Proceedings of the Edinburgh Mathematical Society: page 1 of 15
doi:10.1017/S0013091525000094

ON GROWTH FUNCTIONS OF COXETER GROUPS

SEBASTIAN BISCHOF

Mathematisches Institut, Justus-Liebig-Universität Gießen, Arndtstraße 2, 35392 Gießen, Germany

Email: sebastian.bischof@math.uni-paderborn.de

(Received 5 March 2024)

Abstract Let (W, S) be a Coxeter system of rank n, and let $p_{(W,S)}(t)$ be its growth function. It is known that $p_{(W,S)}(q^{-1}) < \infty$ holds for all $n \leq q \in \mathbb{N}$. In this paper, we will show that this still holds for $q = n - 1$, if (W, S) is 2-spherical. Moreover, we will prove that $p_{(W,S)}(q^{-1}) = \infty$ holds for $q = n - 2$, if the Coxeter diagram of (W, S) is the complete graph. These two results provide a complete characterization of the finiteness of the growth function in the case of 2-spherical Coxeter systems with a complete Coxeter diagram.

Keywords: Coxeter groups; growth function; Poincaré series

2020 Mathematics subject classification: 20F55; 51F15

1. Introduction

One of the most central results in the theory of lattices is Margulis' Normal Subgroup Theorem for irreducible lattices in connected semi-simple Lie groups of real rank ≥ 2 with a finite centre and no non-trivial compact factor [16]. Among all the recent generalizations, let us mention that Bader and Shalom proved a version of the Normal Subgroup Theorem for irreducible cocompact lattices in a product of two locally compact, non-discrete, compactly generated groups [3]. Based on earlier results in [18], Caprace and Rémy applied the Normal Subgroup Theorem to show simplicity for Kac–Moody groups over finite fields of irreducible, non-spherical and non-affine types that are twin building lattices (cf. [10, Theorems 18, 19, 20]). Moreover, it can be used to prove virtual simplicity of certain twin tree lattices with non-trivial commutation relations (cf. [11]).

In [17] and [12], Rémy, and independently Carbone and Garland, proved that certain groups acting on (twin) buildings are lattices. To be more precise: Let (W, S) be a Coxeter system with $|S| < \infty$, and let $\Phi := \Phi(W, S)$ be its associated set of roots (viewed as half-spaces). Let $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD system of type (W, S), i.e. a group G together with a family $(U_\alpha)_{\alpha\in\Phi}$ of subgroups (which we call *root groups*) indexed by the set of roots Φ satisfying some combinatorial axioms (for the precise definition, we refer to [1, Ch. 7,8]). Then, there exists a *twin building* $\Delta = (\Delta_+, \Delta_-, \delta_*)$ such that G acts on Δ. It turns out that under some conditions, $G^\dagger := \langle U_\alpha \mid \alpha \in \Phi \rangle \leq \mathrm{Aut}(\Delta_+) \times \mathrm{Aut}(\Delta_-)$ and $U_+ := \langle U_\alpha \mid \alpha \in \Phi_+ \rangle \leq \mathrm{Aut}(\Delta_-)$ are lattices (cf. [17], [12]) – and in this case $G^\dagger$ is an example of a twin building lattice. Sufficient conditions are that every root group is finite, W is infinite and for $q_{\min} := \min\{|U_\alpha| \mid \alpha \in \Phi\}$, one has $p_{(W,S)}\left(\frac{1}{q_{\min}}\right) < \infty$, where $p_{(W,S)}(t)$ denotes the *growth function* of (W, S). Some authors call $p_{(W,S)}(t)$ the *(spherical) growth series* (cf. [13, Chapter 17] or [14, Chapter VI]) or the *Poincaré series* of (W, S) (cf. [6, Chapter 7.1]). It is clear that $p_{(W,S)}\left(\frac{1}{q_{\min}}\right) < \infty$ holds if $|S| \leq q_{\min}$. It is particularly unsatisfying that the criterion $|S| \leq q_{\min}$ does not apply to Coxeter systems of rank $n \geq 3$ and $q_{\min} = 2$. However, there are examples of Coxeter systems (W, S) of rank $n \geq 3$ with $p_{(W,S)}\left(\frac{1}{2}\right) < \infty$. Note that the growth function $p_{(W,S)}(t)$ applied to q^{-1} with $q \in \mathbb{N}$ and $q \geq 2$ is finite for spherical and affine Coxeter systems (cf. [7, Ch. VI, Exercises § 4, 10]).

Suppose (W, S) is of type $(4, 4, 4)$, that is, $|S| = 3$ and the order of st in W equals 4 for all $s \neq t \in S$. In [5] we constructed uncountably many new examples of RGD systems of type $(4, 4, 4)$ in which every root group has cardinality 2. As the criterion $|S| \leq q_{\min}$ does not apply to such RGD systems, we first asked the question whether $p_{(W,S)}\left(\frac{1}{2}\right) < \infty$ holds. It turns out that this is indeed the case (cf. Theorem A).

Main results

Let (W, S) be a Coxeter system and denote by m_{st} the order of st in W. The Coxeter system is called 2-*spherical* if $m_{st} < \infty$ for all $s \neq t \in S$. The *rank* of (W, S) is given by the cardinality of S. Throughout this paper we assume that all Coxeter systems under consideration are of finite rank. We prove the following (cf. Theorem 5.3):

Theorem A. *Let* (W, S) *be a 2-spherical Coxeter system of rank* n. *Then* $p_{(W,S)}\left(\frac{1}{n-1}\right) < \infty$.

Remark 1. After completion of this project, I was informed by Corentin Bodart that a more general version of Theorem A can be deduced from [2, Theorem 1] and we refer to Remark 3 at the end of the introduction for more details. Our methods of the proof are very different, and most of the results proved in the present paper are also used to prove Theorem C below. Our proofs are Coxeter group theoretic, while the proofs in [2] are for non-elementary word hyperbolic groups.

In view of the examples constructed in [5], Theorem A produces many new examples of lattices in (locally compact) automorphism groups of buildings and in a product of two automorphism groups of buildings. Combining Theorem A with [17, Théorème 1], we obtain that almost all RGD systems of 2-spherical type and rank 3 are twin building lattices:

Corollary B. *Let* (W, S) *be a Coxeter system, and let* $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ *be an RGD system of type* (W, S). *Assume that the following are satisfied:*

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

- *(*W,S*) is* 2*-spherical of rank* 3 *and* W *is infinite.*
- $G = \langle U_\alpha \mid \alpha \in \Phi\rangle$, $|Z(G)| < \infty$ *and* $|U_\alpha| < \infty$ *for all* $\alpha \in \Phi$.

Then, $\mathcal{D}$ *is a twin building lattice.*

Corollary B. (Kac–Moody version). *Let (*W,S*) be a* 2*-spherical Coxeter system of rank* 3 *such that* W *is infinite, and let* $\boldsymbol{G}$ *be the Kac–Moody group (in the sense of [21]) of type (*W,S*). Then* $\mathbf{G}(\mathbb{F}_q)$ *is a twin building lattice, where* $\mathbb{F}_q$ *denotes the finite field with* q *elements.*

Now the question is whether the finiteness still holds for some $q < n-1$. It turns out that in the class of Coxeter systems with $m_{st} \geq 3$ for all $s \neq t \in S$ this will not happen (cf. Theorem 5.5):

Theorem C. *Let (*W,S*) be a Coxeter system of rank* $n \geq 3$ *such that* $m_{st} \geq 3$ *for all* $s \neq t \in S$*. Then,* $p_{(W,S)}\left(\frac{1}{n-2}\right) = \infty$.

Suppose that the Coxeter diagram is 2-spherical, but the Coxeter diagram is not the complete graph. If the number of non-edges in the Coxeter diagram compared to the number of edges is *large*, then it is still possible that $p_{(W,S)}\left(\frac{1}{n-2}\right) < \infty$ holds (cf. [19]). We also remark that Theorem C can be used to exclude certain subdiagrams for twin building lattices, as parabolic subgroups of twin building lattices are again twin building lattices:

Corollary D. *Let (*W,S*) be a Coxeter system, let* $\mathcal{D}$ *be an RGD system of type (*W,S*) with finite root groups and let* $q_{\min} := \min\{|U_\alpha| \mid \alpha \in \Phi\}$*. If* $\mathcal{D}$ *is a twin building lattice, then there does not exist a subdiagram of (*W,S*) with at least* $q_{\min} + 2$ *vertices, whose underlying Coxeter diagram is the complete graph.*

Several remarks of our main results are in order.

Remark 2. The proofs of Theorem A and Theorem C make essential use of a result of Terragni [20, Theorem A]. We recall this result in Subsection 5.1.

Remark 3. A more general version of Theorems A can be deduced from [2, Theorem 1]: Let (W,S) and (W',S') be two Coxeter systems of rank $n \geq 3$. Suppose that (W',S') is of *universal* type, i.e. $m_{st} = \infty$ for all $s \neq t \in S$. Note that $W^{'}$ is word-hyperbolic (cf. [13, Corollary 12.6.3]) and non-elementary (in the sense of [2]; cf. [13, Theorem 8.6.1, 8.7.3]). Suppose that (W,S) is not of universal type. This means that $m_{st} < \infty$ for some $s \neq t \in S$. Let $\pi : W' \to W$ be a canonical homomorphism which induces a bijection between $S^{'}$ and S. Then $N := \ker(\pi)$ is a normal subgroup which is infinite. We now use the notation from [2]. One can show $\lambda(W',S') = n-1$, and by [2, Theorem 1] we have $\lambda(W,S) < \lambda(W',S') = n-1$. We deduce from [14, Chapter VI.C, Observation 50] that $p_{(W,S)}\left(\frac{1}{n-1}\right) < \infty$. This implies that we can replace in Theorem A 2-*spherical* by *non-universal*.

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

Overview

In § 2, we fix notation and recall some basic results. In § 2.2, we define two subsets C_i and D_i of the Coxeter group W, which play a central role in this paper. In § 3, we recall the definition of reflection and combinatorial triangles and prove some results about them. In § 4, we establish some (in-)equalities concerning the cardinalities $|C_i|$ and $|D_i|$. In § 5, we recall a result due to Terragni and prove our main results.

2. Preliminaries

This section is devoted to fixing notation. In § 2.1 which is based on [20], we recall the notion of growth functions in finitely generated groups. In § 2.2 and § 2.3, we recall some basic definitions about Coxeter systems. Moreover, we introduce two sets C_i and D_i which play a central role in this paper. In § 2.4, we recall some basic results about roots and walls in Coxeter systems. § 2.2, § 2.3 and § 2.4 are based on [1, § 5].

2.1. Growth of finitely generated groups

Let G be a finitely generated group, and let $X = X^{-1} \subseteq G\backslash\{1\}$ be a finite, symmetric set of generators. The *length* of $g \in G$ with respect to X is the minimal n such that $g = x_1 \cdots x_n$ with $x_i \in X$; the *length function* will be denoted by $\ell_{(G,X)} : G \to \mathbb{N}$. For $n \in \mathbb{N}$, the *sphere* in $\mathrm{Cay}(G, X)$ centred around 1_G with radius n will be denoted by

$$C_n^{(G,X)} := \left\{g \in G \mid \ell_{(G,X)}(g) = n\right\}.$$

The cardinalities are defined as $c_n^{(G,X)} := |C_n^{(G,X)}|$. The *growth function* of (G, X) is given by

$$p_{(G,X)}(t) := \sum_{n\geq 0} c_n^{(G,X)} t^n \in \mathbb{Z}[[t]].$$

2.2. Coxeter systems

Let W be a group, and let $S \subseteq W$ be a generating set of elements of order 2. For $s, t \in S$, we denote the order of st in W by m_{st}. Then, the pair (W, S) is called *Coxeter system* if the group W admits the presentation

$$W \cong \langle S \mid (st)^{m_{st}} = 1\rangle,$$

where there is one relation for each pair s, t (possibly $s = t$) with $m_{st} < \infty$. Let (W, S) be a Coxeter system, and let $\ell := \ell_{(W,S)}$ be the corresponding length function. The *Coxeter diagram* corresponding to (W, S) is the labelled graph $(S, E(S))$, where $E(S) = \{\{s, t\} \mid m_{st} > 2\}$ and where each edge $\{s, t\}$ is labelled by m_{st} for all $s, t \in S$. The *rank* of the Coxeter system is the cardinality of the set S. Recall from the introduction that in this paper all Coxeter systems under consideration are assumed to be of finite rank.

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

It is well-known that for each $J \subseteq S$, the pair $(\langle J \rangle, J)$ is a Coxeter system (cf. [7, Ch. IV, §1 Theorem 2]). A subset $J \subseteq S$ is called *spherical* if $\langle J \rangle$ is finite. The Coxeter system is called *2-spherical* if $\langle J \rangle$ is finite for all $J \subseteq S$ containing at most 2 elements (i.e. $m_{st} < \infty$ for all $s, t \in S$). Given a spherical subset J of S, there exists a unique element of maximal length in $\langle J \rangle$, which we denote by r_J (cf. [1, Corollary 2.19]).

For $i \in \mathbb{N}$, we define

- $C_i := C_i^{(W,S)} = \{w \in W \mid \ell(w) = i\}$ and $c_i := |C_i| = c_i^{(W,S)}$;
- $D_i := \{w \in C_i \mid \exists! s \in S : \ell(ws) < \ell(w)\}$ and $d_i := |D_i|$.

The set C_i consists of all elements $w \in W$ of length i. The set D_i consists of all elements $w \in W$ of length i whose right descent set contains a single element of S.

2.3. The chamber system $\Sigma(W, S)$

Let (W, S) be a Coxeter system. Defining $w \sim_s w'$ if and only if $w^{-1}w' \in \langle s \rangle$, we obtain a chamber system (for the definition of a chamber system, see [1, Definition 5.21]) with chamber set W and equivalence relations $\sim_s$ for $s \in S$, which we denote by $\Sigma(W, S)$. We call two chambers w, w' *s-adjacent* if $w \sim_s w'$ and *adjacent* if they are s-adjacent for some $s \in S$. A *gallery of length n* from w_0 to w_n is a sequence $(w_0, \ldots, w_n)$ of chambers, where w_i and w_{i+1} are adjacent for each $0 \leq i < n$. A gallery $(w_0, \ldots, w_n)$ is called *minimal* if there exists no gallery from w_0 to w_n of length $k < n$, and we denote the length of a minimal gallery from w_0 to w_n by $\ell(w_0, w_n)$. For $J \subseteq S$, we define the *J-residue* of a chamber $c \in W$ to be the set $R_J(c) := c\langle J \rangle$. A *residue* R is a J-residue for some $J \subseteq S$; we call J the *type* of R, and the cardinality of J is called the *rank* of R. A residue is called *spherical* if its type is a spherical subset of S. Let R be a spherical J-residue. Two chambers $x, y \in R$ are called *opposite in R* if $x^{-1}y = r_J$. Two residues $P, Q \subseteq R$ are called *opposite in R* if for each $p \in P$ there exists $q \in Q$ such that p, q are opposite in R. A *panel* is a residue of rank 1. It is a fact that for every chamber $x \in W$ and every residue R, there exists a unique chamber $z \in R$ such that $\ell(x, y) = \ell(x, z) + \ell(z, y)$ holds for each chamber $y \in R$. The chamber z is called the *projection* or the *gate* of x onto R and is denoted by $z = \operatorname{proj}_R x$.

A subset $\Sigma \subseteq W$ is called *convex* if for any two chambers $c, d \in \Sigma$ and any minimal gallery $(c_0 = c, \ldots, c_k = d)$, we have $c_i \in \Sigma$ for all $0 \leq i \leq k$. Note that residues are convex by [1, Example 5.44(b)].

For two residues R and T, we define $\operatorname{proj}_T R := \{\operatorname{proj}_T r \mid r \in R\}$. By [1, Lemma 5.36(2)], $\operatorname{proj}_T R$ is a residue contained in T. The residues R and T are called *parallel* if $\operatorname{proj}_T R = T$ and $\operatorname{proj}_R T = R$.

2.4. Roots and walls

Let (W, S) be a Coxeter system. A *reflection* is an element of W that is conjugate to an element of S. For $s \in S$ we let $\alpha_s := \{w \in W \mid \ell(sw) > \ell(w)\}$ be the *simple root* corresponding to s. A *root* is a subset $\alpha \subseteq W$ such that $\alpha = v\alpha_s$ for some $v \in W$ and $s \in S$. We denote the set of all roots by $\Phi := \Phi(W, S)$. The set $\Phi_+ := \{\alpha \in \Phi \mid 1_W \in \alpha\}$ is

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

the set of all *positive roots*, and $\Phi_- := \{\alpha \in \Phi \mid 1_W \notin \alpha\}$ is the set of all *negative roots*. For each root $\alpha \in \Phi$, we denote the *opposite root* by $-\alpha$ and we denote the unique reflection which interchanges these two roots by r_α. For $\alpha \in \Phi$, we denote by $\partial\alpha$ (respectively, $\partial^2\alpha$) the set of all panels (respectively, spherical residues of rank 2) stabilized by r_α. Furthermore, we define $\mathcal{C}(\partial\alpha) := \bigcup_{P\in\partial\alpha} P$ and $\mathcal{C}(\partial^2\alpha) := \bigcup_{R\in\partial^2\alpha} R$.

The set $\partial\alpha$ is called the *wall* associated with α. Let $G = (c_0, \ldots, c_k)$ be a gallery with $c_{i-1} \neq c_i$ for each $1 \leq i \leq k$. We say that G *crosses the wall* $\partial\alpha$ if there exists $1 \leq i \leq k$ such that $\{c_{i-1}, c_i\} \in \partial\alpha$. It is a basic fact that a minimal gallery crosses a wall at most once (cf. [1, Lemma 3.69]). Moreover, a gallery which crosses each wall at most once is already minimal.

A pair $\{\alpha, \beta\} \subseteq \Phi$ of roots is called *prenilpotent*, if $\alpha \cap \beta \neq \emptyset \neq (-\alpha) \cap (-\beta)$. For a prenilpotent pair $\{\alpha, \beta\}$ of roots, we will write $[\alpha, \beta] := \{\gamma \in \Phi \mid \alpha \cap \beta \subseteq \gamma \text{ and } (-\alpha) \cap (-\beta) \subseteq (-\gamma)\}$ and $(\alpha, \beta) := [\alpha, \beta] \setminus \{\alpha, \beta\}$. We note that roots are convex (cf. [1, Lemma 3.44]).

Let $(c_0, \ldots, c_k)$ and $(d_0 = c_0, \ldots, d_k = c_k)$ be two minimal galleries from c_0 to c_k, and let $\alpha \in \Phi$. Then, $\partial\alpha$ is crossed by the minimal gallery $(c_0, \ldots, c_k)$ if and only if it is crossed by the minimal gallery $(d_0, \ldots, d_k)$.

Lemma 2.1. *Let* R *be a spherical residue of* $\Sigma(W, S)$ *of rank* 2*, and let* $\alpha \in \Phi$*. Then, exactly one of the following holds:*

(a) $R \subseteq \alpha$;
(b) $R \subseteq (-\alpha)$;
(c) $R \in \partial^2\alpha$.

Proof. It is clear that the three cases are exclusive. Suppose that $R \not\subseteq \alpha$ and $R \not\subseteq (-\alpha)$. Then, there exist $c \in R \cap (-\alpha)$ and $d \in R \cap \alpha$. Let $(c_0 = c, \ldots, c_k = d)$ be a minimal gallery. As residues are convex, we have $c_i \in R$ for each $0 \leq i \leq k$. As $c \in (-\alpha), d \in \alpha$, there exists $1 \leq i \leq k$, with $c_{i-1} \in (-\alpha), c_i \in \alpha$. In particular, $\{c_{i-1}, c_i\} \in \partial\alpha$ and hence $R \in \partial^2\alpha$. □

Lemma 2.2. *Let* R, T *be two spherical residues of* $\Sigma(W, S)$*. Then, the following are equivalent:*

(i) *R, T are parallel;*
(ii) *a reflection of $\Sigma(W, S)$ stabilizes* R *if and only if it stabilizes* T*;*
(iii) *there exist two sequences $R_0 = R, \ldots, R_n = T$ and $T_1, \ldots, T_n$ of residues of spherical type such that for each $1 \leq i \leq n$, the rank of* $\mathrm{T_i}$ *is equal to* $1 + \mathrm{rank}(R)$*, the residues R_{i-1}, R_i are contained and opposite in* $\mathrm{T_i}$ *and moreover, we have* $\mathrm{proj}_{T_i} R = R_{i-1}$ *and* $\mathrm{proj}_{T_i} T = R_i$*.*

Proof. This is [9, Proposition 2.7]. □

Lemma 2.3. *Let $\alpha \in \Phi$ be a root, and let $x, y \in \alpha \cap \mathcal{C}(\partial\alpha)$. Then, there exists a minimal gallery $(c_0 = x, \ldots, c_k = y)$ such that $c_i \in \mathcal{C}(\partial^2\alpha)$ for each $0 \leq i \leq k$. Moreover, for each $1 \leq i \leq k$, there exists $L_i \in \partial^2\alpha$ with $\{c_{i-1}, c_i\} \subseteq L_i$.*

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

Proof. This is a consequence of [8, Lemma 2.3] and its proof. $\square$

Lemma 2.4. *Let $\alpha, \beta \in \Phi, \alpha \neq \pm\beta$ be two roots, and let $R, T \in \partial^2\alpha \cap \partial^2\beta$.*

(a) *The residues* R *and* T *are parallel.*
(b) *If $|\langle J \rangle| = \infty$ holds for all $J \subseteq S$ containing three elements, then* R = T.

Proof. As $R, T \in \partial^2\alpha \cap \partial^2\beta$, there exist panels $P_1, Q_1 \in \partial\alpha$ and $P_2, Q_2 \in \partial\beta$ such that $P_1, P_2 \subseteq R$ and $Q_1, Q_2 \subseteq T$ (as in the proof of Lemma 2.1). By Lemma 2.2, the panels P_i, Q_i are parallel for both $i \in \{1, 2\}$. Now [15, Lemma 17] yields that $P_i, \mathrm{proj}_T P_i$ are parallel, and hence $\mathrm{proj}_T P_1 \in \partial\alpha, \mathrm{proj}_T P_2 \in \partial\beta$ by Lemma 2.2. As $\alpha \neq \pm\beta$, we deduce $\mathrm{proj}_T P_1 \neq \mathrm{proj}_T P_2$, and hence $\mathrm{proj}_T R$ contains the two different panels $\mathrm{proj}_T P_1$ and $\mathrm{proj}_T P_2$. In particular, $\mathrm{proj}_T R$ is not a panel. Since $\mathrm{proj}_T R$ is a residue contained in T, we deduce $\mathrm{proj}_T R = T$. Using similar arguments, we found that $\mathrm{proj}_R T = R$ and R, T are parallel. This proves (a). Moreover, Lemma 2.2 yields $R = T$, as there are no spherical residues of rank 3 by assumption. This finishes the proof. $\square$

3. Reflection and combinatorial triangles in $\Sigma(W, S)$

Reflection triangles and combinatorial triangles were introduced in [8]. A *reflection triangle* is a set $\mathcal{R}$ of three reflections such that the order of tt' is finite for all $t, t' \in \mathcal{R}$, and $\bigcap_{t \in \mathcal{R}} \partial^2\beta_t = \emptyset$, where β_t is one of the two roots associated with the reflection t. Note that $\partial^2\beta_t = \partial^2(-\beta_t)$. A set of three roots $\mathcal{T}$ is called a *combinatorial triangle* (or simply *triangle*) if the following holds:

(CT1) The set $\{r_\alpha \mid \alpha \in \mathcal{T}\}$ is a reflection triangle.
(CT2) For each $\alpha \in \mathcal{T}$, there exists $\sigma \in \partial^2\beta \cap \partial^2\gamma$ such that $\sigma \subseteq \alpha$, where $\{\beta, \gamma\} = \mathcal{T} \backslash \{\alpha\}$.

Lemma 3.1. *Suppose that* (W, S) *is* 2*-spherical and the Coxeter diagram is the complete graph. If $\mathcal{T}$ is a triangle, then $(-\alpha, \beta) = \emptyset$ holds for all $\alpha \neq \beta \in \mathcal{T}$.*

Proof. This is [4, Proposition 2.3]. $\square$

Proposition 3.2. *Assume that* (W, S) *is* 2*-spherical and the Coxeter diagram is the complete graph. Let* R ≠ T *be two residues of rank* 2 *such that $P := R \cap T$ is a panel. If $\ell(1_W, \mathrm{proj}_R 1_W) < \ell(1_W, \mathrm{proj}_T 1_W)$, then $\mathrm{proj}_T 1_W = \mathrm{proj}_P 1_W$.*

Proof. We let $\alpha \in \Phi_+$ be the root with $P \in \partial\alpha$. Let $(c_0 = 1_W, \ldots, c_{k'} = \mathrm{proj}_P c_0)$ be a minimal gallery with $c_k = \mathrm{proj}_R c_0$ for some $0 \leq k \leq k'$ and $c_k, \ldots, c_{k'} \in R$.

We assume that $\mathrm{proj}_T c_0 \neq \mathrm{proj}_P c_0$ holds. Then, we have $k' > \ell(1_W, \mathrm{proj}_T 1_W) > \ell(1_W, \mathrm{proj}_R 1_W) = k$. Let $(d_0 = 1_W, \ldots, d_{m'} = \mathrm{proj}_P d_0)$ be a minimal gallery with $d_m = \mathrm{proj}_T c_0$ for some $0 \leq m \leq m'$ and $d_m, \ldots, d_{m'} \in T$. We let $\beta \in \Phi_+$ be the root with $\{d_m, d_{m+1}\} \in \partial\beta$, and we let $\gamma \in \Phi_+$ be the root with $\{c_k, c_{k+1}\} \in \partial\gamma$. We will show that $\{\alpha, -\beta, -\gamma\}$ is a triangle. Thus, we first show that $\{r_\alpha, r_\beta, r_\gamma\}$ is a reflection triangle. We have $T \in \partial^2\alpha \cap \partial^2\beta$, and, as a minimal gallery crosses a wall at most once, we deduce $\alpha \neq \beta$. Note that the wall $\partial\beta$ is crossed by the minimal gallery $(c_0, \ldots, c_{k'})$.

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

Since $\partial^2\alpha \ni R \neq T \in \partial^2\alpha \cap \partial^2\beta$ and $\alpha \neq \pm\beta$, Lemma 2.4(b) implies $R \notin \partial^2\beta$, and hence $\partial\beta$ is crossed by $(c_0, \ldots, c_k)$. As $k < k'$, we have $\mathrm{proj}_R 1_W \neq \mathrm{proj}_P 1_W$ and hence $\alpha \neq \gamma$. As $\alpha, \gamma \in \Phi_+$, we have $\alpha \neq \pm\gamma$.

Assume that $o(r_\beta r_\gamma) = \infty$. We deduce $\beta \subseteq \gamma$. But $\partial\gamma$ has to be crossed by the gallery $(d_0, \ldots, d_{m'})$. Since $\partial^2\alpha \ni T \neq R \in \partial^2\alpha \cap \partial^2\gamma$ and $\alpha \neq \pm\gamma$, we have $T \notin \partial\gamma^2$ by Lemma 2.4(b) as before. This implies that $(d_0, \ldots, d_m)$ crosses the wall $\partial\beta$ and hence $\gamma \subseteq \beta$. This yields a contradiction, and we have $o(r_\beta r_\gamma) < \infty$.

As $R \in \partial^2\alpha \cap \partial^2\gamma$, Lemma 2.4($b$) implies $\partial^2\alpha \cap \partial^2\gamma = \{R\}$. As $R \notin \partial^2\beta$, we deduce $\partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma = \emptyset$, and hence $\{r_\alpha, r_\beta, r_\gamma\}$ is a reflection triangle.

Now we have to verify (CT2). As $\partial^2\gamma \not\ni T \in \partial^2\alpha \cap \partial^2\beta$ and $P \subseteq T \cap (-\gamma)$, we have $T \subseteq (-\gamma)$ by Lemma 2.1. As $\partial^2\beta \not\ni R \in \partial^2\alpha \cap \partial^2\gamma$ and $P \subseteq R \cap (-\beta)$, we have $R \subseteq (-\beta)$. Let $1 \leq i \leq k$ be such that $\{c_{i-1}, c_i\} \in \partial\beta$. Note that $\{d_m, d_{m+1}\} \in \partial\beta, d_{m+1} \in (-\beta) \cap T \subseteq (-\gamma)$ and $c_i \in (-\beta) \cap \gamma$. By Lemma 2.3 there exists a minimal gallery $(e_0 = d_{m+1}, \ldots, e_z = c_i)$ such that $e_j \in \mathcal{C}(\partial^2\beta)$. As $d_{m+1} \in (-\gamma)$ and $c_i \in \gamma$, there exists $1 \leq p \leq z$ such that $e_{p-1} \in (-\gamma)$ and $e_p \in \gamma$. Again, by Lemma 2.3, there exists $L \in \partial^2\beta$ such that $\{e_{p-1}, e_p\} \subseteq L$, and hence $L \in \partial^2\beta \cap \partial^2\gamma$. As roots are convex and $e_0 = d_{m+1}, e_z = c_i \in \alpha$, we have $e_p \in L \cap \alpha$. As $\{r_\alpha, r_\beta, r_\gamma\}$ is a reflection triangle (and hence $L \notin \partial^2\alpha$), we obtain $L \subseteq \alpha$ by Lemma 2.1. This implies that $\{\alpha, -\beta, -\gamma\}$ is a triangle, and hence $(\alpha, \gamma) = \emptyset$ holds by Lemma 3.1. In particular, $k + 1 = k'$ and $\ell(1_W, \mathrm{proj}_R 1_W) = \ell(1_W, \mathrm{proj}_P 1_W) - 1 \geq \ell(1_W, \mathrm{proj}_T 1_W)$. This is a contradiction to the assumption, and we conclude $\mathrm{proj}_T 1_W = \mathrm{proj}_P 1_W$. □

Corollary 3.3. *Assume that* (W, S) *is* 2*-spherical and that the underlying Coxeter diagram is the complete graph. Suppose* $w \in W$ *and* $s \neq t \in S$ *with* $\ell(ws) = \ell(w) + 1 = \ell(wt)$ *and suppose* $w' \in \langle s, t\rangle$ *with* $\ell(w') \geq 2$*. Then we have* $\ell(ww'r) = \ell(w) + \ell(w') + 1$ *for each* $r \in S\backslash\{s, t\}$*.*

Proof. Suppose $r \in S\backslash\{s, t\}$, and assume that $\ell(ww'r) = \ell(ww') - 1$ holds for some $w' \in \langle s, t\rangle$ with $\ell(w') \geq 2$. Suppose w' starts with s, i.e. $w' = sw''$ for some $w'' \in \langle s, t\rangle$ with $\ell(w'') = \ell(w') - 1$. As $\ell(ww'r) = \ell(ww') - 1$, one easily sees that $\ell(wstr) = \ell(wst) - 1$ and $\ell(wsr) = \ell(ws) - 1$ hold, too. We define $R := R_{\{r,t\}}(ws), T := R_{\{s,t\}}(w)$ and $P := R \cap T = \mathcal{P}_t(ws)$. Clearly, $\mathrm{proj}_T 1_W \neq \mathrm{proj}_P 1_W$. As $m_{rt} \geq 3$, we deduce $\ell(1_W, \mathrm{proj}_R 1_W) < \ell(1_W, \mathrm{proj}_T 1_W)$ and Proposition 3.2 yields a contradiction. □

Lemma 3.4. *Assume that* (W, S) *is* 2*-spherical and that* $m_{st} \geq 4$ *holds for all* $s \neq t \in S$*. Suppose* $w \in W$ *and* $s \neq t \in S$ *with* $\ell(ws) = \ell(w) + 1 = \ell(wt)$*. Then we have* $\ell(w) + 2 \in \{\ell(wsr), \ell(wtr)\}$ *for all* $r \in S\backslash\{s, t\}$*.*

Proof. Assume that $\ell(wsr) = \ell(w) = \ell(wtr)$. Then, $\ell(wr) = \ell(w) - 1$ and $\ell(wrs) = \ell(w) - 2 = \ell(wrt)$. Let R be the $\{r, s\}$-residue containing w. As $m_{rs} \geq 4$, we deduce $\ell(wrsr) = \ell(wrs) - 1$. Let $w' \in \langle s, t\rangle$ be such that $wr = (\mathrm{proj}_R 1_W)w'$. Then, $\ell(w') \geq 2$ and the previous corollary implies $\ell(wrt) = \ell(wr) + 1$, which is a contradiction. This finishes the proof. □

Remark 3.5. Note that Lemma 3.4 is false without the assumption $m_{st} \geq 4$. To see this, one can consider the Coxeter system of type $\tilde{A}_2$.

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

4. Some (in-)equalities

To show the two main results (Theorem 5.3 and 5.5), we will apply the ratio test. In order to do so, we need a few inequalities, which we establish in this and the next section. We recall that for $i \in \mathbb{N}$ we have

- $C_i := \{w \in W \mid \ell(w) = i\}$ and $c_i := |C_i|$;
- $D_i := \{w \in C_i \mid \exists! s \in S : \ell(ws) < \ell(w)\}$ and $d_i := |D_i|$;

Lemma 4.1. *Suppose that the Coxeter diagram of* (W, S) *is the complete graph. Then, for each* $w \in W \backslash \{1_W\}$*, there is either a unique element* $s_w \in S$ *with* $\ell(ws_w) = \ell(w) - 1$*, or else there are exactly two elements* $s_w \neq t_w \in S$ *with* $\ell(ws_w) = \ell(w) - 1 = \ell(wt_w)$*.*

Proof. Let $J \subseteq S$ with $\ell(wj) < \ell(w)$ for each $j \in J$. Then [1, Corollary 2.18] implies that J is spherical. As the underlying Coxeter diagram is the complete graph, it follows that each subset of S containing at least three elements is non-spherical. This finishes the proof. □

Convention 4.2. *In this section, we assume that* (W, S) *is of rank* $n \geq 3$ *and that there exists* $m \geq 3$ *such that* $m_{st} = m$ *holds for all* $s \neq t \in S$*. Moreover, we let* $\mathrm{i} > \mathrm{m}$*.*

Lemma 4.3. $c_i - d_i = \binom{n-2}{2} c_{i-m} + (n-2) d_{i-m}$.

Proof. Let $v \in C_i \backslash D_i$ be an element. By Lemma 4.1, there exist exactly two elements $s \neq t \in S$ with $\ell(vs) = \ell(v) - 1 = \ell(vt)$. We define $R_v := R_{\{s,t\}}(v)$. Then, we consider the mapping

$$f : C_i \backslash D_i \to C_{i-m}, v \mapsto \mathrm{proj}_{R_v} 1_W$$

Note that $C_{i-m} = D_{i-m} \cup C_{i-m} \backslash D_{i-m}$. If $w \in C_{i-m} \backslash D_{i-m}$, Lemma 4.1 implies that there are exactly two elements in S, say $s_w \neq t_w \in S$, which decreases the length of w (as $i > m$). Any other element $r \in S \backslash \{s_w, t_w\}$ increases the length of w. For $n > 3$ and $r_1 \neq r_2 \in S \backslash \{s_w, t_w\}$, we have $f(wr_{\{r_1, r_2\}}) = w$. For $n = 3$, we have $f^{-1}(w) = \emptyset$. In both cases, w has $\binom{n-2}{2}$ many preimages. If $w \in D_{i-m}$ is, there exists a unique $s_w \in S$ which decreases the length of w and (similarly as before) w has $\binom{n-1}{2}$ many preimages. Note that $\binom{n-1}{2} - \binom{n-2}{2} = n - 2$. We conclude

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

$$\begin{aligned}c_i - d_i = |C_i \backslash D_i| &= \sum_{w \in C_{i-m}} |f^{-1}(w)| \\ &= \sum_{w \in C_{i-m} \backslash D_{i-m}} |f^{-1}(w)| + \sum_{w \in D_{i-m}} |f^{-1}(w)| \\ &= \binom{n-2}{2} (c_{i-m} - d_{i-m}) + \binom{n-1}{2} d_{i-m} \\ &= \binom{n-2}{2} c_{i-m} + (n-2) d_{i-m}.\end{aligned}$$

□

Lemma 4.4. $2c_{i+1} - d_{i+1} = (n-2)c_i + d_i$.

Proof. We put $M_i := \{(w, s) \in C_i \times S \mid ws \in C_{i+1}\}$. We prove the claim by showing that both sides of the equation are equal to $|M_i|$.

(a) $2c_{i+1} - d_{i+1} = |M_i|$: We consider the mapping

$$\pi : M_i \to C_{i+1}, (w, s) \mapsto ws.$$

Clearly, π is surjective. We define

$$C_{i+1}^1 := \{w \in C_{i+1} \mid |\pi^{-1}(w)| = 1\} \quad \text{and} \quad C_{i+1}^{>1} := \{w \in C_{i+1} \mid |\pi^{-1}(w)| > 1\}.$$

We show that $C_{i+1}^{>1} = C_{i+1} \backslash D_{i+1}$. Let $\bar{w} \in C_{i+1}^{>1}$ be an element. Then, there exist $(w, s) \neq (w', s') \in \pi^{-1}(\bar{w})$. It follows that $s \neq s'$, and hence $\bar{w} \in C_{i+1} \backslash D_{i+1}$. Now, let $w \in C_{i+1} \backslash D_{i+1}$. By Lemma 4.1, there exist exactly two elements $s_w \neq t_w \in S$, which decreases the length of w. This implies $(ws_w, s_w) \neq (wt_w, t_w) \in \pi^{-1}(w)$. As $|\langle J \rangle| = \infty$ for all $J \subseteq S$ containing three elements, we deduce for every $1 \neq w \in W$ that

$$|\pi^{-1}(w)| \in \{1, 2\}.$$

We infer $C_{i+1}^1 = C_{i+1} \backslash C_{i+1}^{>1} = C_{i+1} \backslash (C_{i+1} \backslash D_{i+1}) = D_{i+1}$ and compute

$$\begin{aligned}|M_i| = \sum_{w \in C_{i+1}} |\pi^{-1}(w)| &= \sum_{w \in D_{i+1}} |\pi^{-1}(w)| + \sum_{w \in C_{i+1} \backslash D_{i+1}} |\pi^{-1}(w)| \\ &= d_{i+1} + 2(c_{i+1} - d_{i+1}) \\ &= 2c_{i+1} - d_{i+1}.\end{aligned}$$

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

(b) $(n-2)c_i + d_i = |M_i|$: For a subset $T \subseteq C_i$, we define

$$M_{i,T} := \{(w, s) \in M_i \mid w \in T\}.$$

For $w \in D_i$ there are exactly $n-1$ elements which increase the length of w. Thus, we have $|M_{i,D_i}| = (n-1)d_i$. For $w \in C_i \backslash D_i$, there are exactly $n-2$ elements in S, which increases the length of w (cf. Lemma 4.1). Thus, we have $|M_{i,C_i \backslash D_i}| = (n-2)(c_i - d_i)$. We conclude

$$|M_i| = |M_{i,C_i \backslash D_i}| + |M_{i,D_i}| = (n-2)(c_i - d_i) + (n-1)d_i = (n-2)c_i + d_i.$$

□

Lemma 4.5. $c_{i+1} \leq (n-1)c_i - (n-2)d_{i-m+1} \leq (n-1)c_i$.

Proof. The last inequality is obvious. Using Lemma 4.3 and 4.4, we deduce the following:

$$c_{i+1} + (n-2)d_{i-m+1} \leq 2c_{i+1} - d_{i+1} = (n-2)c_i + d_i \leq (n-1)c_i.$$

□

Lemma 4.6. *Suppose* $\mathrm{m} > 3$. *Then, the following holds:*

(a) $(n-2)c_i \leq c_{i+1}$;
(b) $(n-2)d_i \leq d_{i+1}$;

Proof. We define $N_i := \{(w, s) \in C_i \times S \mid ws \in D_{i+1}\}$. Then, $N_i \to D_{i+1}, (w, s) \mapsto ws$ is a bijection, and hence $|N_i| = d_{i+1}$. As in the proof of Lemma 4.4, we define for a subset $T \subseteq C_i$:

$$N_{i,T} := \{(w, s) \in N_i \mid w \in T\}.$$

We see that $c_{i+1} \geq d_{i+1} = |N_i| = |N_{i,D_i}| + |N_{i,C_i \backslash D_i}|$. Let $w \in C_i$. We now count pairs $(w, s) \in N_i$. We distinguish the following two cases:

(i) $w \in D_i$: Let $s_w \in S$ be the unique element with $\ell(ws_w) < \ell(w)$. Let $t \in S \backslash \{s_w\}$. Then, $wt \in C_{i+1}$. Suppose $wt \notin D_{i+1}$. Then, there exists $t \neq r \in S$ with $\ell(wtr) < \ell(wt)$. This implies $\ell(wr) < \ell(w)$, and the uniqueness of s_w yields $r = s_w$. Now, let $r \in S \backslash \{s_w, t\}$. Then $wr \in C_{i+1}$. Again, if $wr \notin D_{i+1}$, then s_w would decrease the length of wr. But this is a contradiction to Lemma 3.4. This implies $(w, r) \in N_{i,D_i}$ for all $r \in S \backslash \{s_w, t\}$. This shows (b).

(ii) $w \in C_i \backslash D_i$: Let $s_w \neq t_w \in S$ be the two elements with $\ell(ws_w) = \ell(wt_w) < \ell(w)$. Now let $r \in S \backslash \{s_w, t_w\}$. Then, $wr \in C_{i+1}$. We assume by contradiction that $wr \notin D_{i+1}$. Then, there would exist $u \in S \backslash \{r\}$ with $\ell(wru) = \ell(w)$, and hence $\ell(wu) < \ell(w)$. As s_w and t_w are the only two elements in S with the property that they decrease the length of w, we obtain $u \in \{s_w, t_w\}$. But then, we obtain a contradiction to Corollary 3.3. We conclude $(w, r) \in N_{i,C_i \backslash D_i}$.

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

We infer $c_{i+1} \geq |N_{i,D_i}| + |N_{i,C_i \setminus D_i}| \geq (n-2)d_i + (n-2)(c_i - d_i) = (n-2)c_i$. □

5. Main results

In this section, we prove our main results. In § 5.1, we recall a reduction result due to Terragni. In § 5.2, we use the reduction result to prove convergence of $p_{(W,S)}\left(\frac{1}{n-1}\right)$, where (W, S) is 2-spherical of rank $n \geq 3$. In § 5.3, we use the reduction result to prove divergence of $p_{(W,S)}\left(\frac{1}{n-2}\right)$, where (W, S) is of rank $n \geq 4$ and the underlying Coxeter diagram is the complete graph.

5.1. Reduction step

Let (W, S) and (W', S') be two Coxeter systems. Following [20], we define $(W, S) \preceq (W', S')$ if there exists an injective map $\varphi : S \to S'$ satisfying $m_{st} \leq m'_{\varphi(s)\varphi(t)}$ for all $s, t \in S$.

Theorem 5.1. *Let* (W, S) *and* (W', S') *be two Coxeter systems, and let* $c_n := c_n^{(W,S)}$ *and* $c'_n := c_n^{(W',S')}$. *If* $(W, S) \preceq (W', S')$, *then* $c_n \leq c'_n$.

Proof. This is [20, Theorem A]. □

5.2. Convergence

Lemma 5.2. *Let* (W, S) *be of rank* $n \geq 3$, *and assume that there exists* $m \geq 4$ *such that* $m_{st} = m$ *holds for all* $s \neq t \in S$. *Then, there exists* $k \in \mathbb{R}$ *such that* $\frac{d_i}{c_i} \geq k > 0$ *holds for all* $\mathrm{i} > \mathrm{m}$.

Proof. Using Lemma 4.3 and 4.6, we compute

$$\begin{aligned} 1 = \frac{c_i - d_i + d_i}{c_i} &= \frac{1}{c_i}\left(\binom{n-2}{2} c_{i-m} + (n-2)d_{i-m} + d_i\right) \\ &\leq \frac{1}{c_i}\left(\binom{n-2}{2} \frac{1}{(n-2)^m} c_i + \left(\frac{1}{(n-2)^{m-1}} + 1\right) d_i\right) \\ &= \frac{1}{c_i}\left(\frac{(n-3)}{2(n-2)^{m-1}} c_i + \left(\frac{1}{(n-2)^{m-1}} + 1\right) d_i\right) \\ &\leq \frac{1}{2(n-2)^{m-2}} + \left(\frac{1}{(n-2)^{m-1}} + 1\right) \frac{d_i}{c_i}. \end{aligned}$$

We put

$$k := \left(1 - \frac{1}{2(n-2)^{m-2}}\right) \cdot \left(\frac{1}{(n-2)^{m-1}} + 1\right)^{-1}.$$

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

As $n \geq 3$ and $m \geq 4$, we have $k > 0$. This proves the claim. □

Theorem 5.3. *Let* (W, S) *be* 2*-spherical and of rank* $n \geq 3$*. Then,* $p_{(W,S)}\left(\frac{1}{n-1}\right) < \infty$*.*

Proof. Let $m := \max\{4, m_{st} \mid s, t \in S\}$, and let (W', S') be the Coxeter system of rank n with $m'_{st} = m$ for all $s \neq t \in S'$. Using Theorem 5.1, it suffices to show that

$$p_{(W',S')}\left(\frac{1}{n-1}\right) < \infty.$$

By Lemma 5.2, there exists $k \in \mathbb{R}$ such that $\frac{d_i}{c_i} \geq k > 0$ holds for all $i > m$. We apply the ratio test. We use Lemma 4.5 and compute for $i > 2m - 1$ and $t = \frac{1}{n-1}$:

$$\frac{c_{i+1}t^{i+1}}{c_i t^i} \leq \frac{(n-1)c_i - (n-2)d_{i-m+1}}{(n-1)c_i} \leq 1 - \frac{(n-2)d_{i-m+1}}{(n-1)^m c_{i-m+1}} \leq 1 - \frac{n-2}{(n-1)^m}k < 1.$$

□

5.3. Divergence

In this subsection, we prove that the new lower bound $\frac{1}{n-1}$ for the finiteness of the growth function is optimal for the class of 2-spherical Coxeter systems with a complete Coxeter diagram.

Lemma 5.4. *Let* (W, S) *be* 2*-spherical and of rank* $n \geq 4$*, and assume that the underlying Coxeter diagram is the complete graph. Then* $(n-2)c_i \leq d_i + d_{i+1}$*.*

Proof. For $i = 0$, we have $c_0 = 1$, $d_0 = 0$ and $d_1 = n$, and the claim follows. Thus, we can assume $i > 0$. As in Lemma 4.6, we define $N_i := \{(w, s) \in C_i \times S \mid ws \in D_{i+1}\}$ as well as $N_{i,T} := \{(w, s) \in N_i \mid w \in T\}$ for $T \subseteq C_i$. We consider the mapping

$$\pi : N_i \to D_{i+1}, (w, s) \mapsto ws.$$

As before, π is a bijection and we have $|N_i| = d_{i+1}$. Moreover, we have $N_i = N_{i,D_i} \cup N_{i,C_i \backslash D_i}$ and this union is disjoint. We now count pairs (w, s) in N_i.

We fix $w \in D_i$, and we let $s_w \in S$ be the unique element with $\ell(ws_w) = \ell(w) - 1$. Assume that there are $r, s, t \in S \backslash \{s_w\}$ pairwise distinct with $wr, ws, wt \in C_{i+1} \backslash D_{i+1}$. Similarly, as in Lemma 4.6(b), we deduce $\ell(wzs_w) = \ell(w)$ for each $z \in \{r, s, t\}$. As $m_{pq} \geq 3$ holds for all $p \neq q \in S$, we infer $\ell(ws_w z) = \ell(ws_w) - 1$. As $\{r, s, t\}$ is not spherical, this is a contradiction and we have for a fixed $w \in D_i$ at least $n - 3$ tuples (w, s) in N_i.

We fix $w \in C_i \backslash D_i$, and we let $s_w \neq t_w \in S$ be the two elements with $\ell(ws_w) = \ell(w) - 1 = \ell(wt_w)$. Assume that there is $s \in S \backslash \{s_w, t_w\}$ with $ws \in C_{i+1} \backslash D_{i+1}$. Then, $\ell(w) \in \{\ell(wss_w), \ell(wst_w)\}$. W.l.o.g. we assume $\ell(wss_w) = \ell(w)$. But then Corollary 3.3 implies $\ell(wt_w) = \ell(w) + 1$, which is a contradiction. Thus, we have for a fixed $w \in C_i \backslash D_i$

exactly $n-2$ tuples (w,s) in N_i (cf. Lemma 4.1). This implies that $(n-2)c_i - d_i = (n-3)d_i + (n-2)(c_i - d_i) \leq d_{i+1}$. □

Theorem 5.5. *Let* (W,S) *be of rank* $n \geq 4$*, and assume that the underlying Coxeter diagram is the complete graph. Then,* $p_{(W,S)}\left(\frac{1}{n-2}\right) = \infty$.

Proof. Let (W', S') be the Coxeter system of rank n with $m'_{st} = 3$ for all $s \neq t \in S'$. Using Theorem 5.1, it suffices to show that

$$p_{(W',S')}\left(\frac{1}{n-2}\right) = \infty.$$

As before, we apply the ratio test. Using Lemmas 4.4 and 5.4, we deduce the following for $i > m = 3$ and $t = \frac{1}{n-2}$:

$$\frac{c_{i+1}t^{i+1}}{c_i t^i} = \frac{(n-2)c_i + d_i + d_{i+1}}{2(n-2)c_i} = \frac{1}{2} + \frac{d_i + d_{i+1}}{2(n-2)c_i} \geq \frac{1}{2} + \frac{1}{2} = 1.$$

□

Acknowledgements. I am very grateful to Bernhard Mühlherr for stimulating discussions and interesting questions on the topic. I also thank Corentin Bodart and Pierre-Emmanuel Caprace for valuable remarks on an earlier draft. I thank Corentin Bodart for pointing out reference [2]. Moreover, I would like to thank the anonymous referee for many detailed comments.

References

(1) P. Abramenko and K. S. Brown, Theory and applications. In *Buildings, Volume 248, of Graduate Texts in Mathematics* (Springer, New York, 2008).

(2) G. N. Arzhantseva and I. G. Lysenok, Growth tightness for word hyperbolic groups, *Math. Z.* **241**(3) (2002), 597–611.

(3) U. Bader and Y. Shalom, Factor and normal subgroup theorems for lattices in products of groups, *Invent. Math.* **163**(2) (2006), 415–454.

(4) S. Bischof, On commutator relations in 2-spherical RGD-systems, *Comm. Algebra* **50**(2) (2022), 751–769.

(5) S. Bischof. (2023). PhD thesis, Justus-Liebig-Universität Giessen, *Construction of RGD-systems of type (4,4,4) over* $\mathbb{F}_2$.

(6) A. Björner and F. Brenti, *Combinatorics of Coxeter groups, Volume 231 of Graduate Texts in Mathematics* (Springer, New York, 2005).

(7) N. Bourbaki, *Lie Groups and Lie algebras. Chapters 4–6* (Springer-Verlag, Berlin, 2002). Elements of Mathematics (Berlin). Translated from the 1968 French original by Andrew Pressley.

(8) P. -E. Caprace and B. Mühlherr, Reflection triangles in Coxeter groups and biautomaticity, *J. Group Theory* **8**(4) (2005), 467–489.

(9) P. -E. Caprace and B. Mühlherr, Isomorphisms of Kac-Moody groups which preserve bounded subgroups, *Adv. Math.* **206**(1) (2006), 250–278.

https://doi.org/10.1017/S0013091525000094 Published online by Cambridge University Press

(10) P. -E. Caprace and B. Rémy, Simplicity and superrigidity of twin building lattices, *Invent. Math.* **176**(1) (2009), 169–221.

(11) P. -E. Caprace and B. Rémy, Simplicity of twin tree lattices with non-trivial communication relations. in *Topology and Geometric Group theory, Volume 184, of Springer Proc. Math. Stat.* pp. 143–151 (Springer, [Cham], 2016).

(12) L. Carbone and H. Garland, Lattices in Kac-Moody groups, *Math. Res. Lett.* **6**(3-4) (1999), 439–447.

(13) M. W. Davis, *The geometry and topology of Coxeter groups, Volume 32, of London Mathematical Society Monographs Series* (Princeton University Press, Princeton, NJ, 2008).

(14) P. de la Harpe, Topics in geometric group theory. In *Chicago Lectures in Mathematics* (University of Chicago Press, Chicago, IL, 2000).

(15) A. Devillers, B. Mühlherr and H. Van Maldeghem, Codistances of 3-spherical buildings, *Math. Ann.* **354**(1) (2012), 297–329.

(16) G. A. Margulis, *Discrete subgroups of semisimple Lie groups, Volume 17 of Ergebnisse der Mathematik und Ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]* (Springer-Verlag, Berlin, 1991).

(17) B. Rémy, Construction de réseaux en théorie de Kac-Moody, *C. R. Acad. Sci. Paris Sér. I Math.* **329**(6) (1999), 475–478.

(18) B. Rémy, Integrability of induction cocycles for Kac-Moody groups, *Math. Ann.* **333**(1) (2005), 29–43.

(19) T. Terragni, Data about hyperbolic coxeter systems, (2015), https://arxiv.org/pdf/1503.08764, 86.

(20) T. Terragni, On the growth of a Coxeter group, *Groups Geom. Dyn.* **10**(2) (2016), 601–618.

(21) J. Tits, Uniqueness and presentation of Kac-Moody groups over fields, *J. Algebra* **105**(2) (1987), 542–573.