AUTOMORPHISMS FIXING EVERY SUBNORMAL SUBGROUP OF A FINITE GROUP

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Introduction. We denote by $\mathrm{Aut}_{\mathrm{sn}}(G)$ the set of all automorphisms that fix every subnormal subgroup of G setwise. In a recent paper [7] Robinson showed that the structure of $\mathrm{Aut}_{\mathrm{sn}}(G)$ is quite restricted for a finite group G. Our aim in this paper is to show that more detailed information about the structure of $\mathrm{Aut}_{\mathrm{sn}}(G)$ can be obtained by focussing on its action on $F^*(G)$, the generalised Fitting subgroup of G.

For the remainder of this paper G will denote a finite group and we will write $A = \mathrm{Aut}_{\mathrm{sn}}(G)$. We denote by $E(G)$ the layer of G and recall that $F^*(G)$ is a central product of $E(G)$ and $F(G)$, the Fitting subgroup of G. (See, for example, Section 13 of Huppert and Blackburn [6].) We denote by τ the natural homomorphism of G onto $\mathrm{Inn}(G)$. We set $C = C_A(F^*(G))$. For an arbitrary finite group we obtain the following structure.

THEOREM 1. *For a finite group G we have $E(A) = E(G)^\tau$ and $A/(E(A)C)$ soluble with derived length at most* 3.

Note that C is an abelian normal subgroup of A by Lemma 5 of Robinson [7], Theorem 13.2 of Huppert and Blackburn [6] and Lemma 3 of P. Hall [5], and so we have that $C \leq F(A)$. It then follows that $\zeta(E(A))C$ is abelian (by Huppert and Blackburn [6, Theorem 13.15], with $\zeta(E(A))$ the centre of $E(A)$) and so A is abelian-by-completely reducible-by-soluble of derived length at most 3. We also get a criterion for A to be soluble; it is equivalent to the one given in Corollary 3 of Robinson [7], since the Wielandt subgroup $\omega(G)$ of G is soluble if and only if $E(G) = 1$.

COROLLARY 2. *The group A is soluble if and only if $F^*(G) = F(G)$.*

When A is soluble we can obtain further restrictions on the structure of A. In the finite case, Theorem A of Franciosi and De Giovanni [4] tells us that A is metabelian if G is soluble and Teorema A of Dalle Molle [3] tells us that A is supersoluble if G is soluble. We can extend these results in the following way.

THEOREM 3. *Suppose that G is a finite group with $F^*(G) = F(G)$. Then A is metabelian and supersoluble. Moreover, if π is the set of primes dividing $|F(G)|$, then $|A/F(A)| \leq \prod_{p \in \pi} (p-1)$ and the nilpotency class of $F(A)$ is bounded by $\max_{p\in\pi}\{e_p\}$, where p^{e_p} is the exponent of the Sylow p-subgroup of $F(G) \cap \omega(G)$.*

The proof of Theorem 3 in fact will often give more information about A if there is more information about G; for example if all the Sylow subgroups of $F(G)$ are nonabelian then A is nilpotent.

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We also give examples to show that restrictions on the structure given as a consequence of Theorem 1 and by Theorem 3 are best possible. Robinson [**7**] shows that $A/(A \cap \mathrm{Inn}(G))$ has derived length at most 4 and asks if this is best possible. We have been unable to decide if $A/(A \cap \mathrm{Inn}(G))$ has derived length at most 3.

Proof of Theorem 1. If we denote by Z the centre of $E(G)$, then $E(G)/Z$ is a direct product of nonabelian simple groups. Now A induces a group of automorphisms on $E(G)/Z$, fixing each simple factor setwise. It follows (using Proposition 4 of Robinson [**7**]) that if $\alpha \in A^{(3)}$, the third term of the derived series of A, then α acts on each simple factor and hence on $E(G)/Z$ as an inner automorphism. Thus for some $x \in E(G)$ we have $\beta = \alpha^{-1}x^{\tau}$ acts trivially on $E(G)/Z$. We claim that β acts trivially on $E(G)$. To see this, let p be a prime and $Z_{p'}$ be the Hall p'-subgroup of Z. It is clearly enough to prove that β acts trivially on $E(G)/Z_{p'}$ for an arbitrary prime p. Thus we may assume that Z is a p-group for some prime p. Suppose that q is a prime different to p and let Q be a Sylow q-subgroup of $E(G)$. Then QZ is fixed setwise by β and since Q is the unique Hall p'-subgroup of QZ we must have Q also fixed setwise by β. We also have that Q is fixed elementwise by β modulo Z and hence we conclude that β acts trivially on Q. Since each finite simple group is divisible by a prime other than p, we have $E(G)$ generated by its Sylow q-subgroups, $q \neq p$, and hence β acts trivially on $E(G)$.

Next we observe that if $\alpha \in A^{(3)}$ and $x \in E(G)$, we have $\alpha \in C_A(F(G))$ (since A acts as a group of power automorphisms on $F(G)$) and $x \in C_G(F(G))$ (since $[E(G), F(G)] = 1$ by Huppert and Blackburn [**6**, Theorem 13.15]). Thus we have $\beta \in C_A(F(G))$ and then, by the previous paragraph, $\beta \in C_A(F^*(G) = C$. Thus we have $\alpha \in E(G)^{\tau}C$. It follows immediately that $A^{(3)} \leq E(G)^{\tau}C$. A subnormal quasisimple subgroup of $E(G)^{\tau}$ is also a subnormal subgroup of A and so we have $E(G)^{\tau} \leq E(A)$. (See for example Section 31 of Aschbacher [**1**].) Since $A/E(G)^{\tau}$ is clearly soluble, we have $E(A) \leq E(G)^{\tau}$ also and hence $E(A) = E(G)^{\tau}$. This completes the proof of Theorem 1.

To prove Theorem 3 we first need a lemma which tells us about the action of A on $G/F^*(G)$.

LEMMA 4. *Let G be a finite group with $F(G) = F^*(G)$. If $\alpha \in A$, then we have $g^{-1}g^{\alpha} \in F(G) \cap \omega(G)$, for all $g \in G$.*

Proof. If $g \in G$ and $\alpha \in A$, we put $b = g^{-1}g^{\alpha}$. Then α acts as a power automorphism on $F(G)$ and thus as a universal power automorphism on $F(G)/F(G)'$, by Theorem 3.4.1 of Cooper [**2**]. Since $b^{\tau} = (g^{\tau})^{-1}\alpha^{-1}g^{\tau}\alpha$, if $x \in F(G)$ then x^b is congruent to x modulo $F(G)'$. We then deduce $b \in F(G)$ from the facts that $F^*(G/F(G)') = F(G)/F(G)'$ and that the generalised Fitting subgroup contains its own centraliser.

We also have $b^{\tau} \in A \cap \mathrm{Inn}(G) = \omega(G)^{\tau}$ and hence $b \in \omega(G)$. This completes the proof.

Proof of Theorem 3. Suppose that G is a finite group with $F^*(G) = F(G)$. Following Robinson [**7**] we observe that A' centralises $F(G)$ and hence, by his Lemma 5, also $G/F(G)$. Thus A' is abelian and A is metabelian.

If π is the set of prime divisors of $F(G)$, we choose normal subgroups Q_p, where $p \in \pi$, such that if P_p is the Sylow p-subgroup of $F(G)$ then $Q_p \cap P_p = 1$ and G/Q_p has no normal p'-subgroups. (Such a choice is always possible; we may choose Q_p to be a maximal normal subgroup of G with $Q_p \cap P_p = 1$.) Since Q_p is fixed by A we have a natural homomorphism of A into $\mathrm{Aut}(G/Q_p)$; we denote the image of A by A_p and the kernel by K_p. Note that since $\bigcap_{p\in\pi} Q_p = 1$ we also have $\bigcap_{p\in\pi} K_p = 1$. Thus A is a subdirect product of the A_p and so to prove the remainder of Theorem 3 it will be enough to prove that A_p is supersoluble, with $|A_p/F(A_p)| \le p-1$ and $F(A_p)$ a nilpotent group of class at most e_p. We set $G_p = G/Q_p$.

Denote by F_p the Fitting subgroup of G_p and by W_p the image of $P_p \cap \omega(G)$ in G_p. We have that A_p acts trivially on G_p/F_p and as power automorphisms on F_p. Thus $C_{A_p}(F_p)$ acts trivially on both G_p/F_p and F_p and it follows that $C_{A_p}(F_p)$ is an abelian normal p-subgroup of A_p. Suppose now that F_p is nonabelian. Since A_p acts on F_p as a power automorphism, $A_p/C_{A_p}(F_p)$ is an abelian p-group, by Cooper [**2**, Corollary 5.1.2] and so A_p is a p-group. Next suppose that F_p is abelian. If $W_p \ne F_p$ then any automorphism α in A_p of order prime to p acts trivially on F_p/W_p and since α maps each element of F_p to the same power by Cooper [**2**, Theorem 3.4.1] we must have $\alpha = 1$. It follows that A_p is a p-group. If $W_p = F_p$, then $A_p/C_{A_p}(F_p)$ is a cyclic group of order dividing $p^{e_p-1}(p-1)$; it then follows that A_p has a normal p-subgroup of index dividing $p-1$ and so is clearly supersoluble with $|A_p/F(A_p)| \le p-1$ and moreover $F(A_p)$ has class equal to that of its Sylow p-subgroup, since the Sylow p'-subgroups are cyclic.

We now have the Sylow p-subgroup, B_p say, of A_p acting trivially on G_p/W_p, by Lemma 4. If we set $W_i = W_p^{p^i}$, then B_p acts trivially on W_i/W_{i+1}. Thus B_p stabilises a series of normal subgroups of length $e_p + 1$ and so, by Lemma 3 of Hall [**5**], we have B_p nilpotent of class at most e_p.

Examples. Our first example is a group G with trivial centre for which $\mathrm{Aut}_{\mathrm{sn}}(G)$ is abelian-by-completely reducible-by-soluble of derived length 3. We begin by choosing H to be a simple group whose outer automorphism group is isomorphic to S_4; (for example, as in Robinson [**7**], we may take $H = D_4(3)$). Next we take $SL_3(7)$; (note that the centre of $SL_3(7)$ is of order 3). Now let α be the automorphism of $SL_3(7)$ given by the transpose inverse map; (note that α is an automorphism of order 2 which inverts the central elements of $SL_3(7)$). Let K be the semidirect product of $SL_3(7)$ by $\langle\alpha\rangle$. Then K has trivial centre. We now set $G = H \times K$ and claim $\mathrm{Aut}_{\mathrm{sn}}(G)$ has the required properties. To see this note that $\mathrm{Aut}(H) \times K$ is isomorphic to a subgroup $\mathrm{Aut}(G)$. It is now easy to check that this subgroup fixes each subnormal subgroup of G setwise and that $(\mathrm{Aut}(H) \times K)^{(3)} = \mathrm{Inn}(H) \times SL_3(7)$, which is abelian-by-completely reducible.

For our next example, given a positive integer e and an odd prime p, we construct a group G for which $F(G) \cap \omega(G)$ has exponent p^e, A has $F(A)$ a p-group of class e and $|A/F(A)| = p-1$. Let X be a cyclic group of order p^e. Then $\mathrm{Aut}(X)$ is the group of power automorphisms of X and is cyclic of order $p^{e-1}(p-1)$. If H is the holomorph of X, then H is the semidirect product of X and $\mathrm{Aut}(X)$. We let Y be the subgroup of $\mathrm{Aut}(X)$ of order 2 and then we take for G the subgroup XY of H. The automorphism group of G is then just H and we then have easily that $\mathrm{Aut}(G) = \mathrm{Aut}_{\mathrm{sn}}(G) = H$. Thus $F(H)$ is the extension of X by a cyclic group C of order p^{e-1} and so $|H/F(H)| = p-1$. Finally, it is easy to see that the class of XC is e.

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