This leads to

$$\frac{f'(x)}{f(x)} = \frac{16x^5}{11x^6 - 30x^4 + 20x^2 - 8} = \frac{2x}{x^2 - 2} + \frac{4x - 6x^3}{11x^4 - 8x^2 + 4},$$

which integrates to give

$$f(x) = \frac{x^2 - 2}{(11x^4 - 8x^2 + 4)^{3/22}} \cdot \exp\left[\frac{5}{11\sqrt{7}} \tan^{-1}\left(\frac{11x^2 - 4}{2\sqrt{7}}\right)\right].$$

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A cautionary tale about the pole of polar coordinates

The pole of polar coordinates, given by the point $r = 0$ is usually said to have an undefined argument (just as with the complex number $z = 0$). But, as the following cautionary example shows, this is arguably not the whole story.

The Figure shows the graphs of two curves C_1 and C_2 with respective polar equations $r_1 = 1 + \cos\theta$ and $r_1 = 1 + 2\cos\theta$ for $-\pi < \theta \leqslant \pi$.

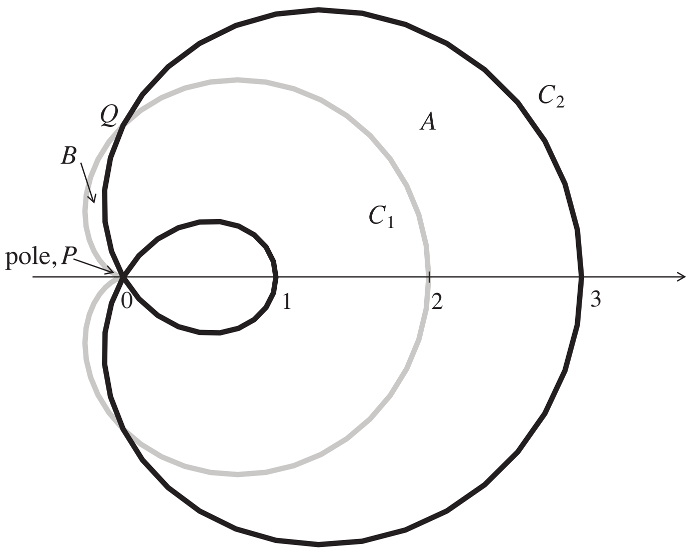

FIGURE: The inner loop of C_2 (which some authors would show dotted) is not involved in this Note.

The point Q is given by $1 + \cos\theta = 1 + 2\cos\theta$, which solves to give $\theta = \frac{1}{2}\pi$, so Q has polar coordinates $\left(1, \frac{1}{2}\pi\right)$. The area of the region labelled A is thus

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$$\frac{1}{2}\int_0^{\pi/2} r_2^2\, d\theta - \frac{1}{2}\int_0^{\pi/2} r_1^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \left(r_2^2 - r_1^2\right) d\theta = \frac{1}{2}\int_0^{\pi/2} \left(2\cos\theta + 3\cos^2\theta\right) d\theta = \frac{3}{8}\pi + 1.$$

Now consider the area of region B. One endpoint is Q with polar coordinates $\left(1, \frac{1}{2}\pi\right)$. The other is given by $r_1 = 0 = r_2$. The left-hand side solves to give $\theta = \pi$ and the right-hand side gives $\theta = \frac{2}{3}\pi$: if you like, the pole P has rival polar coordinates $(0, \pi)$ and $(0, \frac{2}{3}\pi)$ on the two curves. The area of B is then given by two separate integrals that cannot be combined:

$$\frac{1}{2}\int_{\pi/2}^{\pi} r_1^2\, d\theta - \frac{1}{2}\int_{\pi/2}^{2\pi/3} r_2^2\, d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta - \frac{1}{2}\int_{\pi/2}^{2\pi/3} \left(1 + 4\cos\theta + 4\cos^2\theta\right) d\theta$$

$$= \left(\frac{3\pi}{8} - 1\right) - \left(\frac{\pi}{4} + \frac{3\sqrt{3}}{4} - 2\right) = \frac{\pi}{8} - \frac{3\sqrt{3}}{4} + 1.$$

The ambiguity of endpoints exhibited at P can only occur when the curves meet at the pole. This example is a fine one for class discussion and similar ones are also worth investigating: for example, the region corresponding to B for the curves $r_1 = 1 + \sqrt{2}\cos\theta$ and $r_2 = 1 + 2\cos\theta$ has the "π-less" area $\frac{11}{4} - \sqrt{2} - \frac{3\sqrt{3}}{4}$.

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Another appearance of the golden ratio

Professor Anne Watson in her inspirational plenary talk "What school mathematics can be … really" at the 2022 MA Conference mentioned the following problem as a rich one for a discussion of problem-solving strategies and approaches.

Find the area of right-angled triangle ABC, situated in a quadrant of the unit circle as in Figure 1(a).

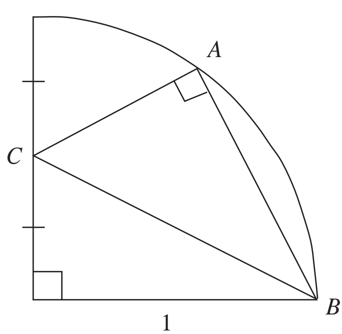

FIGURE 1(a)

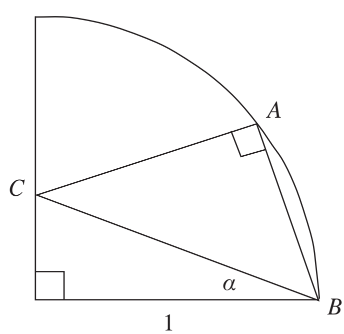

FIGURE 1(b)