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THE MAXIMUM MODULUS OF NORMAL MEROMORPHIC FUNCTIONS AND APPLICATIONS TO VALUE DISTRIBUTION

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Introduction. Let $f(z)$ be a function meromorphic in the unit disc $D = (|z| < 1)$. We consider the maximum modulus

$$M(r,f) = \max_{|z|=r} |f(z)|$$

and the minimum modulus

$$m(r,f) = \min_{|z|=r} |f(z)|.$$

When no confusion is likely, we shall write $M(r)$ and $m(r)$ in place of $M(r,f)$ and $m(r,f)$.

Since every normal holomorphic function belongs to an invariant normal family, a theorem of Hayman [**6**, Theorem 6.8] yields the following result.

THEOREM 1. *If $f(z)$ is a normal holomorphic function in the unit disc D, then*

$$\limsup\, (1 - r) \log M(r) < +\infty, \quad \text{as } r \to 1. \tag{1}$$

This means that for normal holomorphic functions, $M(r)$ cannot grow too rapidly. The main result of this paper (Theorem 5, also due to Hayman, but unpublished) is that a similar situation holds for normal meromorphic functions.

In [**2**; **3**] we considered the distribution of values of meromorphic functions having asymptotic values. In the present paper, with the help of Hayman's theorems we will see how the speed with which the asymptotic value is approached affects the value distribution of the function. We are motivated by the classical theorems in this direction for entire functions.

In addition to asking how quickly $M(r)$ may grow for a normal function, we might also ask how slowly $M(r)$ may grow without being bounded. The latter question, however, is quickly answered. For if $\phi(r)$ is a function which tends to $+\infty$ as slowly as we please, it can be shown, using the Ahlfors Distortion Theorem, that there is a conformal map $f(z)$ of the unit disc for which $M(r)$ tends to $+\infty$ more slowly than $\phi(r)$. The function $f(z)$ is, of course, normal.

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1. Definitions and lemmas. For $z, z' \in D$, we denote by $\rho(z, z')$ the non-Euclidean (hyperbolic) metric. $\rho(z, z') = \frac{1}{2}\log[(1+a)/(1-a)]$, where $a = |(z'-z)/(1-\bar{z}z')|$. We call $\rho(z, z')$ the ρ-distance between z and z'. For $S \subset D$ and $r > 0$, we write

$$\Delta(S, r) = \{z \in D\colon \rho(S, z) \leqq r\}.$$

A sequence $\Delta(n)$ of discs in D is a sequence of (non-Euclidean) *cercles de remplissage* for a function $f(z)$ defined in D provided that the ρ-diameters of the $\Delta(n)$ tend to zero, and the images $f(\Delta(n))$ cover all of the Riemann sphere, with the possible exception of two sets $E(n)$ and $G(n)$ whose spherical diameters tend to zero as $n \to \infty$. The sequence $\{z_n\}$ of centres of the $\{\Delta(n)\}$ is called a sequence of ρ-points for $f(z)$. Sequences of ρ-points were introduced by Lange [**9**], studied by Gavrilov [**4**] under the name of P-points, and by Rung [**10**] under the name of M-points. For the equivalence of P-points and ρ-points, see [**1**, Theorem 4]; and for the equivalence of these to M-points, see [**3**, Theorem 1].

LEMMA 1 [**1**, Theorem 3]. *A meromorphic function is normal in the unit disc D if and only if it possesses no sequence of cercles de remplissage.*

A sequence $\{z_n\}$, $z_n \in D$, is called a boundary sequence if $|z_n| \to 1$, as $n \to \infty$. A continuous curve $\alpha(t)$, $0 \leqq t < 1$, in D, is called a boundary path provided that $|\alpha(t)| \to 1$, as $t \to 1$. A boundary segment is a boundary path which is a straight line segment. We shall call a boundary path α a spiral if $\arg \alpha(t)$ is unbounded. The end of a boundary sequence (path) is the set of all points on the unit circle which are limit points of the boundary sequence (path). We say that a boundary sequence (path) ends in a point z_0 of the unit circle if z_0 is the only point in the end of the sequence (path). A boundary path (segment) α is called a ρ-path (ρ-segment) if there is a sequence of ρ-points on α. A Stolz angle ending at a point z_0 of the unit circle is a triangle contained in D except for one vertex at z_0. A boundary disc ending at z_0 is a disc contained in D except for one point z_0 on the unit circle. For concepts and notation not explicitly defined in this paper, we refer the reader to [**6**].

We will need the following results.

LEMMA 2 (Pick [**9**, Theorem 15.1.3]). *Let $h(\zeta)$ be a function holomorphic and bounded by* 1 *in the unit disc* ($|\zeta| < 1$). *Then*

$$\rho(h(\zeta), h(\zeta')) \leqq \rho(\zeta, \zeta'), \qquad \zeta, \zeta' \in (|\zeta| < 1).$$

LEMMA 3. *Suppose that $h(\zeta)$ is holomorphic and $h'(\zeta)$ is bounded by C in a convex domain K. Then*

$$|h(\zeta') - h(\zeta)| \leqq C|\zeta' - \zeta|, \qquad \zeta, \zeta' \in K.$$

Proof.

$$|h(\zeta') - h(\zeta)| = \left|\int_{\zeta}^{\zeta'} h'(t)\, dt\right| \leqq C|\zeta' - \zeta|.$$

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LEMMA 4 [**10**, Theorem 1]. *Let $F(\zeta)$ be a holomorphic function in the unit disc $(|\zeta| < 1)$ such that*

$$|F(\zeta)| \leqq \exp \frac{-A_n}{1-|\zeta|}, \qquad \zeta \in \gamma_n, \quad n = 1, 2, \ldots,$$

where $\{\gamma_n\}$ is a sequence of Jordan arcs in $(|\zeta| < 1)$ satisfying

$$\tfrac{1}{2} \leqq \min_{\zeta \in \gamma_n} |\zeta| = r_n \to 1, \qquad n \to \infty;$$

$$0 < \liminf \sup_{\zeta, \zeta' \in \gamma_n} \rho(\zeta, \zeta'), \qquad n \to \infty;$$

and $\{A_n\}$ is a sequence of positive numbers. If

$$\lim \frac{\log M(r_n, F)}{A_n} = 0, \qquad n \to \infty,$$

then $F(\zeta) \equiv 0$.

2. Holomorphic functions. It follows from Theorem 1 and Lemma 1 that if $M(r)$ grows quickly for a holomorphic function $f(z)$, then $f(z)$ has a wild distribution of values, in the sense that it possesses a sequence of ρ-points. One might ask just where in the unit disc this sequence of ρ-points is located. It seems natural to expect that the sequence of ρ-points will be in some sense close to the set along which $M(r)$ grows quickly, since it is the latter phenomenon which proclaims the existence of the ρ-points. In general, we will see that this is the case.

THEOREM 2. *If $f(z)$ is a holomorphic function in the unit disc D, satisfying on a boundary sequence $\{z_n\}$, $z_n \in D$, the inequality*

$$(2) \qquad |f(z_n)| \geqq \exp\left(\frac{p(1-|z_n|)}{1-|z_n|}\right), \qquad n = 1, 2, \ldots,$$

where $p(x) \to +\infty$ as $x \to 0$, then each point in the end of $\{z_n\}$ is the limit of a sequence of ρ-points.

Proof. Let z_0 be a point in the end of $\{z_n\}$, and let G be the intersection with D, of a disc about z_0. It is enough to show that there is a sequence of ρ-points in G, for then, by a diagonal process, we can find a sequence of ρ-points tending to z_0.

We may assume that $z_0 = 1$, that $z_n \in G$, $n = 1, 2, \ldots$, and that $z_n \to 1$, as $n \to \infty$. Let $h(\zeta)$ map the unit disc $(|\zeta| < 1)$ conformally onto G with $h(1) = 1$. Then $h(\zeta)$ is actually conformal in a neighbourhood of $\zeta = 1$, and so there is a constant C, $0 < C < +\infty$, such that

$$(3) \qquad \frac{1}{C+\epsilon} \leqq \frac{1-|z|}{1-|\zeta|} \leqq C, \qquad \text{where } z = h(\zeta),$$

provided ζ is sufficiently near 1. To see this, let ζ' be the nearest point of

($|\zeta| = 1$) for which $\arg \zeta = \arg \zeta'$. Then by Lemma 3,

$$1 - |z| \leqq |h(\zeta') - h(\zeta)| \leqq C|\zeta' - \zeta| = C(1 - |\zeta|).$$

Since a similar inequality holds in the opposite direction with C replaced by $1/(C + \epsilon)$, (3) follows.

From (3) and (2), it follows that $f(h(\zeta))$ fails to satisfy (1) in ($|\zeta| < 1$), and so by Lemma 1 there is a sequence of ρ-points for $f(h(\zeta))$ in ($|\zeta| < 1$). By Lemma 2 this corresponds to a sequence of ρ-points for $f(z)$ in G. This completes the proof.

THEOREM 3. *If $f(z)$ is a holomorphic function satisfying the hypotheses of Theorem 2 for some sequence $\{z_n\}$, with z_n approaching the unit circle from within a boundary disc H_0, then in each boundary disc H_1 (strictly) containing H_0, there is a sequence of ρ-points.*

Proof. We may assume that H_0 meets the unit circle at $z = 1$. Let $h(\zeta)$ map the disc ($|\zeta| < 1$) conformally onto the disc H_1, so that $h(1) = 1$, and let $h(\zeta_n) = z_n$, $n = 1, 2, \ldots$. Then, as in the proof of Theorem 2, it follows from Lemma 3 that $1 - |\zeta_n|$ is proportional to the Euclidean distance of z_n from the boundary of H_1, and so it is enough to show that this latter distance is proportional to $1 - |z_n|$.

For fixed $z \in H_0$ we have, denoting the boundary of H_1 by ∂H_1,

$$\frac{1 - |z|}{\operatorname{dist}(|z|, \partial H_1)} \leqq \frac{1 - |z|}{\operatorname{dist}(z, \partial H_1)} \leqq \frac{1 - |z'|}{\operatorname{dist}(z', \partial H_1)}, \tag{4}$$

where z' is a point of ∂H_0 with $|z| = |z'|$. Since the left member of (4) is bounded away from zero, it is enough to show that the right member is bounded.

To see this, let a_0 and a_1 be the respective centres of H_0 and H_1. Then we may write

$$z' = a_0 + (1 - a_0)e^{i\theta_0} = a_1 + r_1 e^{i\theta_1} = re^{i\theta}.$$

Solving for r and r_1 as functions of θ_0 yields

$$\lim_{\theta_0 \to 0} \frac{1 - r^2}{(1 - a_1)^2 - r_1^2} = \frac{a_0}{a_0 - a_1}.$$

But this tells us that $1 - |z'|$ is proportional to $\operatorname{dist}(z', \partial H_1)$, and so $1 - |z_n|$ is proportional to $1 - |\zeta_n|$. Writing $F(\zeta) = f(z)$, we have, by Theorem 1 and Lemma 1, that $F(\zeta)$ has a sequence of ρ-points; and by Lemma 2, then, $f(z)$ has a sequence of ρ-points in H_1. This completes the proof.

COROLLARY. *If $f(z)$ is a function satisfying the hypotheses of Theorem 2 with z_n approaching $z = 1$ non-tangentially, then there is a sequence of ρ-points which is eventually in every boundary disc at $z = 1$.*

We cannot infer, simply under the assumptions of the preceding corollary, that there is a non-tangential sequence of ρ-points, as the following example shows.

Example. Consider the function

$$f(z) = \exp \frac{\log[1/(1-z)]}{1-z}.$$

This function clearly satisfies (2) along the radius $\arg z = 0$. However, a straightforward calculation shows that $f(z)$ tends to ∞ within any Stolz angle ending in $z = 1$. Therefore there is no non-tangential sequence of ρ-points tending to $z = 1$.

In order to obtain non-tangential ρ-points, we must assume a faster rate of growth for $M(r)$.

THEOREM 4. *Let $f(z)$ be a function holomorphic in the unit disc D, and suppose that for some boundary sequence $\{z_n\}$ approaching $z = 1$ non-tangentially, and some $\sigma > 1$,*

$$|f(z_n)| \geqq \exp\left(\frac{p(1-|z_n|)}{(1-|z_n|)^{\sigma}}\right), \qquad n = 1, 2, \ldots,$$

where $p(x) \to +\infty$ as $x \to 0$. Then $f(z)$ has a non-tangential sequence of ρ-points at $z = 1$.

We merely sketch the proof since it is quite similar to the preceding proofs. There is a Stolz angle Δ of opening π/α, $\alpha > 1$, which contains a subsequence of $\{z_n\}$, and so we may assume that the sequence itself is contained in Δ. Let $h(\zeta)$ map the unit disc $(|\zeta| < 1)$ conformally onto Δ so that $h(1) = 1$, and set $h(\zeta_n) = z_n$, $n = 1, 2, \ldots$. Setting $F(\zeta) = f(z)$, we have

$$F(\zeta_n) \geqq \exp \frac{q(1-|\zeta_n|)}{(1-|\zeta_n|)^{\sigma/\alpha}}, \qquad n = 1, 2, \ldots,$$

where $q(x) \to +\infty$ as $x \to 0$. It follows from Theorem 2, that for $1 \leqq \alpha \leqq \sigma$, $F(\zeta)$ has a sequence of ρ-points. Thus, by Lemma 2, $f(z)$ has a sequence of ρ-points in Δ. Since $\sigma > 1$, we may choose $\alpha > 1$, and so the sequence of ρ-points is non-tangential. This completes the proof.

One might think that if a holomorphic function tends to ∞ quickly enough on a boundary sequence $\{z_n\}$, then there would be a sequence of ρ-points at a finite ρ-distance from $\{z_n\}$. The following example, however, shows that this is not the case.

Example. Given any continuous positive function $\phi(r)$, there exists a boundary sequence $\{z_n\}$ and a holomorphic function $f(z)$ satisfying

$$|f(z)| > \phi(|z|), \qquad z \in \Delta(z_n, n), \quad n = 1, 2, \ldots .$$

By Mergelyan's theorem [**13**, p. 367, Theorem 1] one establishes the existence of a sequence of polynomials whose limit is the required function $f(z)$.

In this section we considered the distribution of values of holomorphic functions which approach ∞ rapidly. Similar results hold for a meromorphic function which approaches an omitted value rapidly.

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3. Meromorphic functions. We now present results for meromorphic functions, which are analogous to those given in the preceding section for holomorphic functions.

THEOREM 5 (Hayman, written communication). *If $f(z)$ is a normal meromorphic function in the unit disc D, then*

$$\liminf\, (1-r)\log M(r) < +\infty, \quad \textit{as } r \to 1. \tag{5}$$

Proof. It follows from [**6**, p. 171, Example 1] that if f belongs to an invariant normal family and, in particular, if f is a normal (meromorphic) function, then for $\frac{1}{2} \leqq r < 1$, $|z_0| < 1 - r$, we have

$$S(r, f(z_0 + z)) \leqq \frac{B_1}{[(1-r)^2 - |z_0|^2]^{1/2}},$$

where the left-hand side is the spherical area of the image of the disc $|z - z_0| < 1 - r$ by $f(z)$. Integrating this with respect to r and noting that

$$T_0(r, f(z + z_0)) = \int_0^r S(\rho, f(z_0 + z))\,\frac{d\rho}{\rho},$$

where T_0 is the Ahlfors–Shimizu characteristic, we deduce

$$T(r, f(z_0 + z)) \leqq B_2 \log\frac{1}{1-r}, \tag{6}$$

with the same hypotheses. It then follows from [**5**, p. 365, Theorem 6] that (5) holds. For, otherwise, the theorem asserts that either

$$\frac{T(r,f)}{\log[(1-r)^{-1}]} \to \infty,$$

which contradicts (6) with $z_0 = 0$, or that $T(r, f(z_0 + z))$ is unbounded for fixed r, subject to $|z_0| < 1 - r$, which again contradicts (6) for variable z_0 and fixed r. This concludes the proof.

By considering reciprocals, it follows that Theorem 5 is equivalent to the following.

THEOREM 5′. *If $f(z)$ is a normal meromorphic function in the unit disc D, satisfying*

$$\lim\, (1-r)\log m(r) = -\infty, \tag{7}$$

then $f(z) \equiv 0$.

The above theorem was proved for bounded holomorphic functions by Heins [**7**, Theorem 7.1].

Theorems 5 and 5′ have several interesting corollaries analogous to the results of § 2.

COROLLARY 1. *If $f(z)$ is a meromorphic function in the unit disc D, satisfying* *on a boundary path* α *the inequality*

$$(8) \qquad |f(z)| \geqq \exp\left(\frac{p(1-|z|)}{1-|z|}\right), \qquad z \in \alpha,$$

where $p(x) \to +\infty$ *as* $x \to 0$, *then each point in the end of* α *is the limit of a sequence of* ρ*-points.*

Proof. If the end of α contains more than one point, then this result is already known [**3**, Theorem 2]. If α ends at a single point, then the corollary follows from Theorem 5 in the same way that Theorem 2 follows from Theorem 1.

COROLLARY 2. *If* $f(z)$ *is a meromorphic function in the unit disc* D, *satisfying on a boundary path* α *the inequality*

$$(9) \qquad |f(z)| \leqq \exp\left(-\frac{p(1-|z|)}{1-|z|}\right), \qquad z \in \alpha,$$

where $p(x) \to +\infty$ *as* $x \to 0$, *then either* $f(z) \equiv 0$ *or each point in the end of* α *is the limit of a sequence of* ρ*-points.*

From Corollary 2 and Lemma 1, we have the following result.

COROLLARY 3. *If* $f(z)$ *is a normal function satisfying the hypotheses of Corollary* 2, *then* $f(z) \equiv 0$.

COROLLARY 4. *Under the hypotheses of Corollary* 2, *if* α *approaches* $z = 1$ *from within a boundary disc* H_0, *then in each boundary disc* H_1 (*strictly*) *containing* H_0, *there is a sequence of* ρ*-points.*

The proof is the same as the proof of Theorem 3.

COROLLARY 5. *If* $f(z)$ *satisfies the hypotheses of Corollary* 2 *on a non-tangential boundary path* α *ending at* $z = 1$, *then there is a sequence of* ρ*-points which is eventually in each boundary disc at* $z = 1$.

COROLLARY 6 [**10**, Theorem 7]. *Let* $f(z)$ *be a function meromorphic in* D, *and suppose that for some boundary path* α *tending to* $z = 1$ *non-tangentially and some* $\sigma > 1$,

$$|f(z)| \leqq \exp\left(-\frac{p(1-|z|)}{(1-|z|)^{\sigma}}\right), \qquad z \in \alpha.$$

Then either $f(z) \equiv 0$, *or* $f(z)$ *has a non-tangential sequence of* ρ*-points at* $z = 1$.

The proof is the same as the proof of Theorem 4, using Theorem 5 in place of Theorem 1.

In § 2 it was remarked that there is no rate of growth of $f(z)$ on a boundary sequence $\{z_n\}$ which implies the existence of a sequence of ρ-points at a finite ρ-distance from $\{z_n\}$. However, by considering boundary paths instead of boundary sequences, we obtain the following positive result.

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THEOREM 6. *There exists a continuous positive function* $\lambda(r)$, $0 \leqq r < 1$, *such that if* $f(z)$ *is any meromorphic function in the unit disc* D, *satisfying on a boundary path* α *the inequality*

$$(10) \qquad |f(z)| \leqq \lambda(|z|), \qquad z \in \alpha,$$

then either $f(z) \equiv 0$ *or* α *is a* ρ*-path.*

Proof. Choose $0 = r_0 < r_1 < \ldots < r_n < \ldots < 1$, $r_n \to 1$. Let $S(n, 0)$ be a disc centred at the origin and contained in $(|z| < r_n)$. Let $S(n, j)$, $j = 0, 1, \ldots, j(n)$, be a partition of $(|z| \leqq r_n)$ into a finite number of annular sectors bounded by radial segments and concentric circular arcs. We may assume that the $S(n, j)$ are such that the ρ-diameter of each $S(n, j)$ is less than $1/n$. Let $A(n, p, l)$, $l = 1, 2, \ldots, l(n, p)$, $p = 0, 1, \ldots, n - 3$, be the family of all regions which connect some circle $(|z| = r_p)$ to $(|z| = r_n)$, and which are such that $A(n, p, l)$ is the union of a subfamily of

$$\{S(n, j) : j = 0, 1, \ldots, j(n)\}.$$

Let $K(n, p, l)$ be an arc in $A(n, p, l)$ which connects $(|z| = r_{p+1})$ to $(|z| = r_{p+2})$. Let $h_{n,p,l}$ map the disc $(|\zeta| < 1)$ onto $A(n, p, l)$ via the universal covering surface of $A(n, p, l)$, and let $k(n, p, l)$ be a fixed component of the inverse image of $K(n, p, l)$.

From the Two Constants Theorem and another theorem of Nevanlinna [**10**, Theorem A] we conclude the following. For each $0 < a_n < 1$, there is a constant $m(n, p, l)$ such that if $F(\zeta)$ is holomorphic and bounded by 1 in $(|\zeta| < 1)$ and bounded by $m(n, p, l)$ on a boundary curve γ which meets $(|\zeta| = a_n)$, then $|F(\zeta)| < 1/n$, for ζ in $k(n, p, l)$. Let $m(n) = \inf m(n, p, l)$, where the infimum is taken over all $m(n, p, l)$ with $p = 0, 1, \ldots, n - 3$, $l = 1, 2, \ldots, l(n, p)$. It follows that if $F(\zeta)$ is holomorphic and bounded by 1 in $(|\zeta| < 1)$ and bounded by $m(n)$ on a boundary curve γ which meets $(|\zeta| = a_n)$, then

$$(11) \qquad |F(\zeta)| < 1/n, \qquad \zeta \in k(n, p, l), \quad p = 0, 1, \ldots, n - 3, \quad l = 1, 2, \ldots, l(n, p).$$

The constant a_n will be specified shortly.

For any boundary path β in $(|z| < 1)$, let $p(\beta)$ be the smallest index p such that β meets $(|z| = r_p)$. For $n > p$, let $A(\beta, n)$ be the union of all $S(n, j)$ which meet $\Delta(\beta, 2/n)$. Then $A(\beta, n) = A(n, p, l)$ for some l and some $p < p(\beta)$. We now set

$$a_n = \max\{\min\{|\zeta|; h_{n,p,l}(\zeta) \in \beta, \text{ where } A(\beta, n) = A(n, p, l)\}\},$$

where the maximum is taken over all boundary paths β which meet $(|z| = r_{n-3})$. Since $\beta \cap (|z| = r_{n-2})$ is at a ρ-distance of at least $1/n$ from the boundary of $A(\beta, n)$, it follows that $a_n < 1$.

We may now assume that $m(n)$ is strictly decreasing to zero as $n \to \infty$. Let $\lambda(r)$ be a continuous positive function satisfying

$$\lambda(r) < m(n), \qquad r_{n-1} \leqq r < r_n.$$

We may also assume that $\lambda(r) < \frac{1}{2}$.

Suppose now, to prove the theorem, that α is a boundary path on which $f(z)$ satisfies (10), and suppose that α is not a ρ-path. Then [**3**, Theorem 1] the family

$$f\left(\frac{z + \alpha(t)}{1 + \overline{\alpha(t)}z}\right), \qquad 0 \leqq t < 1,$$

is normal in some neighbourhood of $z = 0$, and hence [**8**, Theorem 15.2.2] equicontinuous. It follows from this and from (10), that for all sufficiently large $n > N(\alpha)$, α meets $(|z| = r_n)$, and

$$|f(z)| < 1, \qquad z \in \Delta(\alpha, 4/n) \cap (|z| \geqq r_n).$$

For each $n > N(\alpha) + 4$, we have $A(\alpha, n) = A(n, p, l)$, for some index p and some index l. By our choice of a_n, we see that some component γ_n of the inverse image of α by $h_{n,p,l}$ meets $(|\zeta| = a_n)$. Also from the definition of $\lambda(r)$,

$$|F_n(\zeta)| \equiv |f(h_{n,p,l}(\zeta))| \leqq m(n), \qquad \zeta \in \gamma_n;$$

and thus (11) holds for $F = F_n$. It follows that

$$|f(z)| < 1/n, \qquad z \in K(n, p, l), \qquad n > p + 3.$$

Since $K(n, p, l)$ is an arc which intersects both $(|z| = r_{p+1})$ and $(|z| = r_{p+2})$, and since $p < p(\alpha)$, we have $f(z) \equiv 0$. This completes the proof.

4. Spiral asymptotic behaviour. We now consider the distribution of values of a meromorphic function which tends quickly to an asymptotic value along a spiral path. As in the previous theorems, there are two types of results depending on whether the asymptotic value is an omitted value or not. The first case is typified by the approach of a holomorphic function to ∞, and the second case is exemplified by the approach of a general meromorphic function to, say 0.

To begin, we note the following consequence of the proof of Theorem 4.

THEOREM 7. *Let $f(z)$ be a function holomorphic in the unit disc D, and suppose that for some spiral α and some $\sigma \geqq 1$,*

$$|f(z)| \geqq \exp\left(\frac{p(1 - |z|)}{(1 - |z|)^\sigma}\right), \qquad z \in \alpha,$$

where $p(x) \to +\infty$ as $x \to 0$. Then in each Stolz angle of opening π/σ, $f(z)$ has a sequence of ρ-points.

COROLLARY. *If $f(z)$ is a holomorphic function in the unit disc D, satisfying eventually, on a spiral α, the inequality*

$$|f(z)| \geqq \exp\,(1 - |z|)^{-\sigma}$$

for each fixed σ, then each boundary segment is a ρ-segment.

For functions approaching zero on a spiral we have an analogue to Theorem 7. However, it seems that we must assume a faster rate of approach since zero may not be an omitted value.

THEOREM 8. *Let $f(z)$ be a function holomorphic in the unit disc D, and suppose that for some spiral α and some $\sigma \geqq 1$, $f(z)$ satisfies*

$$|f(z)| \leqq \exp\left(-\frac{p(1-|z|)}{(1-|z|)^{2\sigma}}\right), \qquad z \in \alpha,$$

where $p(x) \to +\infty$ as $x \to 0$. Then either $f(z) \equiv 0$ or in each Stolz angle of opening π/σ, $f(z)$ has a sequence of ρ-points.

Proof. Suppose that Δ is a Stolz angle of opening π/σ. We may assume that the vertex of Δ is at $z = 1$. As usual, let $h(\zeta)$ map the disc $(|\zeta| < 1)$ conformally onto Δ so that $h(1) = 1$, and let $F(\zeta) = f(z)$. Then since $(1 - |z|)^\sigma$ is proportional to $1 - |\zeta|$, we have

$$(12) \qquad |F(\zeta)| \leqq \exp\left(\frac{-q(1-|\zeta|)/(1-|\zeta|)}{1-|\zeta|}\right), \qquad \zeta \in \gamma_n,$$

where $q(x) \to +\infty$ as $x \to 0$, and $\{\gamma_n\}$ is a sequence of crosscuts of $(|\zeta| < 1)$ which tend to $\zeta = 1$.

Suppose, now, that $F(\zeta)$ is a normal function. Then by Theorem 1,

$$(13) \qquad \lim \frac{\log|F(\zeta)|}{q(1-|\zeta|)/(1-|\zeta|)} = 0, \qquad |\zeta| \to 1.$$

According to Lemma 4, (12) and (13) imply that $F(\zeta) \equiv 0$. Thus $f(z) \equiv 0$.

If, on the other hand, $F(\zeta)$ is not normal, then by Lemma 1, it possesses a sequence of ρ-points. By Lemma 2, $f(z)$ has a sequence of ρ-points in Δ. This completes the proof.

It is surprising to see precisely the same order of growth on $M(r)$ in Theorems 1 and 5 but not in Theorems 7 and 8; perhaps Theorem 8 is not sharp. In addition, it seems that Theorem 8 should hold for meromorphic as well as holomorphic functions. However, in the proof we rely heavily on a theorem of Rung (Lemma 4) which has thus far been proved only for holomorphic functions.

COROLLARY. *If $f(z)$ is a holomorphic function in the unit disc D, satisfying eventually, on a spiral α, the inequality*

$$|f(z)| \leqq \exp(-(1-|z|)^{-\sigma})$$

for each fixed σ, then either $f(z) \equiv 0$ or each boundary segment is a ρ-segment.

It is important to point out that there exist non-constant functions satisfying the respective hypotheses of the theorems and corollaries of this paper (with the exception of those where we assume the function is normal). This follows from a theorem of Schnitzer and Seidel [**11**].

5. Normal families

THEOREM 9. *Let F be a normal family of meromorphic functions in the unit disc D. Then there is a continuous positive function $\lambda(r)$, $0 \leqq r < 1$, such that for each f in F,*

$$\liminf \lambda(r) M(r, f) = 0, \qquad r \to 1. \tag{14}$$

Proof. Set

$$F(n) = \left\{ f \in F : \min_{|z| \leqq \frac{1}{2}} |f(z)| \leqq n \right\}.$$

Then $F = F(1) \cup F(2) \cup \ldots \cup F(n) \cup \ldots$. Now choose

$$\tfrac{1}{2} = r_0 < r_1 < \ldots < r_n < \ldots, \qquad r_n \to 1.$$

For each n, there is a constant $M(n)$ such that for each $f \in F(n)$, there is an r, with $r_{n-1} \leqq r \leqq r_n$ such that $M(r, f) \leqq M(n)$. For otherwise, for each $m = 1, 2, \ldots$, there is a function f_m in $F(n)$ satisfying $M(r, f_m) \geqq m$, for $r_{n-1} \leqq r \leqq r_n$. Now $\{f_m\}$ has a subsequence which converges uniformly to ∞ on $(|z| \leqq r_n)$. However, for each f_m there is a point z_m such that $|f_m(z_m)| \leqq n$, and $|z_m| \leqq \frac{1}{2}$. This is a contradiction, and so $M(n)$ exists, as claimed.

Set $\lambda(r) = 1$, for $0 \leqq r \leqq r_0$, and set $\lambda(r) = (nM(n))^{-1}$, for $r_{n-1} < r \leqq r_n$, $n = 1, 2, \ldots$. Now, if f is in F, then for each $n > N(f)$, f is in $F(n)$. Hence for each $n > N(f)$, there is an r, with $r_{n-1} \leqq r \leqq r_n$ such that $M(r, f) \leqq M(n)$, and so for this r, $\lambda(r) M(r, f) \leqq 1/n$. The theorem follows.

THEOREM 10. *Let F be a normal family of holomorphic functions in the unit disc D. Then there is a continuous positive function $\lambda(r)$, $0 \leqq r < 1$, such that for each f in F,*

$$\limsup \lambda(r) M(r, f) = 0, \qquad r \to 1. \tag{15}$$

Proof. Set

$$F(n) = \{ f \in F : |f(0)| \leqq n \}, \qquad n = 1, 2, \ldots.$$

Then $F = F(1) \cup F(2) \cup \ldots \cup F(n) \cup \ldots$. By an argument similar to that used in Theorem 9, one can show that each family $F(n)$ is uniformly bounded by, say $M(n)$, on $(|z| \leqq r_n)$. The rest of the proof parallels the proof of Theorem 9, and thus we omit the details.

It should be remarked that condition (15), although necessary, is certainly not sufficient for normalcy. For example, the family of functions $(2z)^n$, $n = 1, 2, \ldots$, satisfies (15) but is not normal in the unit disc.

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