions from complex and Riemannian geometry, starting with the appropriate differential operators.

By common practice we identify the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with the space of right-invariant vector fields on $G_{\mathbb{C}}$, where $X \in \mathfrak{g}_{\mathbb{C}}$ corresponds to the differential operator

$$\widetilde{X}u(x) = \frac{d}{dt}\bigg|_{t=0} u(\exp(-tX)x) \quad (x \in G_{\mathbb{C}}, u \in C^{\infty}(G_{\mathbb{C}})).$$

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

If we denote the complex structure on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ by J, the Cauchy–Riemann operators $\overline{\partial}_Z$ and ∂_Z associated to $Z \in \mathfrak{g}_{\mathbb{C}}$ are given by $\overline{\partial}_Z := \widetilde{Z} + i\widetilde{JZ}$ and $\partial_Z := \widetilde{Z} - i\widetilde{JZ}$, respectively.

In this section it will be convenient to replace the left G-invariant metric $\mathbf{g}$ on G used in § 2.1 by a right-invariant one, which we shall denote by the same symbol. Note that the corresponding distance functions d on G are equivalent (see (2.2)). The function

$$K(\exp(JX)g) := \tfrac{1}{2}|X|^2 := \tfrac{1}{2}\mathbf{g_1}(X, X)$$

endows a sufficiently small complex neighborhood VG of G with a right G-invariant Kähler structure. To see this, choose an orthonormal basis $\{X_j\}_{j=1}^{l}$ of $\mathfrak{g}$ with respect to the metric. A straightforward computation results in

$$\partial_{X_i}\overline{\partial}_{X_j}K(\mathbf{1}) = \mathbf{g_1}(X_i, X_j),$$

so that the complex Hessian $(Z_1, Z_2) \mapsto \partial_{Z_1}\overline{\partial}_{Z_2}K(\mathbf{1})$ defines a positive-definite Hermitian form on $\mathfrak{g}_{\mathbb{C}}$. By continuity and invariance, positivity extends to give a Kähler metric on a small neighborhood VG.

The complex Laplacian

$$\Delta_{\mathbb{C}} = \sum_{j=1}^{l} \partial_{X_j}\overline{\partial}_{X_j} = \sum_{j=1}^{l} \widetilde{X_j}^2 + \widetilde{JX_j}^2$$

agrees with the Kähler Laplacian up to first-order terms and maps real-valued functions to real-valued functions. Therefore, the following weak maximum principle holds.

Lemma B.3. *If $u \in C^2(VG)$ is real valued with a local maximum in $z \in VG$, then*

$$\Delta_{\mathbb{C}}u(z) \leqslant 0.$$

As $\Delta_{\mathbb{C}}$ is a trace of the complex Hessian, we may rely on well-known results about plurisubharmonic functions to conclude.

Lemma B.4. *For $u \in \mathcal{O}(VG)$, $\Delta_{\mathbb{C}}u = 0$ and $\Delta_{\mathbb{C}} \log|u| \geqslant 0$.*

So, while it may be less obvious how to control applications of $\Delta_{\mathbb{C}}$ to the Riemannian distance function d on G, $\Delta_{\mathbb{C}}$ annihilates the holomorphically regularized distance function $\tilde{d} := e^{-\Delta_{\mathbf{g}}}d$ from [GKL]. This is going to be useful in the proof of Theorem B.1, and the following lemma, which is shown as in [GKL, Lemma 4.3], collects the key properties of $\tilde{d}$.

Lemma B.5. (a) *The function $\tilde{d}$ extends to a function in $\mathcal{O}(UG)$ for some neighborhood U of $\mathbf{1} \in G_{\mathbb{C}}$.*

(b) *For all $U' \Subset U$, $\sup_{zg \in U'G} |\tilde{d}(zg) - d(g)| < \infty$ and $\widetilde{X_j}\tilde{d}$ as well as $\widetilde{JX_j}\tilde{d}$ are bounded on $U'G$ for all j.*

Before finally coming to the proof of Theorem B.1, we introduce an equivalent representation of $\mathcal{A}(G)$ based on geometrically more convenient neighborhoods. If we define for $n \in \mathbb{N}$, $\nu \in \mathbb{N}_0$ the neighborhoods

$$\widetilde{V}_n := \left\{ \exp(JX) \in G_{\mathbb{C}} \mid |X| < \frac{1}{n} \right\},$$

$$\Omega_n^{\nu} := \left\{ \exp(JX) \in G_{\mathbb{C}} \mid |X| < \frac{1}{n + (\nu+2)^{-1}} \right\}$$

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

and associated subspaces of $\mathcal{A}(G)$,

$$\widetilde{\mathcal{A}(G)}_n := \left\{ f \in \mathcal{O}(\widetilde{V}_n G) \mid \forall \nu \in \mathbb{N} : p_{n,\nu}(f) := \sup_{g \in G, z \in \Omega_n^\nu} |f(zg)| e^{\nu d(g)} < \infty \right\},$$

then $\mathcal{A}(G)$ is again an inductive limit $\lim_{n\to\infty} \widetilde{\mathcal{A}(G)}_n$ of Fréchet–Montel spaces. Condition (B.1) translates into

$$\sup_{zg \in \Omega_m^\mu G} |f(zg)| e^{\mu d(g)} \leqslant C \left(\sup_{zg \in \Omega_k^\kappa G} |f(zg)| e^{\kappa d(g)} + \sup_{zg \in \Omega_n^\nu G} |f(zg)| e^{\nu d(g)} \right) \tag{B.2}$$

for $f \in \widetilde{\mathcal{A}(G)}_n$.

To show this, let n sufficiently large, $0 \not\equiv f \in \widetilde{\mathcal{A}(G)}_n$, $m = n+1$, $\nu = 0$, $k > m$, and $\mu \in \mathbb{N}$, and consider

$$u(z) = \log |f(z)| + N(z) D(z)$$

on $\widetilde{V}_n G \backslash \widetilde{V}_{k+1} G$, where we choose $N(\exp(JX)g) = N(\exp(JX)) = \bar{\nu}(|X|^{-2\alpha} - (n+\frac{1}{2})^{2\alpha})$ and $D(z) = D_0 + \operatorname{Re} \tilde{d}(z)$ for some $\bar{\nu}, \alpha, D_0 > 0$. First note that $\Delta_{\mathbb{C}} u > 0$ if D_0 and α are sufficiently large. Indeed, by Lemma B.4, it is enough to show that $\Delta_{\mathbb{C}}(N(z)D(z)) > 0$. But, $\Delta_{\mathbb{C}} D = 0$, so that

$$\Delta_{\mathbb{C}}(N(z)D(z)) = (\Delta_{\mathbb{C}} N(z)) D(z) + 2 \sum_{j=1}^{l} \{ \widetilde{X_j} N(z) \widetilde{X_j} D(z) + \widetilde{JX_j} N(z) \widetilde{JX_j} D(z) \}.$$

With $D \geqslant 1$ on $\widetilde{V}_n G$ for large D_0 by Lemma B.5, we only have to show that

$$\Delta_{\mathbb{C}} N(z) > \overline{D} \max_{j=1,\dots,l} \{ |\widetilde{X_j} N(z)| |\widetilde{JX_j} N(z)| \}$$

on $\widetilde{V}_n G$ for large n and $\overline{D} = 2 \sup\{ |\widetilde{X_j} D|, |\widetilde{JX_j} D| : j = 1, \dots, l \}$. By G-invariance, it is sufficient to do so in $z = \exp(\varepsilon JX)$ close to $\varepsilon = 0$. The Baker–Campbell–Hausdorff formula implies that

$$\exp(tJX_j) \exp(\varepsilon JX) = \exp(\varepsilon JX + tJX_j + \mathcal{O}(\varepsilon t^2) + \mathcal{O}(\varepsilon^2 t)) \cdot \exp(\tfrac{1}{2} \varepsilon t [JX_j, JX]),$$

so that

$$\begin{aligned} \widetilde{JX_j} N(\exp(\varepsilon JX)) &= \frac{d}{dt}\bigg|_{t=0} N(\varepsilon JX + tJX_j + \mathcal{O}(\varepsilon t^2) + \mathcal{O}(\varepsilon^2 t)) \\ &= -2\alpha \bar{\nu} \varepsilon^{-1-2\alpha} \frac{\mathbf{g_1}(X_j, X)}{\mathbf{g_1}(X, X)^{\alpha+1}} + \mathcal{O}(\varepsilon^{-2\alpha}). \end{aligned}$$

Similarly,

$$(\widetilde{JX_j})^2 N(\exp(\varepsilon JX)) = 2\alpha \bar{\nu} \varepsilon^{-2-2\alpha} \frac{2(\alpha+1)\mathbf{g_1}(X_j, X)^2 - \mathbf{g_1}(X_j, X_j)\mathbf{g_1}(X, X)}{\mathbf{g_1}(X, X)^{\alpha+2}}$$

up to terms of order $\varepsilon^{-1-2\alpha}$. Summing over j establishes the assertion for large α and small ε and hence for large n.

For $\kappa \geqslant 0$, set $S_n := \sup_{\partial \Omega_n^0 G} u$ and $S_k^\kappa := \sup_{\partial \Omega_k^\kappa G} u$. Because $u(z)$ is bounded from above and $\leqslant \max\{S_k^\kappa, S_n\}$ on $\partial \Omega_k^\kappa G \cup \partial \Omega_n^0 G$, the maximum principle, Lemma B.3, assures that

$$u(z) \leqslant \max\{S_k^\kappa, S_n\}$$

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press

in $\Omega_n^0 G \backslash \Omega_k^\kappa G$, or

$$|f(z)|e^{N(z)D(z)} \leqslant e^{\max\{S_k^\kappa, S_n\}} \leqslant \sup_{w \in \partial\Omega_k^\kappa G} |f(w)|e^{N(w)D(w)} + \sup_{w \in \partial\Omega_n^0 G} |f(w)|e^{N(w)D(w)}.$$

As $\widetilde{V}_m \Subset \Omega_n^0$, we may choose $\bar{\nu}$ such that $N|_{\widetilde{V}_m G \backslash \widetilde{V}_{k+1} G} \geqslant \mu$. Setting $\kappa := \sup_{\widetilde{V}_k G \backslash \widetilde{V}_{k+1} G} N \geqslant \mu$, we obtain

$$\begin{aligned} \sup_{z \in \Omega_m^\mu G} |f(z)|e^{\mu D(z)} &\leqslant \sup_{z \in \Omega_k^\kappa G} |f(z)|e^{\kappa D(z)} + \sup_{z \in \partial\Omega_n^0 G} |f(z)| \\ &\leqslant \sup_{z \in \Omega_k^\kappa G} |f(z)|e^{\kappa D(z)} + \sup_{z \in \Omega_n^0 G} |f(z)|. \end{aligned}$$

Lemma B.5 implies that $d(z) - C \leqslant D(z) \leqslant d(z) + C$ for some $C > 0$, and Theorem B.1 follows.

Remark B.6. It would be interesting to better understand the topology of $\mathcal{A}(G)^N/I$ for a stepwise closed, $\mathcal{A}(G)$-invariant subspace I. Because $\widetilde{\mathcal{A}(G)}_n$ is even Fréchet–Schwarz, the quotients $\widetilde{\mathcal{A}(G)}_n^N/(I \cap \widetilde{\mathcal{A}(G)}_n^N)$ are Fréchet–Montel and one might hope to verify condition (B.1) as before. However, adapting the above proof requires strong assumptions on I, and general Hausdorff quotients $\mathcal{A}(G)^N/I$ are likely to be incomplete: for a convex domain $\Omega \subset \mathbb{R}^n$, the space of test functions $\mathcal{D}(\Omega)$ is isomorphic to a similar weighted space of holomorphic functions by Paley–Wiener's theorem. However, given any non-surjective differential operator A on $\mathcal{D}'(\Omega)$, the quotient of $\mathcal{D}(\Omega)$ by the image of A^t will be incomplete.

References

BK J. Bernstein and B. Krötz, *Smooth Fréchet globalizations of Harish-Chandra modules*, submitted, http://service.ifam.uni-hannover.de/∼kroetz/bk21.pdf.

BD01 J. Bonet and P. Domanski, *Parameter dependence of solutions of partial differential equations in spaces of real analytic functions*, Proc. Amer. Math. Soc. **129** (2001), 495–503.

Cas89 W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. **41** (1989), 385–438.

DS79 S. Dierolf and U. Schwanengel, *Examples of locally compact noncompact minimal topological groups*, Pacific J. Math. **82** (1979), 349–355.

Gar60 L. Gårding, *Vectors analytiques dans le représentations des groupes de Lie*, Bull. Soc. Math. France **88** (1960), 73–93.

GKL H. Gimperlein, B. Krötz and C. Lienau, *Analytic factorization of Lie group representations*, submitted.

Glo02 H. Glöckner, *Infinite-dimensional Lie groups without completeness restrictions*, in *Geometry and analysis on Lie groups*, Banach Center Publications, vol. 55 (2002), 43–59.

Goo69 R. Goodman, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. **143** (1969), 55–76.

Gro53a A. Grothendieck, *Sur certains espaces de fonctions holomorphes, I*, J. Reine Angew. Math. **192** (1953), 35–64.

Gro53b A. Grothendieck, *Sur certains espaces de fonctions holomorphes, II*, J. Reine Angew. Math. **192** (1953), 77–95.

Gro73 A. Grothendieck, *Topological vector spaces*, Notes on Mathematics and its Applications (Gordon and Breach Science Publishers, New York–London–Paris, 1973) (Translated from the French by Orlando Chaljub).

Har53 Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I*, Trans. Amer. Math. Soc. **75** (1953), 185–243.

Har66 Harish-Chandra, *Discrete series for semisimple Lie groups. II*, Acta Math. **116** (1966), 1–111.

Jar81 H. Jarchow, *Locally convex spaces* (Teubner, Stuttgart, 1981).

Kas08 M. Kashiwara, *Equivariant derived category and representation of real semisimple Lie groups*, in *Representation theory and complex analysis*, Lecture Notes in Mathematics, vol. 1931 (Springer, New York, 2008), 137–234.

KS94 M. Kashiwara and W. Schmid, *Quasi-equivariant $\mathcal{D}$-modules, equivariant derived category, and representations of reductive Lie groups*, in *Lie theory and geometry*, Progress in Mathematics, vol. 123 (Birkhäuser, Boston, MA, 1994), 457–488.

Kuc04 J. Kucera, *Reflexivity of inductive limits*, Czechoslovak Math. J. **54** (2004), 103–106.

LW73 J. Lepowsky and N. Wallach, *Finite- and infinite-dimensional representations of linear semisimple groups*, Trans. Amer. Math. Soc. **184** (1973), 223–246.

MV97 R. Meise and D. Vogt, *Introduction to functional analysis* (Oxford University Press, Oxford, 1997).

Nel59 E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615.

Net64 J. B. Neto, *Spaces of vector valued real analytic functions*, Trans. Amer. Math. Soc. **112** (1964), 381–391.

Sch85 W. Schmid, *Boundary value problems for group invariant differential equations*, in *The mathematical heritage of Élie Cartan (Lyon, 1984)*. Astérisque, Numero Hors Serie (1985), 311–321.

Wal88 N. Wallach, *Real reductive groups I* (Academic Press, New York, 1988).

War72 G. Warner, *Harmonic analysis on semi-simple Lie groups* (Springer, Berlin, 1972).

Wen96 J. Wengenroth, *Acyclic inductive spectra of Fréchet spaces*, Studia Math. **120** (1996), 247–258.

Heiko Gimperlein gimperlein@math.ku.dk
Leibniz Universität Hannover, Institut für Analysis, Welfengarten 1, D-30167, Hannover, Germany

Current address: Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

Bernhard Krötz kroetz@math.uni-hannover.de
Leibniz Universität Hannover, Institut für Analysis, Welfengarten 1, D-30167, Hannover, Germany

Henrik Schlichtkrull schlicht@math.ku.dk
Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

https://doi.org/10.1112/S0010437X11005392 Published online by Cambridge University Press