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ON DEFECT RELATIONS OF MOVING HYPERPLANES

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§ 1. Introduction

The defect relation $\sum_{j=1}^{q} \delta(f, H_j) \leq n + 1$ gives the best-possible estimate, where f is a linearly non-degenerate holomorphic curve in $P^n(C)$ and $H_1, \cdots, H_q$ are hyperplanes in $P^n(C)$ which are in general position. However, the case of moving hyperplanes has ever got only $n(n + 1)$ instead of $n + 1$ (Stoll [4]) and it has not yet been known whether this bound is best-possible or not. In this paper we shall give some particular cases which have the bound $n + 1$.

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§ 2. Holomorphic curves and moving hyperplanes

In this paper, we fix one homogeneous coordinate system of the n-dimensional complex projective space $P^n(C)$ and denote it by the notation $w = (w_0 : \cdots : w_n)$.

A hyperplane H in $P^n(C)$ is an $(n - 1)$-dimensional projective subspace of $P^n(C)$, i.e., it is given by $H = \{w \in P^n(C) \mid \sum_{j=0}^{n} a_j w_j = 0\}$, where $(a_0, \cdots, a_n) \in C^{n+1} - \{0\}$. We call the vector $(a_0, \cdots, a_n)$ a representation of H. Let H_j be hyperplanes in $P^n(C)$ with representations $a^j = (a_0^j, \cdots, a_n^j)$ $(j = 1, \cdots, q)$. If any $\min(q, n + 1)$ elements of $a^1, \cdots, a^q$ are linearly independent over C, $H_1, \cdots, H_q$ are said to be in general position.

We call a holomorphic mapping $f: C \to P^n(C)$ a holomorphic curve in $P^n(C)$. A representation of f is a holomorphic mapping $\tilde{f} = (f_0, \cdots, f_n)$: $C \to C^{n+1}$ which satisfies $\tilde{f}^{-1}(0) \neq C$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ for all $z \in C - \tilde{f}^{-1}(0)$. Then we write $f = (f_0 : \cdots : f_n)$. If $\tilde{f}^{-1}(0) = \varnothing$, then the representation $\tilde{f}$ is said to be reduced.

DEFINITION 2.1. A moving hyperplane H^M in $P^n(C)$ is a mapping of

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C into the set of all hyperplanes in $P^n(C)$ given by $H^M(z) = \{w \in P^n(C) | \sum_{j=0}^n a_j(z)w_j = 0\}$ $(z \in C)$, where $(a_0, \cdots, a_n)$ is a reduced representation of some holomorphic curve g in $P^n(C)$. We call a representation and a reduced representation of g a representation and a reduced representation of H^M, respectively.

DEFINITION 2.2. Let H_j^M be moving hyperplanes in $P^n(C)$ $(j = 1, \cdots, q)$. $H_1^M, \cdots, H_q^M$ are said to be in general position if there exists a point z_0 of C such that hyperplanes $H_1^M(z_0), \cdots, H_q^M(z_0)$ in $P^n(C)$ are in general position.

DEFINITION 2.3. Let f be a holomorphic curve in $P^n(C)$ with a representation $(f_0, \cdots, f_n)$ and let K be an extension field of C. We say that f is non-degenerate over K if $f_0, \cdots, f_n$ are linearly independent over K. In particular, f is said to be linearly non-degenerate if it is non-degenerate over C.

§ 3. Characteristic functions, counting functions and defects

We define the norm $\|z\|$ of $z = (z_1, \cdots, z_m) \in C^m$ by $\|z\|^2 = \sum_{j=1}^m |z_j|^2$.

DEFINITION 3.1. The characteristic function of a holomorphic curve f in $P^n(C)$ with a reduced representation $\tilde{f}$ is defined for $0 < s < r$ by

$$T(f; r, s) = \frac{i}{2\pi} \int_s^r \frac{dt}{t} \int_{|z| \le t} \partial\bar{\partial} \log \|\tilde{f}\|^2 .$$

This definition does not depend on the choice of $\tilde{f}$. We see that $T(f; r, s)$ is non-negative and that if f is non-constants, then $T(f; r, s) \to \infty$ monotonically as $r \to \infty$. Furthermore we can easily verify that

$$T(f; r, s) = \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{f}(re^{i\theta})\| \, d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{f}(se^{i\theta})\| \, d\theta . \tag{3.2}$$

DEFINITION 3.3. The counting function of zeros for a meromorphic function $F \not\equiv 0$ on C is defined for $0 < s < r$ by

$$N(F; r, s) = \int_s^r n(F; t) \frac{dt}{t} ,$$

where $n(F; t)$ is the sum of zero orders of F in $\{z \in C | |z| \le t\}$.

By the definition, $N(F; r, s)$ is non-negative, and Jensen's formula shows that

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(3.4) $$N(F; r, s) - N(1/F; r, s) = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\,d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|F(se^{i\theta})|\,d\theta .$$

In the situation of Definition 2.1, we define the characteristic function of H^M by $T(H^M; r, s) := T(g; r, s)$. And we define the counting function of H^M for a holomorphic curve f by $N(f, H^M; r, s) := N((\tilde{f}, \tilde{g}); r, s)$, where $\tilde{f} = (f_0, \cdots, f_n)$ and $\tilde{g} = (a_0, \cdots, a_n)$ are reduced representations of f and g, respectively, and $(\tilde{f}, \tilde{g}) := \sum_{j=0}^{n} a_j f_j$, if $(\tilde{f}, \tilde{g}) \not\equiv 0$. This assumption holds if f is non-degenerate over a field containing all a_j/a_k with $a_k \not\equiv 0$. This definition does not depend on the choice of $\tilde{f}$ and $\tilde{g}$. By (3.2), (3.4) and Schwarz's inequality, we get

(3.5) $$N(f, H^M; r, s) \leq T(f; r, s) + T(H^M; r, s) + O(1), \quad r \longrightarrow \infty .$$

If either f or g is not constant, the defect of H^M for f is defined by

$$\delta(f, H^M) = \liminf_{r\to\infty}\left(1 - \frac{N(f, H^M; r, s)}{T(f; r, s) + T(H^M; r, s)}\right)$$

which does not depend on s. By (3.5), we see $0 \leq \delta(f, H^M) \leq 1$. The moving hyperplane H^M is said to be of lower order than f if $T(H^M; r, s) = o(T(f; r, s))$ as $r \to \infty$. Then

$$\delta(f, H^M) = \liminf_{r\to\infty}\left(1 - \frac{N(f, H^M; r, s)}{T(f; r, s)}\right).$$

The definitions of counting functions and defects of (not-moving) hyperplanes are the same as those of moving hyperplanes. However, for convenience sake, we consider that the category of moving hyperplanes contains not-moving hyperplanes.

Let f be a holomorphic curve in $P^n(C)$. We denote by K_f the set of all meromorphic functions g which satisfy the condition that $T(g; r, s) = o(T(f; r, s))$ as $r \to \infty$. If a representation $(f_0, \cdots, f_n)$ satisfies that $f \not\equiv 0$ for each j and that each f_j/f_k $(j \neq k)$ is not constant, then we set $\tilde{K}_f = \bigcap_{j\neq k} K_{f_j/f_k}$. Now, we present two lemmas without proofs.

LEMMA 3.6 ([4, Lemma 5.3]). *The sets K_f and $\tilde{K}_f$ are fields.*

LEMMA 3.7 ([1, Proposition 5.9]). *A holomorphic curve $f = (f_0 : \cdots : f_n)$ in $P^n(C)$ is rational, i.e., all f_j/f_k with $f_k \not\equiv 0$ are rational if and only if*

$$T(f; r, s) = O(\log r) \qquad \text{as } r \longrightarrow \infty .$$

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PROPOSITION 3.8. *Let f be a non-constant holomorphic curve and let g be a holomorphic curve in $P^n(C)$ with a reduced representation $(g_0, \cdots, g_n)$. Assume that $g_j/g_k \in K_f$ if $g_k \not\equiv 0$. Then, $T(g; r, s) = o(T(f; r, s))$ as $r \to \infty$.*

Proof. Without loss of generality, we may assume that $g_0 \not\equiv 0$. Since the representation $(g_0, \cdots, g_n)$ is reduced, for each point p where g_0 vanishes there is some g_j with $g_j(p) \neq 0$. Hence, we have

$$\begin{aligned} N(g_0; r, s) &\leq \textstyle\sum_{j=1}^{n} N(g_j/g_0, \infty; r, s) \\ &\leq \textstyle\sum_{j=1}^{n} T(g_j/g_0; r, s) + O(1) \\ &= o(T(f; r, s)) \qquad (r \longrightarrow \infty) \end{aligned}$$

and

$$\begin{aligned} T(g; r, s) &= \frac{1}{4\pi}\int_0^{2\pi} \log (1 + \textstyle\sum_{j=1}^{n} |g_j(re^{i\theta})/g_0(re^{i\theta})|^2) d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \log |g_0(re^{i\theta})| d\theta + O(1) \\ &\leq \textstyle\sum_{j=1}^{n} T(g_j/g_0; r, s) + N(g; r, s) + O(1) \\ &= o(T(f; r, s)) \qquad (r \longrightarrow \infty) . \end{aligned}$$

Q.E.D.

In this paper, we treat non-rational holomorphic curves f and we use a notation $S(f, r)$ for representing a quantity with a property that

$$\lim_{r\to\infty, r \notin E} S(f; r)/T(f; r, s) = 0$$

for a set $E \subset (0, \infty)$ of finite Lebesgue measure.

§ 4. Defect relations

First, we give the known defect relations.

THEOREM 4.1 (See, for example, [3, Chapter 3]). *Let f be a linearly non-degenerate holomorphic curve in $P^n(C)$ and let $H_1, \cdots, H_q$ be hyperplanes in $P^n(C)$ which are in general position. Then*

$$\textstyle\sum_{j=1}^{q} \delta(f, H_j) \leq n + 1 .$$

THEOREM 4.2 ([4, Theorem 6.19]). *Let f be a holomorphic curve in $P^n(C)$ and let $H_1^M, \cdots, H_q^M$ be moving hyperplanes in $P^n(C)$ with lower orders than f which are in general position. Let $(a_0^j, \cdots, a_n^j)$ be reduced representations of H_j^M $(j = 1, \cdots, q)$ and K be the smallest extension field* *of* C *which contains all* a_k^j/a_m^j $(1 \leq j \leq q,\ 0 \leq k \leq n,$ *and* $m \in \{k \mid a_k^j \not\equiv 0\})$. *Assume that* f *is non-degenerate over* K. *Then*

$$\sum_{j=1}^{q} \delta(f, H_j^M) \leq n(n+1)\,.$$

THEOREM 4.3 (2, Theorem 3.4]). *Let* f *be a holomorphic curve in* $P^n(C)$ *and let* $H_0^M, \cdots, H_{n+1}^M$ *are moving hyperplanes in* $P^n(C)$ *with lower orders than* f *which are in general position. Let* $(a_0^j, \cdots, a_n^j)$ *be reduced representations of* H_j^M $(j = 0, \cdots, n+1)$ *and let* K *be the smallest extension field of* C *which contains all* a_k^j/a_m^j $(0 \leq j,\ k \leq n$ *and* $m \in \{k \mid a_k^j \not\equiv 0\})$. *Assume that* f *is non-degenerate over* K. *Then*

$$\sum_{j=0}^{n+1} \delta(f, H_j^M) \leq n+1\,.$$

The main purpose of this paper is to prove the following:

THEOREM 4.4. *Let* f *be a linearly non-degenerate holomorphic curve in* $P^n(C)$ *with a reduced representation* $(f_0, \cdots, f_n)$ *and let* $H_1^M, \cdots, H_q^M$ *be moving hyperplanes in general position in* $P^n(C)$. *Let* $(a_0^j, \cdots, a_n^j)$ *be reduced representations of* H_j^M $(1 \leq j \leq q)$. *Assume that the following three conditions are satisfied*:

(C1) $a_k^j/a_m^j \in \tilde{K}_f$ *if* $a_m^j \not\equiv 0$;

(C2) f *is non-degenerate over* $\tilde{K}_f$;

(C3) $N(f_j; r, s) = S(f; r)$ $(j = 0, \cdots, n)$.

Then

$$\sum_{j=1}^{q} \delta(f, H_j^M) \leq n+1\,.$$

§ 5. Second main theorems

The next second main theorem is well-known and Theorem 4.1 is its corollary.

THEOREM 5.1 (See, for example, [3, Chapter 3]). *In the same situation of Theorem* 4.1, *the inequality*

$$(q-n-1)T(f; r, s) \leq \sum_{j=1}^{q} N(f, H_j; r, s) + S(f; r) \tag{5.2}$$

holds for $0 < s < r$.

The next lemma will be proved by the same method of Theorem 4.3. For a proof, see [5, pp. 313–333].

LEMMA 5.3. *In the same situation of Theorem* 4.3, *the inequality*

$$T(f; r, s) \leq \sum_{j=0}^{n+1} N(f, H_j^M; r, s) + S(f; r) \tag{5.4}$$

holds for $0 < s < r$.

§6. Proof of Theorem 4.4

Before begining to prove Theorem 4.4, we show the following lemma.

LEMMA 6.1. *Let* f *be as in Theorem* 4.4. *Let* H^M *be a moving hyperplane in* $P^n(C)$ *with a reduced representation* $(a_0, \cdots, a_n)$. *Assume that* $a_j/a_k \in \tilde{K}_f$ *if* $a_k \not\equiv 0$. *If* $a_{j_0} \not\equiv 0, \cdots, a_{j_k} \not\equiv 0$ *and* $a_j \equiv 0$ *for* $j \neq j_0, \cdots, j_k$, *we give a hyperplane* $H = \{w \in P^n(C) \mid w_{j_0} + \cdots + w_{j_k} = 0\}$ *in* $P^n(C)$. *Then*

$$N(f, H; r, s) = N(f, H^M; r, s) + S(f; r)\,. \tag{6.2}$$

Proof. For simplicity, we may assume that $j_0 = 0, \cdots, j_k = k$. In the case of $k = 0$, the conclusion is evident since $N(f_0; r, s) = o(T(f; r, s))$ $(r \to \infty)$ by Proposition 3.8. Hence we assume that $k \geq 1$.

Let $h := (f_0 : \cdots : f_k)$ be a holomorphic curve in $P^k(C)$ and let L^M be a moving hyperplane in $P^k(C)$ with a reduced representation $(a_0, \cdots, a_k)$. Furthermore, we consider the hyperplanes $L_j := \{w \in P^k(C) \mid w_j = 0\}$ $(j = 0, \cdots, k)$ and $L := \{w \in P^k(C) \mid \sum_{j=0}^k w_j = 0\}$ in $P^k(C)$. Note that L^M has a lower order than h. We get by Theorem 5.1 and Lemma 5.3.

$$T(h; r, s) \leq N(h, L^M; r, s) + S(f; r)$$

and

$$T(h; r, s) \leq N(h, L; r, s) + S(f; r)\,.$$

Here we used the fact $N(h, L_j; r, s) = S(f; r)$ $(j = 0, \cdots, k)$. By (3.5) and the above inequalities, we have $T(h; r, s) = N(h, L; r, s) + S(f; r)$ and $T(h; r, s) = N(h, L^M; r, s) + S(f; r)$. Since $N(h, L; r, s) + o(T(f; r, s)) = N(f, H; r, s)$ and $N(h, L^M; r, s) = N(f, H^M; r, s) + o(T(f; r, s))$ $(r \to \infty)$, we obtain (6.2). Q.E.D.

Proof of Theorem 4.4. There exists a point z_0 of C such that $a_k^j(z_0) \neq 0$ if $a_k^j \not\equiv 0$ and that $H_1^M(z_0), \cdots, H_q^M(z_0)$ are in general positions. Then by Lemma 6.1, we have

$$N(f, H_j^M(z_0); r, s) = N(f, H_j^M; r, s) + S(f; r) \qquad (j = 1, \cdots, q)\,.$$

On the other hand, we have by Theorem 5.1,

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$$(q-n-1)T(f;r,s) \leq \sum_{j=1}^{q} N(f, H_j^M(z_0); r, s) + S(f;r)\,.$$

Hence we obtain

$$(q-n-1)T(f;r,s) \leq \sum_{j=1}^{q} N(f, H_j^M; r, s) + S(f;r)\,.$$

Therefore we have the defect relation

$$\sum_{j=1}^{q} \delta(f, H_j^M) \leq n+1\,. \qquad \text{Q.E.D.}$$

§ 7. Further result

In this section, we give a generalization of Theorem 4.4.

Before stating it, we show next lemmas.

LEMMA 7.1. *Let g be a linearly non-degenerate holomorphic curve in $P^m(C)$ with a reduced representation $(g_0, \cdots, g_m)$. Assume that $N(g_j; r, s) = S(g_k/g_l)$ for any distinct k and l. Then g is non-degenerate over $\tilde{K}_g$.*

Proof. Assume that $g_0, \cdots, g_m$ are linearly dependent over $\tilde{K}_g$. So there exist $a_0, \cdots, a_m \in \tilde{K}_g$ such that some $a_j \not\equiv 0$ and that $a_0 g_0 + \cdots + a_m g_m \equiv 0$. Without loss of generality, we may assume that $a_j \not\equiv 0$ $(0 \leq j \leq k+1)$ and $a_j \equiv 0$ $(k+2 \leq j \leq m)$, where $k+1 \leq m$, and that $g_0, \cdots, g_{k+1}$ are linearly independent over $\tilde{K}_g$. If $k=0$, we can immediately lead a contradiction. So, let $k \geq 1$.

Consider the holomorphic curve $h = (g_0 : \cdots : g_k)$ in $P^k(C)$ and moving hyperplanes

$$H_j^M(z) = \{w \in P^k(C) \mid w_j = 0\} \qquad (0 \leq j \leq k)$$

and H_{k+1}^M with a representation $(a_0, \cdots, a_k)$ in $P^k(C)$. They are in general position and of lower order than h. By the assumption and the relation $a_0 g_0 + \cdots + a_k g_k = -a_{k+1} g_{k+1}$, we see that $\delta(g, H_j^M) = 1$ $(0 \leq j \leq k+1)$. This contradicts to Theorem 4.3. Hence we complete the proof of this lemma. Q.E.D.

LEMMA 7.2. *Let f be a linearly non-degenerate holomorphic curve in $P^n(C)$ with a reduced representation $\tilde{f} = (f_0, \cdots, f_n)$ and let g be a linearly non-degenerate holomorphic curve in $P^m(C)$ with a reduced representation $\tilde{g} = (g_0, \cdots, g_m)$. Assume that there are relations*

$$f_j = \sum_{k=0}^{m} a_k^j g_k, \qquad a_k^j \in C \ (0 \leq j \leq n) \tag{7.3}$$

and that for each $k = 0, \cdots, m$, there is a $j(k)$ such that $a_k^{j(k)} \neq 0$. Moreover, if $N(g_j; r, s) = S(g; r)$ for $j = 0, \cdots, m$, then

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$$T(g; r, s) = T(f; r, s) + S(g; r).$$

Proof. By (7.3), we have the inequality $\|\tilde{f}\| \leq C\|\tilde{g}\|$ for some $C > 0$. Therefore we get

$$T(f; r, s) \leq T(g; r, s) + O(1). \tag{7.4}$$

Now, we can choose $b_0, \cdots, b_n \in \boldsymbol{C}$ such that $c_k := \sum_{j=0}^{n} a_k^j b_j \neq 0$. Consider hyperplanes

$$H = \{w \in P^n(\boldsymbol{C}) \mid \sum_{j=0}^{n} b_j w_j = 0\}$$

in $P^n(\boldsymbol{C})$ and

$$L_k = \{w \in P^m(\boldsymbol{C}) \mid w_k = 0\} \qquad (0 \leq k \leq m),$$
$$L = \{w \in P^m(\boldsymbol{C}) \mid \sum_{k=0}^{m} c_k w_k = 0\}$$

in $P^m(\boldsymbol{C})$. Then by Theorem 5.1, we have

$$T(g; r, s) \leq \sum_{k=0}^{m} N(g, L_k; r, s) + N(g, L; r, s) + S(g; r)$$
$$= N(g, L; r, s) + S(g; r).$$

Since $\sum_{k=0}^{m} c_k g_k = \sum_{j=0}^{n} b_j f_j$, we have

$$N(g, L; r, s) = N(f, H; r, s).$$

Hence, we get by (3.5)

$$T(g; r, s) \leq N(f, H; r, s) + S(g; r)$$
$$\leq T(f; r, s) + S(g; r).$$

Consequently, by (7.4), we obtain

$$T(g; r, s) = T(f; r, s) + S(g; r). \qquad \text{Q.E.D.}$$

The generalization of Theorem 4.4 is the following:

THEOREM 7.5. *Let f be a linearly non-degenerate holomorphic curve in $P^n(\boldsymbol{C})$ with a reduced representation $\tilde{f} = (f_0, \cdots, f_n)$ given by $f_j = \sum_{k=1}^{m_j} f_k^j$, where $f_1^j, \cdots, f_{m_j}^j$ are entire functions which are linearly independent over $\boldsymbol{C}$ $(j = 0, \cdots, n)$. Let H_j^M be as in Theorem 4.4. Assume that f is non-degenerate over $\tilde{K}_f$ and that $N(f_k^j; r, s) = S(f_k^j/f_m^l; r)$ if f_k^j/f_m^l is not constant. Then*

$$\sum_{j=1}^{q} \delta(f, H_j^M) \leq n + 1.$$

Proof. Choose $g_0, \cdots, g_m$ from f_k^j $(1 \leq k \leq m_j,\ 0 \leq j \leq n)$ such that

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$\{g_0, \cdots, g_m\}$ is a base of the vector space over $\boldsymbol{C}$ spanned by f_k^j $(1 \leq k \leq m_j,\ 0 \leq j \leq n)$. Let g be a holomorphic curve in $P^n(\boldsymbol{C})$ with a reduced representation $\tilde{g} = (g_0/h, \cdots, g_m/h)$, where h is an entire function such that $g_0/h, \cdots, g_m/h$ are entire functions without common zero. By Lemma 7.1, g is non-degenerate over $\tilde{K}_g$. It is easy to check that $\tilde{K}_f \subset \tilde{K}_g$.

We define entire functions b_k^j $(1 \leq j \leq q,\ 0 \leq k \leq m)$ by the equations

$$a_0^j f_0 + \cdots + a_n^j f_n = b_0^j g_0 + \cdots + b_m^j g_m \not\equiv 0 \qquad (1 \leq j \leq q)\,.$$

Since b_k^j are linear combinations of $a_0^j, \cdots, a_n^j$ with complex coefficients, we see that $b_l^j/b_k^j \in \tilde{K}_g$ if $b_k^j \not\equiv 0$. Let d_j be a common factor of $a_0^j, \cdots, a_m^j$ and let L_j^M be a moving hyperplane in $P^m(\boldsymbol{C})$ with a reduced representation $b_j = (b_0^j/d_j, \cdots, b_m^j/d_j)$. Set $a_j = (a_0^j, \cdots, a_n^j)$. Then $(\tilde{f}, a_j) = hd_j(\tilde{g}, b_j)$. Hence we have

$$N(f, H_j^M; r, s) = N(g, L_j^M; r, s) + N(hd_j; r, s)\,. \tag{7.6}$$

We choose z_0 of $\boldsymbol{C}$ such that $b_k^j(z_0) \neq 0$ if $b_k^j \not\equiv 0$ and $H_1^M(z_0), \cdots, H_q^M(z_0)$ are in general position. Then by Lemma 6.1, we get

$$N(g, L_j^M(z_0); r, s) = N(g, L_j^M; r, s) + S(g; r)\,. \tag{7.7}$$

Furthermore we have

$$N(g, L_j^M(z_0); r, s) + N(h; r, s) = N(f, H_j^M(z_0); r, s) \tag{7.8}$$

by $(\tilde{f}, a_j(z_0)) = hd_j(z_0)(\tilde{g}, b_j(z_0))$. Since $N(d_j; r, s) = o(T(f; r, s))$ by Proposition 3.8, $N(h; r, s)$ is $S(f; r)$ and $S(g; r)$ is $S(f; r)$ by Lemma 7.2, we obtain

$$N(f, H_j^M(z_0); r, s) = N(f, H_j^M; r, s) + S(f; r)$$

by (7.6), (7.7) and (7.8). Hence using Theorem 5.1, we have

$$\begin{aligned}(q - n - 1)T(f; r, s) &\leq \textstyle\sum_{j=1}^q N(f, H_j^M(z_0); r, s) + S(f; r)\\ &\leq \textstyle\sum_{j=1}^q N(f, H_j^M; r, s) + S(f; r)\,.\end{aligned}$$

Therefore we obtain the defect relation

$$\textstyle\sum_{j=1}^q \delta(f, H_j^M) \leq n + 1\,. \qquad \text{Q.E.D.}$$

The most typical case of Theorem 4.4 is that $f_j = \exp h_j$, where h_j are entire functions, and a_k^j are polynomials.

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