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A NOTE ON M-IDEALS OF COMPACT OPERATORS

BY

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ABSTRACT. Suppose X and Y are closed subspaces of $(\Sigma X_n)_p$ and $(\Sigma Y_n)_q$ $(1 < p \leqq q < \infty, \dim X_n < \infty, \dim Y_n < \infty)$, respectively. If $K(X, Y)$, the space of the compact linear operators from X to Y, is dense in $L(X, Y)$, the space of the bounded linear operators from X to Y, in the strong operator topology, then $K(X, Y)$ is an M-ideal in $L(X, Y)$.

1. **Introduction.** Since Alfsen and Effros [1] introduced the notion of an M-ideal, an interesting problem has been determining those Banach spaces X and Y for which $K(X, Y)$, the space of compact linear operators from X to Y, is an M-ideal in $L(X, Y)$, the space of bounded linear operators from X to Y. It is well known that if X is c_0, $l^p (1 < p < \infty)$ or a Hilbert space, then $K(X)$ is an M-ideal in $L(X)$ [6, 13] while $K(l^1)$ and $K(l^\infty)$ are not M-ideals in the corresponding spaces of operators [13]. Several authors proved that $K(l^p, l^q)$ when $1 < p \leqq q < \infty$ is an M-ideal in $L(l^p, l^q)$ [6, 9, 12] and $k(X, c_0)$ is an M-ideal in $L(X, c_0)$ for every Banach space X [8, 12, 13].

Harmand and Lima [5] proved that if X is a Banach space for which $K(X)$ is an M-ideal in $L(X)$ then there exists a net $\{T_\alpha\}$ in $K(X)$ such that

(i) $T_\alpha \longrightarrow I_X$ strongly

(ii) $\|T_\alpha\| \leqq 1$ for all α

(iii) $T_\alpha^* \longrightarrow I_{X^*}$ strongly

(iv) $\|I_X - T_\alpha\| \longrightarrow 1$.

Thus, if $K(X)$ is an M-ideal in $L(X)$, then X satisfies the metric compact approximation property. A strong converse of this is also true if X is a closed subspace of $l^p (1 < p < \infty)$ [3].

Recently Werner [15] proved that for a closed subspace Y of a c_0-sum of finite dimensional Banach spaces $K(X, Y)$ is an M-ideal in $L(X, Y)$ for every Banach space X if and only if Y satisfies the metric compact approximation property.

Cho [4] observed that if X and Y are Banach spaces and $K(X, Y)$ is an M-ideal in $L(X, Y)$ then the closed unit ball of $K(X, Y)$ is dense in the closed unit ball of $L(X, Y)$

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in the strong operator topology, and the converse is also true if Y is a closed subspace of a c_0-sum of finite dimensional Banach spaces and X is a reflexive Banach space.

The purpose of this paper is to prove the analogue of a result of Cho [4] for closed subspaces X and Y of l^p and l^q $(1 < p \leqq q < \infty)$, respectively. In Theorem 5 we will show that if X and Y are closed subspaces of $(\Sigma X_n)_p$ and $(\Sigma Y_n)_q$ $(1 < p \leqq q < \infty, \dim X_n < \infty, \dim Y_n < \infty)$, respectively, and if $K(X,Y)$ is dense in $L(X,Y)$ in the strong operator topology, then $K(X,Y)$ is an M-ideal in $L(X,Y)$. Thus, if either X or Y has the compact approximation property then $K(X,Y)$ is an M-ideal in $L(X,Y)$.

The general approach to proving our main theorem is greatly inspired by a paper of Cho and Johnson [3].

2. **Notation and preliminaries.** A closed subspace J of a Banach space X is called an L-summand if there exists a projection P on X such that $PX = J$ and $\|x\| = \|Px\| + \|x - Px\|$ for every x in X. In this case we write $X = j \oplus_1 J'$ where $J' = (I - P)X$. A closed subspace J of a Banach space X is called an M-ideal in X if J^0, the annihilator of J in X^*, is an L-summand in X^*.

If X and Y are Banach spaces, $L(X,Y)$ (resp. $K(X,Y)$) will denote the space of all bounded linear operators (resp. compact linear operators) from X to Y. If $X = Y$, then we simply write $L(X)$ (resp. $K(X)$).

A Banach space X is said to have a finite dimensional Schauder decomposition (F. D. D. in short) $\{X_n\}_{n=1}^\infty$ if every x in X can be uniquely written as $x = \Sigma x_n$ where each $x_n \in X_n$ and each X_n is a finite dimensional subspace of X. For each n the partial sum projection P_n on X is defined by

$$P_n\left(\sum_{i=1}^{\infty} x_i\right) = \sum_{i=1}^{n} x_i \text{ where } x_i \in X_i.$$

By the uniform boundedness principle we have $\sup_n \|P_n\| < \infty$. A Banach space X with a F. D. D. $\{X_n\}_{n=1}^\infty$ is called the l^p-sum of $\{X_n\}_{n=1}^\infty$ and is written $X = (\Sigma X_n)_p$ if $\|\Sigma x_n\| = (\Sigma \|x_n\|^p)^{1/p}$ for every $x = \Sigma x_n \in X$ with $x_n \in X_n$.

If X is a Banach space, B_X will denote the closed unit ball of X. A Banach space X is said to have the compact approximation property (resp. metric compact approximation property) if the identity operator on X is in the closure of $K(X)$ (resp. $B_{K(X)}$) with respect to the topology of uniform convergence on compact sets in X.

3. ***M*-ideals.** As was mentioned earlier, if X is a Banach space for which $K(X)$ is an M-ideal in $L(X)$, then X has the metric compact approximation property, equivalently $B_{K(X)}$ is dense in $B_{L(X)}$ in the topology of uniform convergence on compact sets in X. For a pair of Banach spaces X and Y we have an analogous conclusion.

THEOREM 1. *If X and Y are banach spaces and $K(X,Y)$ is an M-ideal in $L(X,Y)$, then $B_{K(X,Y)}$ is dense in $B_{L(X,Y)}$ in the topology of uniform convergence on compact sets in X.*

PROOF. Suppose $K(X,Y)$ is an M-ideal in $L(X,Y)$ and suppose $L(X,Y)^* = K(X,Y)^0 \oplus_1 J$ for a subspace J of $L(X,Y)^*$. Then the map $\phi \longrightarrow \phi + K(X,Y)^0$ defines an isometry from J onto $L(X,Y)^*/K(X,Y)^0$ and hence the map $\phi \longrightarrow \phi|_{K(X,Y)}$ defines an isometry from J onto $K(X,Y)^*$ via $L(X,Y)^*/K(X,Y)^0$.

Let Q be the projection from $L(X,Y)^*$ onto J. Then $\phi \in L(X,Y)^*$ is in the range of Q if and only if the restriction of ϕ to $K(X,Y)$ has the same norm as ϕ. If $T \in L(X,Y) \subseteq L(X,Y)^{**}$ with $\|T\| \leqq 1$, then for $\phi = \phi_1 + \phi_2$ in $L(X,Y)^*$ with $\phi_1 \in K(X,Y)^0$ and $\phi_2 \in J$ we have $(Q^*T)\phi = TQ(\phi_1 + \phi_2) = T\phi_2$. Thus $Q^*T \in K(X,Y)^{00} = J^* = K(X,Y)^{**}$.

Since $Q^*T \in K(X,Y)^{**}$ and $\|Q^*T\| \leqq 1$, by the Goldstine's theorem there is a net $\{K_\alpha\}$ in $B_{K(X,Y)}$ such that $K_\alpha \longrightarrow Q^*T$ in the weak*-topology induced by $K(X,Y)^*$. Since for each $x \in X$ and each $y^* \in Y^*$ $y^* \otimes x$ is in the range of Q, we have

$$y^*(K_\alpha x) = K_\alpha(y^* \otimes x) \longrightarrow (Q^*T)(y^* \otimes x) = y^*(Tx).$$

This shows that T is in the closure of $B_{K(X,Y)}$ in the weak operator topology and hence in the strong operator topology. □

The above theorem is essentially due to Werner [15] although he restricted attention to the case $X = Y$ and the identity map on X.

LEMMA 2. *Suppose E is a Banach space which has a F. D. D. $\{X_n\}_{n=1}^\infty$ with the partial sum projections $\{P_n\}_{n=1}^\infty$. Suppose X is a reflexive subspace of E and Y is a Banach space. Then for a given $\epsilon > 0$ and $T \in B_{K(X,Y)}$ there exists a positive integer m such that if $x \in B_X$ and $\|P_m x\| \leqq \epsilon$, then $\|Tx\| \leqq 2\alpha\epsilon$ where $\alpha = \sup_n \|P_n\|$.*

PROOF. If the statement were false, then there would exist a sequence $\{x_k\}$ in B_X such that $\|P_k x_k\| \leqq \epsilon$ and $\|Tx_k\| > 2\alpha\epsilon$. Since B_X is weakly compact, by passing to a subsequence if necessary we may assume $x_k \longrightarrow x$ weakly. Since T and P_j are compact, $P_j x_k \longrightarrow P_j x$ and $Tx_k \longrightarrow Tx$ in norm as $k \longrightarrow \infty$. If $k > j$, $\|P_j x_k\| = \|P_j P_k x_k\| \leqq \alpha\|P_k x_k\| \leqq \alpha\epsilon$. Thus $\|P_j x\| \leqq \alpha\epsilon$ for all j. Since $P_j x \longrightarrow x$, $\|x\| \leqq \alpha\epsilon$ and hence $\|Tx\| \leqq \alpha\epsilon$. This is impossible since $\|Tx_k\| > 2\alpha\epsilon$ and $\|Tx_k\| \longrightarrow \|Tx\|$. □

LEMMA 3. [3]. *Suppose $\{P_n\}_{n=1}^\infty$ is a sequence in $K(X)$ for a Banach space X which converges strongly to the identity map on X and K is a weakly compact subset of X. Then for any $\epsilon > 0$ and a positive integer m there exists $n = n(m, \epsilon) > m$ such that*

$$\sup_{x \in K} \min_{m<k<n} d(P_k x, K) < \epsilon$$

where $d(x, K) = \inf\{\|x - z\|; z \in K\}$.

PROPOSITION 4. *Let X be a separable reflexive Banach space and Y a closed subspace of $Z = (\Sigma Y_n)_q (1 < q < \infty)$. If $K(X,Y)$ is dense in $L(X,Y)$ in the strong operator topology, then for any $T \in B_{L(X,Y)}$ there exist sequences $\{K_n\}_{n=1}^\infty$ in $K(X,Y)$ and $\{R_n\}_{n=1}^\infty$ in $B_{K(X,Z)}$ such that $\|(T - R_n)x\| \leqq \|Tx\|$ for all $x \in X$, $\|R_n - K_n\| \longrightarrow 0$*

and $Q_n(T - R_r) = 0$ for all r and with $r \geqq n$, where $\{Q_n\}_{n=1}^{\infty}$ is the partial sum projections of Z.

PROOF. Let $T \in B_{L(X,Y)}$ and let $\{T_n\}_{n=1}^{\infty}$ be a sequence in $K(X,Y)$ such that $T_n \longrightarrow T$ strongly. Since $Q_nT \longrightarrow T$ strongly, $T - Q_nT \longrightarrow 0$ strongly in $K(X,Z)$ and for some $\alpha > 0$ $\|T_n - Q_nT\| < \alpha$ for all n.

We claim that $T_n - Q_nT \longrightarrow 0$ weakly in $K(X,Z)$. Since B_X with the weak topology and B_{Z^*} with the weak*-topology are compact Hausdorff, the product space $\Omega = B_X \times B_{Z^*}$ is a compact Hausdorff space. Let $C(\Omega)$ be the space of all continuous scalar valued functions on Ω with the supremum norm. To each $S \in K(X,Z)$ we assign a function h_S on Ω defined by $h_S(x,z^*) = z^*(Sx)$ for $(x,z^*) \in \Omega$. Suppose $\{(z_\tau, z_\tau^*)\}$ is a net in Ω converging to (x,z^*). Then

$$|h_S(X_\tau, z_\tau^*) - h_S(x,z^*)| = |z_\tau^*(Sx_\tau) - z^*(Sx)|$$
$$\leqq \|S^*(z_\tau^* - z^*)\| \, \|x_\tau\| + |S^*z^*(x_\tau - x)|.$$

Since S^* is compact and weak*-to-weak continuous, $\|S^*(z^* - z^*)\| \longrightarrow 0$, and since $S^*z^* \in X^*, S^*z^*(x_\tau - x) \longrightarrow 0$. Hence h_S is continuous on Ω. Since $\|S\| = \sup |z^*(Sx)| = \sup |h_S(x,z^*)|$ where the supremum is taken over Ω, $\|S\| = \|h_S\|$ and hence the map $S \longrightarrow h_S$ defines an isometry from $K(X,Z)$ to $C(\Omega)$. Thus by the Hahn-Banach theorem and the Riesz representation theorem for every $\phi \in K(X,Z)^*$ there exists a regular Borel measure μ on Ω such that

$$\phi(S) = \int_\Omega z^*(Sx)d\mu(x,z^*) \text{ for all } S \in K(X,Y).$$

As a sequence in $C(\Omega), T_n - Q_nT \longrightarrow 0$ pointwise on Ω.

By the bounded convergence theorem

$$\phi(T_n - Q_nT) = \int_\Omega z^*(T_n - Q_nT)zd\mu(x,z^*) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus $T_n - Q_nT \longrightarrow 0$ weakly in $K(X,Z)$.

Since $T_n - Q_nT \longrightarrow 0$ weakly in $K(X,Z)$, there exist sequences $\{K_n\}$ in $K(X,Y)$ and $\{R_n\}$ in $B_{K(X,Z)}$ such that

$$K_n = \sum_{k=a_n+1}^{a_{n+1}} \lambda_k T_k, \quad R_n = \sum_{k=a_n+1}^{a_{n+1}} \lambda_k Q_k T$$

and $\|R_n - K_n\| \longrightarrow 0$, where $\lambda_k \geqq 0$,

$$\sum_{k=a_n+1}^{a_{n+1}} \lambda_k = 1$$

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and $\{a_n\}$ is a strictly increasing sequence of positive integers. From the construction of R_n it is obvious that $\|(T-R_n)x\| \leqq \|Tx\|$ for all $x \in X$ and $Q_n(T-R_r)=0$ for all $r>n$. □

Now we are ready to prove the main results. We will use the following characterization of M-ideals due to Lima [7]. A closed subspace J of a Banach space X is an M-ideal in X if and only if for any $\epsilon > 0$, for any $x \in B_X$ and for any $y_i \in B_J (i=1,2,3)$, there exists $y \in J$ such that $\|x+y_i-y\| < 1+\epsilon$ for $i=1,2,3$.

THEOREM 5. *Suppose X and Y are closed subspaces of $(\Sigma X_n)_p$ and $Z=(\Sigma Y_n)_q$, respectively* ($1<p\leqq q<\infty$, $\dim X_n<\infty$, $\dim Y_n<\infty$). *If $K(X,Y)$ is dense in $L(X,Y)$ in the strong operator topology, then $K(X,Y)$ is an M-ideal in $L(X,Y)$.*

PROOF. Let $S_1, S_2, S_3 \in B_{K(X,Y)}$ and $T \in B_{L(X,Y)}$. We will show that for a given $\eta>0$, there exists $K \in K(X,Y)$ such that $\|S_i+T-K\| < 1+\eta\,(i=1,2,3)$. Let $\{P_n\}_{n=1}^\infty$ and $\{Q_n\}_{n=1}^\infty$ be the partial sum projections of $(\Sigma X_n)_p$ and $(\Sigma Y_n)_q$, respectively. Using $\{Q_n\}_{n=1}^\infty$, we choose sequences $\{K_n\}_{n=1}^\infty$ in $K(X,Y)$ and $\{R_n\}_{n=1}^\infty$ in $B_{K(X,Y)}$ as in Proposition 4 so that $\|K_n-R_n\| \longrightarrow 0$, $Q_n(T-R_r)=0$ for $r>n$ and $\|(T-R_n)x\| \leqq \|Tx\|$ for all $x \in X$.

Fix $0<\epsilon<1$. By Lemma 2, Proposition 4, and the compactness of the norm closure of $\bigcup_{i=1}^3 S_i(B_X)$ we can choose m so that

$$\text{(i)} \qquad \|S_i-Q_mS_i\| < \epsilon \text{ for } i=1,2,3, \text{ and } \|R_n-K_n\| < \epsilon \text{ for } n \geqq m$$

$$\text{(ii)} \qquad \text{if } x \in B_X \text{ and } \|P_mx\| \leqq \epsilon \text{ then } \|S_ix\| \leqq 2\epsilon \text{ for } i=1,2,3.$$

By Lemma 3, we choose $N>m$ so that for every $x \in B_X$ there exists $k=k(x)$ $(m \leqq k < N)$ such that $d(P_kx, B_X) < \epsilon$. For $x \in X$ with $\|x\|=1$, let $k=k(x)$ and pick $x_1 \in B_X$ with $\|P_kx-x_1\| \leqq \epsilon$. Set $x_2=x-x_1$. Then we get

$$\text{(iii)} \qquad \|(I-P_k)x_1\| \leqq \epsilon, \|P_kx_2\| \leqq \epsilon \text{ and } \|x_2\| \leqq \|(I-P_k)x\|+\epsilon.$$

Choose $r>N$ so that

$$\text{(iv)} \qquad \|(T-R_r)x\| \leqq 4\epsilon \text{ for every } x \text{ in the set } A=\{x \in X : \|x\| \leqq 1 \text{ and } \|(I-P_N)x\| \leqq \epsilon\}.$$

This is possible since A has a 3ϵ-net, $\|T-R_n\| \leqq 1$ and $T-R_n \longrightarrow 0$ strongly. By (i), we have

$$\|S_i+T-K_r\| < \|Q_mS_i+T-R_r\|+2\epsilon.$$

For $x \in X$ with $\|x\| = 1$, we write $x = x_1 + x_2$ as in (iii). Then for $i = 1, 2, 3$,

$$\begin{aligned}
&\|Q_m S_i x + (T - R_r)x\|^q \\
&= \|Q_m S_i x_1 + Q_m S_i x_2 + (T - R_r)x_1 + (T - R_r)x_2\|^q \\
&< (\|Q_m S_i x_1 + (T - R_r)x_2\| + 4\epsilon + 4\epsilon)^q \text{ by (ii)–(iv)} \\
&= \|Q_m S_i x_1\|^q + \|(T - R_r)x_2\|^q + f(\epsilon) \quad (f(\epsilon) \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0) \text{ by Proposition 4} \\
&\leqq \|x_1\|^q + \|x_2\|^q + f(\epsilon) \\
&\leqq \|x_1\|^p + (1+\epsilon)^{q-p}\|x_2\|^p + f(\epsilon) \text{ since } \|x_1\| \leqq 1 \text{ and } \|x_2\| \leqq 1 + \epsilon \\
&\leqq (\|P_k x\| + \epsilon)^p + (1+\epsilon)^{q-p}(\|(I - P_k)x\| + \epsilon)^p + f(\epsilon) \text{ by (iii) and } \|P_k x - x_1\| \leqq \epsilon \\
&= \|P_k x\|^p + \|(I - P_k)x\|^p + g(\epsilon) \quad (g(\epsilon) \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0) \\
&= 1 + g(\epsilon).
\end{aligned}$$

Thus for $i = 1, 2, 3$,

$$\|S_i + T - K_r\| \leqq (1 + g(\epsilon))^{1/q} + 2\epsilon.$$

Now choose $\epsilon > 0$ so that $(1 + g(\epsilon))^{1/q} + 2\epsilon < 1 + \eta$ and let $K = K_r$. □

COROLLARY 6. *Suppose X and Y are as in Theorem 5. If either X of Y has the compact approximation property, then $K(X, Y)$ is an M-ideal in $L(X, Y)$.*

PROOF. Suppose Y has the compact approximation property. Let $T \in L(X, Y)$. If K is a compact subset of X, then $T(K)$ is a compact subset of Y. Hence for any $\epsilon > 0$, there exists a compact operator S from Y to Y such that $\|Sy - y\| < \epsilon$ for all $y \in T(K)$. Thus $\|STx - Tx\| < \epsilon$ for all $x \in K$. Since $ST \in K(X, Y)$, $K(X, Y)$ is dense in $L(X, Y)$ in the strong operator topology. By Theorem 5, $K(X, Y)$ is an M-ideal in $L(X, Y)$. The proof of the other case is similar. □

Cho and Johnson [3] proved that if X is a separable reflexive Banach space which has the compact approximation property, then X has the metric compact approximation property. Theorem 5 gives a short proof of this for closed subspace X of $(\Sigma X_n)_p$ $(1 < p < \infty, \dim X_n < \infty)$.

COROLLARY 7. *If X is a closed subspace of $(\Sigma X_n)_p$ $(1 < p < \infty$, $\dim X_n < \infty)$ which has the compact approximation property, then X has the metric compact approximation property.*

PROOF. By Theorem 5, $K(X)$ is an M-ideal in $L(X)$ and hence satisfies the metric compact approximation property. □

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