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SUBMODULES OF COMMUTATIVE C^*-ALGEBRAS

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Abstract. In this paper we generalise a result of Izuchi and Suárez (K. Izuchi and D. Suárez, Norm-closed invariant subspaces in L^∞ and H^∞, *Glasgow Math. J.* **46** (2004), 399–404) on the shift invariant subspaces of $L^\infty(\mathbb{T})$ to the non-commutative setting. Considering these subspaces as $C(\mathbb{T})$-modules contained in $L^\infty(\mathbb{T})$, we show that under some restrictions, a similar description can be given for the $\mathfrak{B}$-submodules of $\mathfrak{A}$, where $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{B}$ is a commutative C^*-subalgebra of $\mathfrak{A}$. We use this to give a description of the $\mathbb{M}_n(\mathfrak{B})$-submodules of $\mathbb{M}_n(\mathfrak{A})$.

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1. Introduction. Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. A subspace S of $L^p(\mathbb{T})$ is said to be *shift invariant* if for every $f \in S$ we have that the function $z \mapsto zf(z)$ is also in S. As is usual, particular importance over the years has been placed on the cases $p = 2$ and $p = \infty$. These subspaces, as well as arising naturally in an abundance of purely operator theoretic contexts, have proved important in the study of linear time invariant systems in control theory. A very lucid account of this is given in [**7**, Chap. 3].

Shift invariant subspaces come in two forms. If S is shift invariant and, in addition, we have that the function $z \mapsto \bar{z}f(z)$ is in S whenever $f \in S$, then S is called *doubly invariant* or *2-invariant*, otherwise it is called *simply invariant* or *1-invariant*. Equivalently, simply invariant subspaces are the shift invariant subspaces S such that $zS \neq S$ and doubly invariant subspaces are those with $zS = S$.

When $p < \infty$, the classification of the closed doubly invariant subspaces is given by Wiener's theorem [**7**, Theorem 3.1.1]. The classification of the closed simply invariant subspaces is slightly more difficult and is the content of the Beurling–Helson theorem [**7**, Theorem 3.1.2]. These theorems were then used to derive analogous results for the weak-* closed shift invariant subspaces of $L^\infty(\mathbb{T})$. Although this provided a satisfactory classification of these subspaces, much less is know in general about the norm closed shift invariant subspaces of $L^\infty(\mathbb{T})$. The most significant progress made so far are the results of Izuchi and Suárez in [**5**]. In their paper the authors characterised the maximal norm closed simply invariant subspaces of $L^\infty(\mathbb{T})$ and all norm closed doubly invariant subspaces of $L^\infty(\mathbb{T})$. In this paper, we will only be considering the latter. For completeness, we will present this result of Izuchi and Suárez [**5**], but first some definitions will be required.

As usual $\Delta(L^\infty(\mathbb{T}))$ is the spectrum of $L^\infty(\mathbb{T})$ and $\hat{f} \in C(\Delta(L^\infty(\mathbb{T})))$ is the Gelfand transform of $f \in L^\infty(\mathbb{T})$. We regard each bounded Borel measure μ on $\Delta(L^\infty(\mathbb{T}))$ as a linear functional on $L^\infty(\mathbb{T})$ and so we write $\ker \mu$ for the collection of all $f \in L^\infty(\mathbb{T})$

such that

$$\int_{\Delta(L^\infty(\mathbb{T}))} \hat{f}\, \mathrm{d}\mu = 0.$$

Let z denote the identity function on $\mathbb{T}$. Then for each $\lambda \in \mathbb{T}$ we define $\mathcal{F}_\lambda \subseteq \Delta(L^\infty(\mathbb{T}))$ to be the set of all characters $\varphi \in \Delta(L^\infty(\mathbb{T}))$ such that $\hat{z}(\varphi) = \lambda$. Then $\Delta(L^\infty(\mathbb{T})) = \cup_{\lambda\in\mathbb{T}}\mathcal{F}_\lambda$. Define Π to be the set of all bounded Borel measures μ on $\Delta(L^\infty(\mathbb{T}))$ such that $\mathrm{supp}\mu \subseteq \mathcal{F}_\lambda$ for some $\lambda \in \mathbb{T}$. We are now in a position to state the theorem.

THEOREM 1. ([**5**]). *A closed subspace S of $L^\infty(\mathbb{T})$ is doubly invariant if and only if there is some collection of measures $\Lambda \subseteq \Pi$ such that*

$$S = \bigcap_{\mu\in\Lambda} \ker \mu.$$

We aim to show that this result is, in fact, a special case of more general results describing some of the modules of certain commutative C^*-algebras. Before proceeding, we will first fix some notation that will be adopted throughout, most of which is standard.

1.1. Notation. Let $\mathcal{H}$ be a Hilbert space. I will denote the identity in $\mathcal{B}(\mathcal{H})$. For a subalgebra $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$, we denote by $(\mathfrak{A})_1$ its closed unit ball, $\overline{\mathfrak{A}}^w$ its weak closure, $Z(\mathfrak{A})$ its centre and $\mathfrak{A}'$ its commutant in $\mathcal{B}(\mathcal{H})$ – that is $\mathfrak{A}' = \{T \in \mathcal{B}(\mathcal{H}) : TA = AT \quad \text{for every} \quad A \in \mathfrak{A}\}$. Given $\psi \in \mathfrak{A}^*$ and $A \in \mathfrak{A}$, we will write $A\psi$ to denote the functional $B \mapsto \psi(AB)$ on $\mathfrak{A}$. $\mathbb{M}_n(\mathfrak{A})$ will denote the algebra of all $n \times n$ matrices with entries in $\mathfrak{A}$, although we will simply write $\mathbb{M}_n$ rather than $\mathbb{M}_n(\mathbb{C})$. E_{ij} denotes the element of $\mathbb{M}_n$ with i, jth entry equal to 1 and all other entries 0.

2. A non-commutative generalisation. It is easily observed that the problem of determining the closed doubly invariant subspaces of $L^\infty(\mathbb{T})$ can be thought of as one of determining the closed $C(\mathbb{T})$-modules contained in $L^\infty(\mathbb{T})$. It is then natural to ask, if rather than $C(\mathbb{T})$ and $L^\infty(\mathbb{T})$ we have two C^*-algebras $\mathfrak{B}$ and $\mathfrak{A}$ with $\mathfrak{B} \subseteq \mathfrak{A}$, whether we can still give a description of the closed left $\mathfrak{B}$-submodules of $\mathfrak{A}$. We show that under some restrictions on the algebras, a similar description can be given which generalises the result of Izuchi and Suárez [**5**]. In particular, we will always require $\mathfrak{B}$ to be commutative. In stating and proving the main results, we use many aspects from the non-commutative theory of antisymmetric algebras developed many years ago by Szymanski in [**8**, **9**]. We begin by recalling a basic definition from this theory.

Let $\mathcal{H}$ be a Hilbert space and let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be an operator algebra. A projection $P \in \mathfrak{A}'$ is called $\mathfrak{A}$-*antisymmetric* if for every $A \in \mathfrak{A}$ such that $PA = PA^*$, there exists some $r \in \mathbb{R}$ such that $PA = rP$. An $\mathfrak{A}$-antisymmetric projection P is *maximal* if whenever Q is an $\mathfrak{A}$-antisymmetric projection such that $Q \geq P$, with the standard ordering of projections, we have $Q = P$. It is a straightforward application of Zorn's Lemma to show that every $\mathfrak{A}$-antisymmetric projection is dominated by a maximal one. We denote by $\mathcal{M}(\mathfrak{A})$ the set of all maximal $\mathfrak{A}$-antisymmetric projections. It was shown in [**9**] that if $\mathfrak{A}$ acts non-degenerately on $\mathcal{H}$, $\mathcal{M}(\mathfrak{A})$ is contained in the centre of $\overline{\mathfrak{A}}^w$ and that the elements of $\mathcal{M}(\mathfrak{A})$ are all orthogonal. For a general operator algebra, indeed even for a C^*-algebra $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$, there need not be any $\mathfrak{A}$-antisymmetric projections (consider, for example, $\mathfrak{A} = \mathcal{B}(\mathcal{H}) = \mathbb{M}_2$).

We say that $\mathcal{M}(\mathfrak{A})$ is *full* or that $\mathfrak{A}$ has a *full set of antisymmetric projections* if $\mathcal{M}(\mathfrak{A})$ is non-empty and

$$\sum_{P \in \mathcal{M}(\mathfrak{A})} P = I,$$

where the sum converges in the strong operator topology. It is easily verified that a necessary (but not sufficient) condition for $\mathfrak{A}$ to have a full set of antisymmetric projections is that $\mathfrak{A}$ is commutative.

Throughout the remainder of this section, we fix a Hilbert space $\mathcal{H}$ and two C^*-subalgebras $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{B}(\mathcal{H})$ with $\mathfrak{B} \subseteq \mathfrak{A}$. We shall also assume the following:

(1) $\mathfrak{B}$ is commutative with a full set of antisymmetric projections.
(2) $\mathfrak{B}$ (and hence $\mathfrak{A}$) acts non-degenerately on $\mathcal{H}$.

We will say that a functional $\psi \in \mathfrak{A}^*$ is *antisymmetrically supported* if for each $P \in \mathcal{M}(\mathfrak{B})$ either $P\psi = \psi$ or $P\psi = 0$. We say that a set $\Lambda \subseteq \mathfrak{A}^*$ is antisymmetrically supported if every $\psi \in \Lambda$ is antisymmetrically supported.

We can now give a generalisation of Theorem 1.

THEOREM 2. *If every norm continuous linear functional on $\mathfrak{A}$ is ultraweakly continuous, then $M \subseteq \mathfrak{A}$ is a closed left $\mathfrak{B}$-module if and only if there exists an antisymmetrically supported set $\Lambda \subseteq \mathfrak{A}^*$ such that*

$$M = \bigcap_{\psi \in \Lambda} \ker \psi.$$

In order to prove Theorem 2 we will require the following lemma. This is a non-commutative analogue of a result in the commutative theory of uniform algebras, a detailed account of which can be found in [**2**].

LEMMA 3. *Assume that every norm continuous linear functional on $\mathfrak{A}$ is ultraweakly continuous and let $M \subseteq \mathfrak{A}$ be a closed left $\mathfrak{B}$-module.*

(a) *For every $\psi \in \mathfrak{A}^*$ we have that if $\psi \in M^\perp$ then $\psi \in (PM)^\perp$ for each $P \in \mathcal{M}(\mathfrak{B})$.*
(b) *If $A \in \mathfrak{A}$ and $PA \in PM$ for every $P \in \mathcal{M}(\mathfrak{B})$ then $A \in M$.*

Proof. (a) Fix $\psi \in M^\perp$ and $P \in \mathcal{M}(\mathfrak{B})$. Since $P \in \overline{\mathfrak{B}}^w$, there exists a net $(B_\lambda) \subseteq \mathfrak{B}$ with $B_\lambda \to P$ in the ultraweak topology. As every $\psi \in \mathfrak{A}^*$ is ultraweakly continuous, for each $A \in M$ we have $\psi(PA) = \lim_\lambda \psi(B_\lambda A) = 0$.

(b) Fix $\psi \in M^\perp$, $A \in (\mathfrak{A})_1$ and suppose that $PA \in PM$ for every $P \in \mathcal{M}(\mathfrak{B})$. The requirement that $\mathcal{M}(\mathfrak{B})$ is full then ensures that the sum

$$\sum_{P \in \mathcal{M}(\mathfrak{B})} PA$$

converges in the strong operator topology (and hence in the weak operator topology) to A. Since ψ is ultraweakly continuous, it is weakly continuous on $(\mathfrak{A})_1$

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[**6**, Proposition 7.4.5], and so

$$\psi(A) = \sum_{P \in \mathcal{M}(\mathfrak{B})} \psi(PA).$$

It then follows from part (a) that $\psi(A) = 0$. □

We can now proceed to prove Theorem 2.

Proof of Theorem 2. Let $M \subseteq \mathfrak{A}$ be a closed left $\mathfrak{B}$-module. Define $\Lambda \subseteq M^{\perp}$ to be the set of all $\psi \in M^{\perp}$, which are antisymmetrically supported. Firstly, we have the trivial inclusion

$$M \subseteq \bigcap_{\psi \in \Lambda} \ker \psi.$$

We also see that if $\psi \in (PM)^{\perp}$ for some $P \in \mathcal{M}(\mathfrak{B})$ then $P\psi \in M^{\perp}$. Since P is a projection, we also clearly have that $P\psi$ is antisymmetrically supported and so $P\psi \in \Lambda$. So if $\psi(A) = 0$ for every $\psi \in \Lambda$, then $\psi(PA) = 0$ for every $\psi \in (PM)^{\perp}$. It then follows from Lemma 3(b) that $A \in M$, and therefore

$$M = \bigcap_{\psi \in \Lambda} \ker \psi.$$

Conversely, fix $\psi \in \mathfrak{A}^*$ and $P \in \mathcal{M}(A)$ such that $P\psi = \psi$. Then for each $B \in \ker \psi$ and $A \in A$,

$$\psi(AB) = \psi(PAB) = \lambda\psi(B) = 0$$

for some $\lambda \in \mathbb{C}$. So $\ker \psi$ is a left $\mathfrak{B}$-module, and hence an intersection of such things will also be a left $\mathfrak{B}$-module. □

Example 4. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis (e_n). We denote by $\mathcal{K}(\mathcal{H})$ and $\mathcal{D}_0(\mathcal{H})$ the algebras of compact operators and compact diagonal operators (with respect to the basis (e_n)) respectively. It is straightforward to verify that the $\mathcal{D}_0(\mathcal{H})$-antisymmetric projections are rank 1 projections onto subspaces spanned by the basis vectors and that these are in fact maximal so that $\mathcal{M}(\mathcal{D}_0(\mathcal{H})) = \{P_n : n \in \mathbb{N}\}$, where P_n is the projection onto the subspace spanned by e_n. Since every continuous linear functional on $\mathcal{K}(\mathcal{H})$ is induced by a trace class operator, it has an extension to $\mathfrak{B}(\mathcal{H})$, which is ultraweakly continuous ([**11**, p. 96]). Fix $S \in \mathcal{S}_1(\mathcal{H})$, where $\mathcal{S}_1(\mathcal{H})$ denotes the trace class operators on $\mathcal{H}$. We will use $\hat{S}$ to denote the functional $T \mapsto \mathrm{tr}ST$. An elementary calculation shows that $P_n\hat{S} = \hat{S}$ if and only if $e_k \in \ker S$ for every $k \neq n$. Equivalently, $P_n\hat{S} = \hat{S}$ if and only if the matrix for S only has non-zero entries in the nth column. Then $T \in \ker S$ if and only if the n, nth entry of ST is 0. Consequently, every $\mathcal{D}_0(\mathcal{H})$-submodule of $\mathcal{K}(\mathcal{H})$ can be constructed by starting with some collection $\{S_\lambda\} \subseteq \mathcal{S}_1(\mathcal{H})$, each member of which will only have non-zero entries in one column, the n_λth column say, and then taking all $T \in \mathcal{K}(\mathcal{H})$ such that the n_λ, n_λth entry of $S_\lambda T$ vanishes for all λ.

Corollary 5. *Let X be a closed subalgebra of $\mathfrak{A}$ containing $\mathfrak{B}$. If every norm continuous linear functional on $\mathfrak{A}$ is ultraweakly continuous, then M is a closed left $\mathfrak{B}$-submodule of X if and only if there exists an antisymmetrically supported set $\Lambda \subseteq \mathfrak{A}^*$*

such that

$$M = \bigcap_{\psi \in \Lambda} \ker \psi \cap X.$$

Proof. Fix $P \in \mathcal{M}(\mathfrak{B})$ and suppose that we have $\psi \in \mathfrak{A}^*$ with $P\psi = \psi$. For every $B \in \mathfrak{B}$ and $A \in X$ we have that $BA - \lambda A \in \ker \psi$, where $\lambda \in \mathbb{C}$ is such that $PB = \lambda P$. If we also have that $A \in \ker \psi$, then we must have that $BA \in \ker \psi$. Hence, $\ker \psi \cap X$ is a closed left $\mathfrak{B}$-submodule of X. Conversely, every closed left $\mathfrak{B}$-submodule of X is trivially a closed left $\mathfrak{B}$-submodule of $\mathfrak{A}$, and so the result follows from Theorem 2. □

EXAMPLE 6. Let X be a closed subalgebra of $L^\infty(\mathbb{T})$ which contains $H^\infty(\mathbb{T})$. Such algebras are called *Douglas algebras* and a detailed account of these is given in [**3**, Chap. 9]. If we further suppose that X *strictly* contains $H^\infty(\mathbb{T})$, then by Theorems 1.4 and 2.2 of [**3**] we have that X contains $C(\mathbb{T})$. Corollary 5 then implies that the closed shift invariant subspaces of X are all of the form $S \cap X$, where S is a closed shift invariant subspace of $L^\infty(\mathbb{T})$.

We now wish to extend Theorem 2 to give a description of the $\mathbb{M}_n(\mathfrak{B})$-submodules of $\mathbb{M}_n(\mathfrak{A})$. However, we will first consider $\mathfrak{A}^n = \mathfrak{A} \oplus \cdots \oplus \mathfrak{A}$ acting on $\mathcal{H}^n$. We can regard $\mathfrak{B}$ as a subalgebra of $\mathfrak{A}^n$ by identifying $B \in \mathfrak{B}$ with $(B, \dots, B) \in \mathfrak{A}^n$. It is clear that if every bounded linear functional on $\mathfrak{A}$ is ultraweakly continuous then the same is true for $\mathfrak{A}^n$. The definition of antisymmetrically supported elements and subsets of $\mathfrak{A}^*$ extends to $\mathfrak{A}^{*n}$ without change, but noting that if $\psi = (\psi_1, \dots, \psi_n) \in \mathfrak{A}^{*n}$ then $P\psi = (P\psi_1, \dots, P\psi_n)$. We are now left with the task of determining the maximal $\mathfrak{B}$-antisymmetric projections in $\mathcal{B}(\mathcal{H}^n)$. It should be noted that despite considering $\mathfrak{B}$ acting on $\mathcal{H}^n$ we will reserve $\mathcal{M}(\mathfrak{B})$ exclusively for denoting the maximal $\mathfrak{B}$-antisymmetric projections in $\mathcal{B}(\mathcal{H})$. Let $Q \in \mathcal{B}(\mathcal{H}^n)$ be any maximal $\mathfrak{B}$-antisymmetric projection. Since Q is contained in the weak closure of $\mathfrak{A}^n$, we can write $Q = (Q_1, \dots, Q_n)$, where each Q_j acts on $\mathcal{H}$. Then it is easy to check that each Q_j is a maximal $\mathfrak{B}$-antisymmetric projection in $\mathcal{B}(\mathcal{H})$. Suppose there are indices j and k with $Q_j \neq Q_k$. Then there is some $B \in \mathfrak{B}$ and distinct complex numbers λ_j and λ_k with $BQ_j = \lambda_j B_j$ and $BQ_k = \lambda_k Q_k$. Then $B(Q_j, Q_k) = (BQ_j, BQ_k) = (\lambda_j Q_j, \lambda_k Q_k) \neq \lambda(Q_j, Q_k)$ for any $\lambda \in \mathbb{C}$. It follows that any projection in $\mathcal{B}(\mathcal{H}^n)$ having a subprojection equivalent to (Q_j, Q_k) cannot be $\mathfrak{B}$-antisymmetric. So in particular Q is not $\mathfrak{B}$-antisymmetric. We conclude from this that all the Q_j are equal and hence $Q = (P, \dots, P)$ for some $P \in \mathcal{M}(\mathfrak{B})$.

We will now turn our attention to the left $\mathbb{M}_n(\mathfrak{B})$-submodules of $\mathbb{M}_n(\mathfrak{A})$. We occasionally identify $\mathbb{M}_n(\mathfrak{A})$ and $\mathbb{M}_n(\mathfrak{B})$ with $\mathfrak{A} \otimes \mathbb{M}_n$ and $\mathfrak{B} \otimes \mathbb{M}_n$, respectively, when it is convenient to do so. Before continuing, let us agree on a useful convention. We will regard elements of $\mathfrak{A}^{*n}$ as column vectors and if $A = (A_{ij}) \in \mathbb{M}_n(\mathfrak{A})$, $\lambda = (\lambda_{ij}) \in \mathbb{M}_n$ and $\psi \in \mathfrak{A}^{*n}$ then the 'products' $A\psi$ and λA are the usual ones; however, in this instance we interpret terms, such as $A_{ij}\psi_k$, to mean $\psi_k(A_{ij})$. With this understood, we define for each $\psi \in \mathfrak{A}^{*n}$ a linear map $R_\psi : \mathbb{M}_n(\mathfrak{A}) \to \mathbb{C}^n$ by setting $R_\psi A = A\psi$ for every $A \in \mathbb{M}_n(\mathfrak{A})$.

THEOREM 7. *If every norm continuous linear functional on $\mathfrak{A}$ is ultraweakly continuous then $M \subseteq \mathbb{M}_n(\mathfrak{A})$ is a closed left $\mathbb{M}_n(\mathfrak{B})$-module if and only if there is an antisymmetrically supported set $\Lambda \subseteq \mathfrak{A}^{*n}$ such that*

$$M = \bigcap_{\psi \in \Lambda} \ker R_\psi.$$

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Proof. Assume that $M \subseteq \mathbb{M}_n(\mathfrak{A})$ is a closed left $\mathbb{M}_n(\mathfrak{B})$-module. For $i = 1, \dots, n$, let M_i be the set of all $(A_{i1}, \dots, A_{in}) \in \mathfrak{A}^n$ such that $A = (A_{ij}) \in M$. It is clear that each M_i is a closed left $\mathfrak{B}$-module and so there exist antisymmetrically supported sets $\Lambda_1, \dots, \Lambda_n \subseteq \mathfrak{A}^{*n}$ such that

$$M_i = \bigcap_{\psi \in \Lambda_i} \ker \psi.$$

We claim that all the Λ_i are equal. Fix i and j with $1 \leq i, j \leq n$. Since M is a left $\mathbb{M}_n(\mathfrak{B})$-module, we have in particular that for every $B \in \mathfrak{B}$ and every $A \in M$, $(B \otimes E_{ji})A \in M$. The jth row of $(B \otimes E_{ji})A$ is $(BA_{i1}, \dots, BA_{in})$. It follows that $\mathfrak{B}M_i \subseteq M_j$. Suppose there is some $A \in M_i \setminus M_j$. Then there is some $\psi \in \Lambda_j$ and $P \in \mathcal{M}(\mathfrak{B})$ such that $P\psi = \psi$ and $\psi(A) \neq 0$. However, if we choose a net $(B_\lambda) \subseteq \mathfrak{B}$ with $B_\lambda \to P$ in the ultraweak topology (which we can always do as $P \in \overline{\mathfrak{B}}^w$), then we see that

$$\psi(A) = \psi(PA) = \lim_\lambda \psi(B_\lambda A) = 0.$$

This proves that such an A cannot exist and hence $M_i \subseteq M_j$. Swapping i and j in the previous analysis gives $M_i = M_j$. Once we set $\Lambda = \Lambda_i$ for some (and hence all) i, the inclusion

$$M \subseteq \bigcap_{\psi \in \Lambda} \ker R_\psi$$

follows immediately.

Given any $A \in \cap \ker R_\psi$ there must exist $A^{(1)}, \dots, A^{(n)} \in M$ such that for each $j = 1, \dots, n$, $E_{jj}A^{(j)} = E_{jj}A$. Again, using that M is a left $\mathbb{M}_n(\mathfrak{B})$-module, we have that $(B \otimes E_{jj})A^{(j)} \in M$ for every $B \in \mathfrak{B}$. Since M is ultraweakly closed, it follows that $P \otimes E_{jj}A^{(j)} \in M$ for every $P \in \mathcal{M}(\mathfrak{B})$. As $\mathcal{M}(\mathfrak{B})$ is full, the sum

$$\sum_{P \in \mathcal{M}(\mathfrak{B})} P \otimes E_{jj}A^{(j)}$$

converges in the strong operator topology to $E_{jj}A^{(j)}$, and since all the partial sums are bounded, it also converges in the ultraweak topology. So $E_{jj}A^{(j)} \in M$. Writing

$$A = \sum_{j=1}^{n} E_{jj}A^{(j)},$$

we see that

$$M = \bigcap_{\psi \in \Lambda} \ker R_\psi.$$

The converse is straightforward. Fix $\psi \in \mathfrak{A}^{*n}$ with $P\psi = \psi$ for some $P \in \mathcal{M}(\mathfrak{B})$. If $A \in \ker R_\psi$ and $B \in \mathbb{M}_n(\mathfrak{B})$ then there is a matrix $\lambda = (\lambda_{ij}) \in \mathbb{M}_n$ such that

$$BA\psi = \lambda A\psi = 0.$$

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It follows that for any antisymmetrically supported set $\Lambda \subseteq \mathfrak{A}^{*n}$,

$$\bigcap_{\psi \in \Lambda} \ker R_\psi$$

is a left $\mathbb{M}_n(\mathfrak{B})$-module. □

3. The case where $\mathfrak{B} \subseteq Z(\mathfrak{A})$. In the following we fix two unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, as before with $\mathfrak{B} \subseteq \mathfrak{A}$, but here we insist that $\mathfrak{B} \subseteq Z(\mathfrak{A})$ and that $\mathfrak{B}$ contains the identity in $\mathfrak{A}$. As usual, $\widehat{\mathfrak{A}}$ will denote the spectrum of $\mathfrak{A}$ (i.e. the set of equivalence classes of irreducible representations of $\mathfrak{A}$) equipped with the usual topology. Let Ψ be the reduced atomic representation of $\mathfrak{A}$. That is,

$$\Psi = \bigoplus_{[\pi] \in \widehat{\mathfrak{A}}} \pi,$$

where each $\pi : \mathfrak{A} \to \mathcal{H}_\pi$ is a representative of the equivalence class $[\pi] \in \widehat{\mathfrak{A}}$. From now on we work only with this set of representatives and make no reference to the equivalence classes. We will show that this representation ensures that $\mathfrak{B}$ has a full set of antisymmetric projections and each $P \in \mathcal{M}(\mathfrak{B})$ has a particularly simple form.

For $\pi \in \widehat{\mathfrak{A}}$, let E_π denote the projection in $\Psi(\mathfrak{A})'$ defined by setting

$$\rho(E_\pi) = \begin{cases} I \text{ if } & \rho = \pi \\ 0 \text{ if } & \rho \neq \pi \end{cases}$$

for every $\rho \in \widehat{\mathfrak{A}}$. Since $\mathfrak{B}$ is contained in the centre of $\mathfrak{A}$, we have that for any irreducible representation π of $\mathfrak{A}$, $\pi(\mathfrak{B}) = \mathbb{C}I$. Since every irreducible representation of $\mathfrak{B}$ extends to an irreducible representation of $\mathfrak{A}$ (on a necessarily larger Hilbert space), we see that the map $\pi \mapsto \pi_{\mathfrak{B}}$, where $\pi_{\mathfrak{B}}(A) = (\pi(A)\xi|\xi)$ for any unit vector $\xi \in \mathcal{H}_\pi$, defines a continuous surjection of $\widehat{\mathfrak{A}}$ onto $\Delta(\mathfrak{B})$. Following the ideas of Izuchi and Suárez [**5**] we will define for each $\varphi \in \Delta(\mathfrak{B})$, the fibre above φ to be the set

$$\mathcal{F}_\varphi = \{\pi \in \widehat{\mathfrak{A}} : \pi_{\mathfrak{B}} = \varphi\}.$$

Set

$$P_\varphi = \sum_{\pi \in \mathcal{F}_\varphi} E_\pi$$

with the sum converging in the strong operator topology. Since for each $B \in \mathfrak{B}$ there is some $\lambda \in \mathbb{C}$ with $\pi(B) = \lambda I$ for every $\pi \in \mathcal{F}_\varphi$, it is clear that the projection P_φ is $\mathfrak{B}$-antisymmetric. The fact that

$$\sum_{\varphi \in \Delta(\mathfrak{B})} P_\varphi = I$$

follows from the surjectivity of the map $\pi \mapsto \pi_{\mathfrak{B}}$. So to show that each P_φ is maximal it is only necessary to show that for any two distinct characters $\varphi, \chi \in \Delta(\mathfrak{B})$ there exists some $B \in \mathfrak{B}$ and distinct complex numbers λ_1 and λ_2 such that $\Psi(B)P_\varphi = \lambda_1 P_\varphi$ and $\Psi(B)P_\chi = \lambda_2 P_\chi$. This follows easily since by the Gelfand representation there

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must exist $B \in \mathfrak{B}$ with $\varphi(B) = 1$ and $\chi(B) = 0$, and so $\Psi(B)P_\varphi = \varphi(B)P_\varphi = P_\varphi$ and $\Psi(B)P_\chi = \chi(B)P_\chi = 0$.

We are still not in a position to apply Theorem 2 because we have not shown that each $\psi \in \mathfrak{A}^*$ is ultraweakly continuous on $\Psi(\mathfrak{A})$, and indeed this is in general not the case. Despite this, we will show, using the idea of Glicksberg in [**4**] for the commutative case, that the conclusions of Lemma 3 still hold. Before doing this, however, we must start by fixing some terminology. If $\psi \in \mathfrak{A}^*$, the *null space* of ψ is the ideal $\mathcal{N}(\psi) \subseteq \mathfrak{A}$ consisting of all $A \in \mathfrak{A}$ such that $\psi(BAC) = 0$ for every $B, C \in \mathfrak{A}$ and the *support* of ψ is the projection

$$S_\psi = \sum_{\mathcal{N}(\psi) \subseteq \ker \pi} E_\pi.$$

So a functional $\psi \in \mathfrak{A}^*$ is antisymmetrically supported if and only if $S_\psi \leq P_\varphi$ for some $\varphi \in \Delta(\mathfrak{B})$. Since for every $A \in \mathfrak{A}$ and $\pi \in \widehat{\mathfrak{A}}$ with $\mathcal{N}(\psi) \subseteq \ker \pi$, $\pi((I - S_\psi)A) = 0$, it follows from [**1**, Proposition 2.11.2] that the norm of $(I - S_\psi)A$ in $\mathfrak{A}/\mathcal{N}(\psi)$ is 0 and so $S_\psi \psi = \psi$.

Lemma 8. *Assume $\mathfrak{B} \subseteq Z(\mathfrak{A})$ and let $M \subseteq \mathfrak{A}$ be a closed left $\mathfrak{B}$-module. If $A \in \mathfrak{A}$ and $P_\varphi A \in P_\varphi M$ for every $\varphi \in \Delta(\mathfrak{B})$ then $A \in M$.*

Proof. Let $\psi \in (\mathfrak{A}^*)_1$ be an extreme point of $(M^\perp)_1$. We will show that S_ψ is a $\mathfrak{B}$-antisymmetric projection.

Let us first note that a projection $P \in \Psi(\mathfrak{A})'$ is $\mathfrak{B}$-antisymmetric if and only if the ideal $\mathfrak{B} \cap (I - P)\mathfrak{B}$ is maximal in $\mathfrak{B}$. This is because if P is $\mathfrak{B}$-antisymmetric then for each $B \in \mathfrak{B}$ there is some $\lambda(B) \in \mathbb{C}$ such that $P\Psi(B) = \lambda(B)P$, so the map $B \mapsto \lambda(B)$ is character on $\mathfrak{B}$ with kernel $\mathfrak{B} \cap (I - P)\mathfrak{B}$. Conversely, if $\mathfrak{B} \cap (I - P)\mathfrak{B}$ is maximal, then $P\mathfrak{B} \simeq \mathfrak{B}/(\mathfrak{B} \cap (I - P)\mathfrak{B}) \simeq \mathbb{C}$ and so $P\mathfrak{B} = \mathbb{C}P$.

Let $\tau : \mathfrak{B} \to \mathfrak{B}/(\mathfrak{B} \cap (I - S_\psi)\mathfrak{B})$ be the quotient map. Choose a positive element $B \in (\mathfrak{B})_1$ such that $\tau(B) \neq 0$ and $\tau(B)$ is not invertible. Then by [**1**, Proposition 2.11.2] and the Gelfand–Naimark theorem, there must exist some $\pi \in \widehat{\mathfrak{A}}$ with $\mathcal{N}(\psi) \subseteq \ker \pi$ and $\pi(B) = 0$. This implies that ψ and $B\psi$ are linearly independent, otherwise there would be some non-zero $\lambda \in \mathbb{C}$ with $(B - \lambda I)\psi = 0$, and since $B \in Z(\mathfrak{A})$, $B - \lambda I \in \mathcal{N}(\psi) \subseteq \ker \pi$. We also have for any $A, C \in (\mathfrak{A})_1$,

$$\begin{aligned}
|B\psi(A) + (I - B)\psi(C)| &= |\psi(BA + (I - B)C)| \\
&\leq \|S_\psi(BA + (I - B)C)\| \\
&= \sup_{E_\pi \leq S_\psi} \|\pi(BA + (I - B)C)\| \\
&\leq \sup_{E_\pi \leq S_\psi} (\pi_{\mathfrak{B}}(B)\|A\| + (1 - \pi_{\mathfrak{B}}(B))\|C\|) \leq 1.
\end{aligned}$$

Consequently, we have $1 = \|\psi\| \leq \|B\psi\| + \|(I - B)\psi\| \leq 1$. Writing

$$\psi = \|B\psi\| \left(\frac{B\psi}{\|B\psi\|} \right) + \|(I - B)\psi\| \left(\frac{(I - B)\psi}{\|(I - B)\psi\|} \right)$$

we have expressed ψ as a nontrivial convex sum of elements in $\mathfrak{A}^* \cap \mathcal{M}^\perp$, which is a contradiction. We conclude that every non-zero positive element of $\mathfrak{B}/(\mathfrak{B} \cap (I - S_\psi)\mathfrak{B})$ is invertible. It follows from the Gelfand–Mazur theorem that $\mathfrak{B}/(\mathfrak{B} \cap (I - S_\psi)\mathfrak{B})$ has co-dimension 1, which completes the proof. □

From this we can state versions of Theorems 2 and 7 for this setting.

THEOREM 9. *If* $\mathfrak{B} \subseteq Z(\mathfrak{A})$ *then* $M \subseteq \mathfrak{A}$ *is a closed left* $\mathfrak{B}$*-module if and only if there exists an antisymmetrically supported set* $\Lambda \subseteq \mathfrak{A}^*$ *such that*

$$M = \bigcap_{\psi \in \Lambda} \ker \psi.$$

THEOREM 10. *If* $\mathfrak{B} \subseteq Z(\mathfrak{A})$ *then* $M \subseteq \mathbb{M}_n(\mathfrak{A})$ *is a closed left* $\mathbb{M}_n(\mathfrak{B})$*-module if and only if there is an antisymmetrically supported set* $\Lambda \subseteq \mathfrak{A}^{*n}$ *such that*

$$M = \bigcap_{\psi \in \Lambda} \ker R_\psi.$$

The proofs of Theorems 9 and 10 are almost identical to those of Theorems 2 and 7 and so we omit them. There is however one difference that should be pointed out: The appeal to the ultraweak continuity of bounded linear functionals in the proof of Theorem 7 is not necessary for Theorem 10 because $\mathfrak{B}$ contains the identity.

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