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A HYBRID BOUNDARY INTEGRAL / TAYLOR SERIES APPROACH TO SOME NONLINEAR EQUATIONS FROM FLUID MECHANICS

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Abstract

We show that a combination of Taylor series and boundary integral methods can lead to an effective scheme for solving a class of nonlinear partial differential equations. The method is illustrated through its application to an equation from two dimensional fluid mechanics.

1. Introduction

The boundary integral method is now a well established technique for the solution of linear partial differential equations [1], but its application to nonlinear equations has been less widespread. This is not surprising as the nonlinear equations require to be treated as a sequence of linear inhomogeneous equations [2, 3]. Further, the boundary integral formulation for an inhomogeneous equation includes body integral terms, these requiring a far more complicated quadrature procedure than the boundary integrals.

It is the purpose of the present paper to develop a boundary integral approach to nonlinear equations that avoids the use of body integrals. We first develop a method for solving linear inhomogeneous equations and then extend this method for the solution of nonlinear equations. In essence, we find a particular solution to the inhomogeneous equation and then use a boundary integral method to find the correct eigensolution. However, since a closed form particular solution can rarely be achieved, we resort to a patchwork of local series solutions.

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We demonstrate our approach for a class of differential equations that arises in the study of plane incompressible viscous flow. In Section 2 we present a boundary integral formulation for the linear homogeneous equation, in Section 3 we describe our approach to the inhomogeneous equation and then, in Section 4, we describe extensions to the nonlinear case. The considerations of Section 4 apply to a wide class of nonlinearities, but in Section 5 we consider that which arises in the study of inertial effects. For this nonlinearity, we consider the simulation of a stick-slip flow.

2. Integral equations for the homogeneous problem

It was shown in [4] that the plane creeping flows of a Newtonian fluid could be described in terms of a complex potential χ that satisfied

$$\partial^2\chi/\partial\bar{z}^2 = 0 \tag{1}$$

where $z(= x_1 + ix_2)$ is a complex coordinate. The stress components σ_{ij} are related to χ by

$$\sigma_{11} = \partial^2 Re\chi/\partial x_2^2,$$

$$\sigma_{22} = \partial^2 Re\chi/\partial x_1^2,$$

$$\sigma_{12} = -\partial^2 Re\chi/\partial x_1 \partial x_2,$$

and the velocity components u_i by

$$u_1 = -\frac{1}{2\eta}\frac{\partial Im\chi}{\partial x_2},$$

$$u_2 = \frac{1}{2\eta}\frac{\partial Im\chi}{\partial x_1},$$

where η is the fluid viscosity.

Consider the problem of finding the complex derivatives of χ ($\partial\chi/\partial z$ and $\partial\chi/\partial\bar{z}$) within a region R having boundary C. We will assume boundary conditions such that, at each point of C, some linear combination of $\partial\chi/\partial z$, $\partial\chi/\partial\bar{z}$, $\partial\bar{\chi}/\partial z$ and $\partial\bar{\chi}/\partial\bar{z}$ is specified, the derivatives being related to physical quantities through

$$2\partial\chi/\partial z = f_1 - 2\eta u_1 + i(2\eta u_2 - f_2),$$

$$2\partial\chi/\partial\bar{z} = f_1 + 2\eta u_1 + i(f_2 + 2\eta u_2),$$

where $f_i(= \partial\phi/\partial x_i)$ are the integrated components of stress. The problem can be recast in terms of the integral equation [4]

$$\frac{1}{2i}\oint_C\left(\frac{\partial\chi}{\partial z}\frac{\partial\chi_\pm}{\partial\bar{z}} + \frac{\partial\chi_\pm}{\partial z}\frac{\partial\chi}{\partial\bar{z}}\right)dz + \frac{1}{2i}\oint_C\frac{\partial\chi}{\partial\bar{z}}\frac{\partial\chi_\pm}{\partial\bar{z}}\,d\bar{z} = \alpha\left(\frac{\partial\chi}{\partial z} \pm \frac{\partial\chi}{\partial\bar{z}}\right)_{|z=z_0} \tag{2}$$

where the integrals are defined in terms of their principal values,

$$\alpha = \begin{cases} \pi & \text{for points inside } C \\ \pi/2 & \text{for points on } C \text{ where the curve is smooth} \\ 0 & \text{for points outside } C \end{cases}$$

and $\chi_\pm$ are the kernels

$$\chi_\pm = \frac{\bar{z} - \bar{z}_0}{z - z_0} \pm \ln(z - z_0).$$

For a given set of boundary conditions, suitable combinations of equations (2) will determine the unknown boundary values. Further, once these boundary values are known, suitable combinations of (2) will also provide values within region R.

In all but a few circumstances, it will be necessary to solve the integral equations by numerical methods. We consider a discretisation for which the boundary is approximated by a set of linear segments with the derivatives of χ taken to be constant on each segment. Then, for the segment from z_i to z_{i+1}, we will have

$$\int_{z_i}^{z_{i+1}} \left(\frac{\partial \chi}{\partial z} \frac{\partial \chi_\pm}{\partial \bar{z}} + \frac{\partial \chi_\pm}{\partial z} \frac{\partial \chi}{\partial \bar{z}} \right) dz + \int_{z_i}^{z_{i+1}} \frac{\partial \chi}{\partial \bar{z}} \frac{\partial \chi_\pm}{\partial \bar{z}} \, d\bar{z}$$

approximated by

$$\left(\frac{\partial \chi}{\partial z} \pm \frac{\partial \chi}{\partial \bar{z}} \right) \ln\left(\frac{z_{i+1} - z_0}{z_i - z_0} \right) + \frac{\partial \chi}{\partial \bar{z}} \left(\frac{\bar{z}_{i+1} - \bar{z}_0}{z_{i+1} - z_0} - \frac{\bar{z}_i - \bar{z}_0}{z_i - z_0} \right)$$

with $\partial\chi/\partial z$ and $\partial\chi/\partial\bar{z}$ represented by their values at the mid point of the segment. After applying this approximation to equations (2), collocation at the segment midpoints provides a finite set of linear equations that determines the approximate values of the unknown boundary fields.

3. The inhomogeneous problem

We now consider the solution of

$$\partial^2\chi/\partial\bar{z}^2 = f(z, \bar{z}) \tag{3}$$

where f is a function of z and $\bar{z}$. Let region R be divided into M subregions R_I with boundaries $C_I (I = 1 \text{ to } M)$. Then, in the subregion R_I, we expand f in Taylor series about some point z_I

$$f(z, \bar{z}) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} a_{jm}^I (\bar{z} - \bar{z}_I)^j (z - z_I)^m.$$

We assume that the Taylor series about z_I has a disc of convergence which contains R_I. Then

$$\chi^I = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} a_{jm}^I (\bar{z} - \bar{z}_I)^{j+2} (z - z_I)^m / (j^2 + 3j + 2)$$

will be a local solution to (3), valid in R_I. However, each local solution χ^I will only be unique up to an eigensolution $\delta\chi^I$ that satisfies $\partial^2 \delta\chi^I / \partial \bar{z}^2 = 0$ and hence its integral equivalent

$$\frac{1}{2i} \oint_{C_I} \left(\frac{\partial \delta\chi^I}{\partial z} \frac{\partial \chi_\pm}{\partial \bar{z}} + \frac{\partial \chi_\pm}{\partial z} \frac{\partial \delta\chi^I}{\partial \bar{z}} \right) dz + \frac{1}{2i} \oint_{C_I} \frac{\partial \delta\chi^I}{\partial \bar{z}} \frac{\partial \chi_\pm}{\partial \bar{z}} \, d\bar{z} = \alpha \left(\frac{\partial \delta\chi^I}{\partial z} \pm \frac{\partial \delta\chi^I}{\partial \bar{z}} \right)_{|z=z_0}. \tag{4}$$

Providing we choose each $\delta\chi^I$ correctly, $\chi^I + \delta\chi^I$ will yield a local representation of χ in region R_I. Now consider a point z_0 that does not lie on more than one of the boundary curves C_I. On summing relations (4) for I from 1 to M, we obtain

$$\begin{aligned} &\frac{1}{2i} \oint_{C} \left(\frac{\partial \chi}{\partial z} \frac{\partial \chi_\pm}{\partial \bar{z}} + \frac{\partial \chi_\pm}{\partial z} \frac{\partial \chi}{\partial \bar{z}} \right) dz + \frac{1}{2i} \oint_{C} \frac{\partial \chi}{\partial \bar{z}} \frac{\partial \chi_\pm}{\partial \bar{z}} \, d\bar{z} \\ &\quad - \frac{1}{2i} \sum_{I=1}^{M} \oint_{C_I} \left(\frac{\partial \chi^I}{\partial z} \frac{\partial \chi_\pm}{\partial \bar{z}} + \frac{\partial \chi_\pm}{\partial z} \frac{\partial \chi^I}{\partial \bar{z}} \right) dz - \frac{1}{2i} \sum_{I=1}^{M} \oint_{C_I} \frac{\partial \chi^I}{\partial \bar{z}} \frac{\partial \chi_\pm}{\partial \bar{z}} \, d\bar{z} \\ &\quad = \alpha \left(\frac{\partial \chi}{\partial z} \pm \frac{\partial \chi}{\partial \bar{z}} \right)_{|z=z_0} - \alpha \left(\frac{\partial \chi^J}{\partial z} \pm \frac{\partial \chi^J}{\partial \bar{z}} \right)_{|z=z_0}, \end{aligned} \tag{5}$$

where $z_0 \in R_J$, $\alpha = \pi$ for allowable points in R and $\alpha = \pi/2$ for points on C where the curve is smooth. It is important to note that interior unknowns (those not on C) have cancelled, with the effect that equations (5) and (2) will only differ by some terms involving known quantities. In essence, we have traded the problem of evaluating some difficult body integrals for that of performing some Taylor expansions and then some contour integrals. The solution procedure is now virtually identical to that for the homogeneous problem.

4. The Nonlinear Problem

Consider the nonlinear equation

$$\partial^2 \chi / \partial \bar{z}^2 = f(\partial \chi / \partial z, \partial \chi / \partial \bar{z}, \partial \bar{\chi} / \partial z, \partial \bar{\chi} / \partial \bar{z}, \ldots), \tag{6}$$

where f is an analytic function of the various complex derivatives $\partial^{i+j} \chi / \partial z^i \partial \bar{z}^j$ and $\partial^{i+j} \bar{\chi} / \partial z^i \partial \bar{z}^j$. Starting with an initial approximation χ_0, we shall find χ as the limit of an iterative procedure for which the $(i+1)th$ iterate χ_{i+1} satisfies

$$\partial^2\chi_{i+1}/\partial\bar{z}^2 = f(\partial\chi_i/\partial z, \dots), \tag{7}$$

subject to the given boundary conditions. Providing the necessary Taylor expansions of $f(\partial\chi_i/\partial z, \dots)$ exist, equation (7) can be solved using the techniques of the last section. These techniques provide, through (5), the derivatives of χ_{i+1} to any order. Consequently, it is possible to produce Taylor expansions for the various derivatives of χ_{i+1} and $\bar{\chi}_{i+1}$. Further, since f is analytic in its arguments, it is also possible to produce the corresponding expansions for $f(\partial\chi_{i+1}/\partial z, \dots)$. These expansions will enable the next stage of iteration and so on until a suitable criterion for convergence has been satisfied.

The expansion of the various derivatives of χ_{i+1} and $\bar{\chi}_{i+1}$ is best achieved by expanding (5) and then differentiating the result. However, this requires the expansion of several contour integrals. Fortunately, once (5) has been discretised, the integrals reduce to linear combinations of the functions $\ln(z_B - z_0)$ and $(\bar{z}_B - \bar{z}_0)/(z_B - z_0)$, with z_B ranging over the vertices of the polygons that approximate the various C_I. These functions have the expansions

$$\ln(z_B - z_0) = \ln(z_B - z_I) - \sum_{n=1}^{\infty} (z_0 - z_I)^n / \{n(z_B - z_I)^n\}$$

and

$$(\bar{z}_B - \bar{z}_0)/(z_B - z_0) = ((\bar{z}_B - \bar{z}_I) - (\bar{z}_0 - \bar{z}_I)) \sum_{n=0}^{\infty} (z_0 - z_I)^n/(z_B - z_I)^{n+1}.$$

5. An Example

We consider the 2D steady state flows of an incompressible Newtonian fluid. These flows can be described in terms of a complex potential χ that satisfies

$$\partial^2\chi/\partial\bar{z}^2 + (\rho_0/4\eta^2)(\partial Im\chi/\partial\bar{z})^2 = 0,$$

where η is the viscosity and ρ_0 the density. The stress components σ_{ij} are related to χ by

$$\sigma_{11} = \frac{\partial^2 Re\chi}{\partial x_2^2} + \frac{\rho_0}{4\eta^2}\left(\frac{\partial Im\chi}{\partial x_2}\right)^2,$$

$$\sigma_{22} = \frac{\partial^2 Re\chi}{\partial x_1^2} + \frac{\rho_0}{4\eta^2}\left(\frac{\partial Im\chi}{\partial x_1}\right)^2,$$

$$\sigma_{12} = -\frac{\partial^2 Re\chi}{\partial x_1 \partial x_2} - \frac{\rho_0}{4\eta^2}\frac{\partial Im\chi}{\partial x_1}\frac{\partial Im\chi}{\partial x_2}$$

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and the velocity components by

$$u_1 = -\frac{1}{2\eta}\frac{\partial Im\chi}{\partial x_2},$$

$$u_2 = \frac{1}{2\eta}\frac{\partial Im\chi}{\partial x_1}.$$

As an example, the methods of Section 4 have been applied to the boundary value problem of Figure 1, a stick-slip flow. For discretisation purposes, the boundary was divided into 160 equal-length segments. Further, full use was made of symmetry in order to reduce the number of boundary unknowns. The interior was subdivided into twelve non-overlapping unit squares and, for all squares without a stick-slip change on the perimeter, the expansion point was located at the centroid. The field χ will become singular at a change from stick to slip and so this point will lie on the circle of convergence for the Taylor series. Consequently, any stick-slip change must be located at the corner of a square if a suitable expansion point is to exist. This is the case for the above subdivision, the four squares with a singular corner being those having the origin as a corner. For these squares, the respective expansion points were chosen to be (–0.6, 0.4), (0.6, 0.4), (–0.6, –0.4) and (0.6, –0.4) in order to ensure that non-singular points were contained within the relevant disc of convergence. Internal contour integrals were evaluated using the same approximate procedure as for integrals on C, the perimeter of each square being divided into 40 equal length segments for this purpose. A further consideration concerns the truncation of the Taylor series, a necessity for any practical implementation. In the case of the present simulation, for squares without a singular corner, it was found sufficient to include terms in $z^i\bar{z}^j$ with $i + j \leqslant 8$. However, for squares with a singular corner, at least terms with $i + j \leqslant 12$ were required.

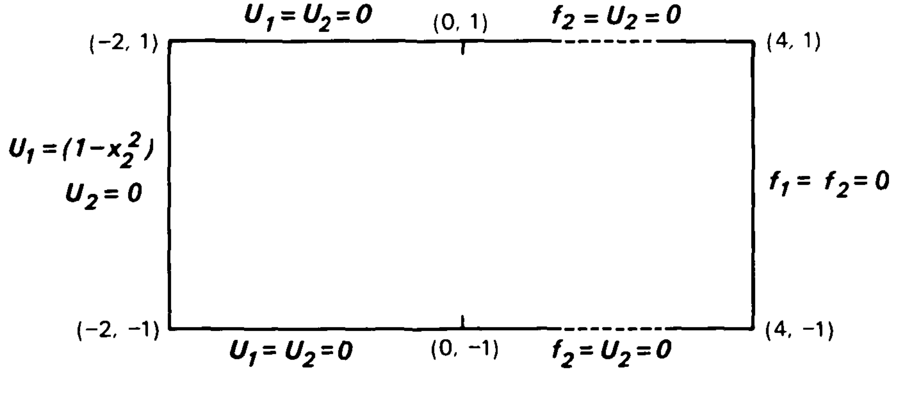

Figure 1

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Figures 2 and 3 show the velocity fields from simulations of a flow with Reynolds number zero (the unbroken contours) and a flow with Reynolds number 10 (the broken contours). The Reynolds number is defined to be $\rho_0 LU/\eta$ where L is a length scale and U a velocity scale ($L = U = 1$ and $\eta = \frac{1}{2}$ for the present problem). In order to attain convergence at Reynolds numbers above 7, the iterative scheme required a small amount of damping. However, for Reynolds numbers above 12, convergence ceased.

The above simulations were performed using a modified version of our scheme. At each stage of iteration, instead of choosing local solutions χ^I such that terms in $\bar{z}^i(i < 2)$ be zero, these terms were taken from the expansion of $\delta\chi^I$ at the previous iteration. Consequently, as the iteration proceeded, each $\delta\chi^I$ tended to zero and each χ^I tended to a local series representation of χ. These local representations of χ provide a convenient way of calculating field values and were used in the production of Figures 2 and 3.

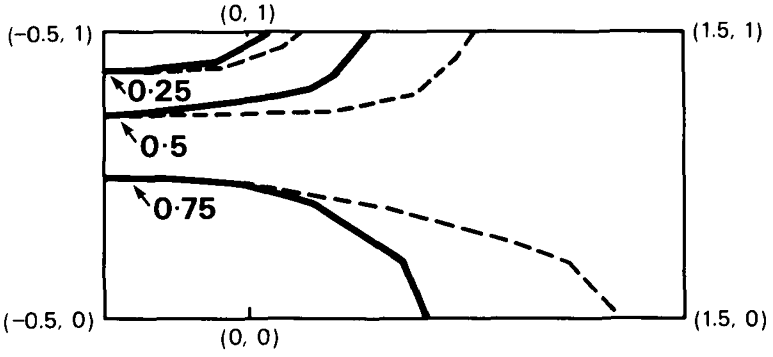

Figure 2

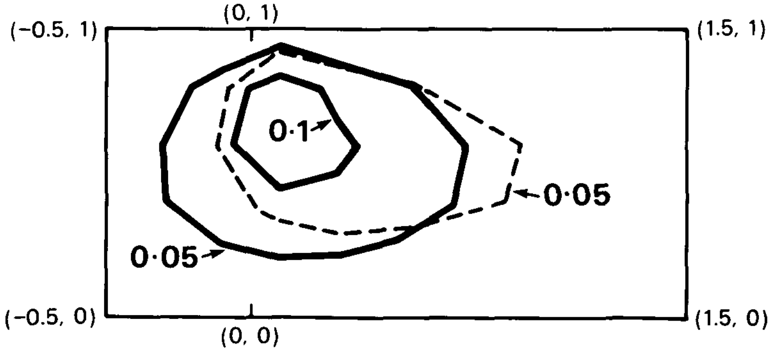

Figure 3

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6. Discussion

We have shown that, by use of Taylor series, it is possible to avoid the use of body integrals in the boundary integral solution of an inhomogeneous partial differential equation. Further, by employing iterative techniques, the method can be extended to enable the solution of nonlinear equations. In nature, our approach can be regarded as a regional method [5].

The method is extremely effective for regular points of the solution, but problems can arise in the vicinity of a singular point. As mentioned in Section 5, singular points will necessitate great care in the choice of expansion regions. Further, as for other methods that do not give special consideration to singularities, errors will become appreciable in the immediate vicinity of a singularity. However, the use of Taylor series enables a possible remedy. For many problems, the nature of the singularity will be known. Consequently, we may remove that part of the Taylor series which corresponds to the singularity and replace it by the correct singular behaviour. Initial experiments with this approach have given promising results.

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