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EMBEDDING INTO GROUPS WITH WELL-DESCRIBED LATTICES OF SUBGROUPS

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A thrifty embedding scheme of an arbitrary set of groups in a simple infinite group with a given outer automorphism group is presented. One of the applications of this scheme is the existence (assuming CH) of an uncountable group G in which all proper subgroups are countable such that G contains every countable group.

1. Introduction

By a famous embedding theorem of Higman, Neumann and Neumann [**1**], every countable group can be embedded in a 2-generator group. But this embedding construction contains a lot of subgroups other than the embedding group and its conjugates, and there is little information about the automorphism group of the resulting group. On the other hand, the method of graded diagrams developed by Ol'shanskii has given an approach to constructing of difficult examples of groups such as, for example, non-abelian infinite groups all of whose proper subgroups are finite (see [**8**]). This technique was extended in [**3**] to diagrams over free products and applied to quotient groups of free products. As a result, a theorem was proved in [**3**] on embeddability of every countable set $\{G_\mu\}_{\mu\in I}$ of countable groups without involutions in a simple 2-generator infinite group G in which every proper subgroup is either a cyclic group or contained in a subgroup conjugate to one of the embedding groups G_μ, and the generalizations of this theorem to the case of arbitrary sets $\{G_\mu\}_{\mu\in I}$ of groups without involutions were given in [**4, 5, 6**]. These constructions have given an opportunity to obtain minimal extensions of the subgroup lattices of the resulting groups G in comparison with the subgroup lattices of the embedding groups which was used in [**3, 4, 5, 6**] and [**8**] for solution of some famous problems, in particular, a well-known problem about the existence of uncountable Artinian groups.

On the other hand, it is easy to see that these results can not be extended to the case of groups $\{G_\mu\}_{\mu\in I}$ with involutions, since any involution $h\in G_\mu$ together with any conjugate involution ghg^{-1}, $g\in G\setminus G_\mu$, must generate in G a dihedral subgroup. Ol'shanskii [**7**] proved that by making such exemptions, one might avoid mentioning the

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absence of involutions from G_μ in the statement of the theorem in [**3**]. (It also leads to the loss of the property that the resulting group G may chosen to be torsion (of finite exponent) if all embedding groups are torsion (of finite exponent).) By combining the ideas from [**6**] and [**7**], we obtain the following embedding scheme of an arbitrary set of groups into a simple infinite group with a "well-described" lattice of subgroups and a given outer automorphism group.

Let $\{G_i\}_{i\in I}$ be an arbitrary set of nontrivial groups. We denote by Ω^1 the *free amalgam* of the groups G_i, $i \in I$, that is, the set $\bigcup\limits_{i\in I} G_i$ with $G_i \cap G_j = 1$ whenever $i \neq j$. We say that the mapping $g : \Omega^1 \to G$ is an *embedding* of Ω^1 into G if g is injective and its restriction to every G_i is a homomorphism.

Let $\Omega = \Omega^1 \setminus \{1\} = \{a_j, j \in J\}$. Then a mapping $f : 2^\Omega \setminus \{\emptyset\} \to 2^\Omega$ is called *generating* on the set Ω if the following conditions hold:

1) if $C \subseteq G_i$ for some $i \in I$, then $f(C) = gp\{C\} \setminus \{1\}$;

2) if $C \not\subseteq G_i$ for each $i \in I$ and $C = \{a, b\} \subseteq \Omega$, where a and b are involutions (such a subset C will be called *dihedral*), then $f(C) = C$;

3) if C is a finite non-dihedral subset of Ω and $C \not\subseteq G_i$ for each $i \in I$, then $f(C) = B$, where B is an arbitrary countable subset of Ω such that $C \subseteq B$ and if D is a finite subset of B, then $f(D) \subseteq B$;

4) if C is an infinite subset of Ω, then $f(C) = \bigcup\limits_{A\in T} f(A)$, where T is the set of all finite subsets of C.

For example, a generating mapping f on Ω can be defined in the following way: if $C \in 2^\Omega \setminus \{\emptyset\}$ and $C = \bigcup\limits_{i\in I} C_i$, where $C_i = C \cap G_i$, $i \in I$, then $f(C) = \big(\bigcup\limits_{i\in I} gp\{C_i\}\big) / \{1\}$.

We denote by $G(1)$ the free product of groups $G_i, i \in I$. A group G having a presentation

$$G = \langle G(1) || R = 1; R \in D\rangle \tag{1.1}$$

is called *(diagrammatically) aspherical ((diagrammatically) atoroidal)* if every diagram on the sphere (torus) over (1.1) is either non-reduced or consists entirely of 0-cells. (All necessary information about diagrams can be found in [**8**].)

Let $G = gp\{\Omega\}, f$ an arbitrary generating mapping on Ω. We say that X is a *minimal* word of the group G if it follows from $X = Y$ in G that $|X| \leqslant |Y|$, where $|Z|$ denotes the length of the word Z. Let W be the set of all non-empty words over the alphabet Ω written in the *normal form*, that is, every element X in W is written in the form $X_1 \dots X_k$, where each X_l, $1 \leqslant l \leqslant k$, is a nontrivial element of $G_{\mu(l)}$, $\mu(l) \in I$, and $\mu(l) \neq \mu(l+1)$ for $l = 1, \dots, k-1$. Then a mapping $F : 2^W \setminus \{\emptyset\} \to 2^\Omega$ is defined in the following way: if $C \subseteq W$ and $C \neq \emptyset$, then let V be the set of all letters occuring in the expressions of words of C. Then we set $F(C) = f(V)$.

The main result of this paper is the following embedding scheme.

THEOREM A. *Let $g_i : G_i \to H$ be a set of arbitrary homomorphisms of the groups G_i into a group H with kernels N_i, $i \in I$, such that a system of subgroups $\{g_i(G_i)\}_{i\in I}$ generates H, let $\{N_j\}_{j\in I_1}$, $I_1 \subseteq I$, be the set of nontrivial groups of the set $\{N_i\}_{i\in I}, \Omega_1^1$ the free amalgam of the groups N_j, $j \in I_1$, and also let f be an arbitrary generating mapping on Ω such that $f(C)\cap\Omega_1^1 \neq \emptyset$ if $C \not\subseteq G_i$ for each $i\in I$ and C is non-dihedral. If the set $\{N_j\}_{j\in I_1}$ contains either three groups or two groups of which one has order at least 3, then the free amalgam Ω^1 of the groups G_i can be embedded in an aspherical atoroidal group $G = gp\{\Omega\}$ with the following properties:*

1) the free amalgam Ω_1^1 is embedded in a normal simple infinite subgroup L of G such that $G/L \cong H$;

2) if $X \in G$ and X is not conjugate in G to an element of one of the groups G_i, $i \in I$, then X is of infinite order;

3) every subgroup M of G is either a cyclic group or infinite dihedral (if one of the groups G_i, $i \in I$, has involutions), or $M \cap L = 1$ and the homomorphic image of M in $H \cong G/L$ has an element of infinite order, or if M is not cyclic or infinite dihedral, then M is conjugate in G to an extension $G_{C,H'}$ of a group H' by a normal subgroup L_C (that is, $G_{C,H'}/L_C \cong H'$), where $H' \leqslant H$ and $L_C \leqslant L$. In what follows, using the notation L_C, we assume that every element of L_C is a minimal word of G, and $C = F(L_C \setminus \{1\})$ or $C = \emptyset$ in the case $L_C = \{1\}$;

4) $L_C = R_C \cap L$, where $R_C = gp\{C\}$ if $C \in 2^\Omega \setminus \{\emptyset\}$ or $R_C = \{1\}$ in the case $C = \emptyset$, and if $C \not\subseteq G_i$ for each $i \in I$, then $G_{C,H'} \leqslant R_C$, L_C is a simple group, $N_G(L_C) = R_C$ and $C_G(L_C) = \{1\}$;

5) if $C \not\subseteq G_i$ for each $i \in I$, then $\operatorname{Aut} L_C \cong R_C$ and $\operatorname{Out} L_C \cong R_C/L_C$ (in particular, $\operatorname{Aut} L \cong G$ and $\operatorname{Out} L \cong H$), and if $g \in (G_i \cap C)\setminus\Omega_1^1$, $i \in I$, then the mapping $g : L_C \to g^{-1}L_C g$ is a regular automorphism of L_C (that is, $g(a) = a$ if and only if $a = 1$) if and only if there is no $c \in G_i \cap C \cap \Omega_1$, where $\Omega_1 = \Omega_1^1 \setminus \{1\}$, such that $[g,c] = 1$;

6) if $C \not\subseteq G_i$ for each $i \in I$, then for each $a \in C \cap \Omega_1$, we have that $L_C = gp\{cbab^{-1}c^{-1}; b,c \in C\}$ (in particular, $L = gp\{cbab^{-1}c^{-1}; b,c \in \Omega\}$, where a is an arbitrary element of Ω_1);

7) if X is a minimal nontrivial word of the group G, then $X \in R_C$ if and only if $F(\{X\}) \subseteq f(C)$;

8) if $\{G_j\}_{j\in J}$, $J \subseteq I$, is a set of all groups having nontrivial intersections with a subgroup R_C of G and $X \in Z^{-1}R_C Z$, where Z is of minimal length among all words in $R_C Z$ and $G_j Z$, then $F(\{Z\}) \subseteq F(\{X\})$;

9) if $C \not\subseteq G_i$ for each $i \in I$ and M is a subgroup of G in which every element is a minimal word in G, then $gp\{L_C, M\} \cap L = L_{C_1}$, where $C_1 = F(C \cup (M \setminus \{1\}))$;

10) if $N_s = \{1\}$ for some $s \in I$ and the homomorphism $g_j : G_j \to H$ is trivial for each $j \in I \setminus \{s\}$, then G is the semidirect product of H and L;

11) if a subgroup M of G is contained in some group G_i, $i \in I$, then $N_G(M) = N_{G_i}(M)$ and $C_G(M) = C_{G_i}(M)$;

12) if a subgroup M of G is infinite dihedral and not conjugate in G to a subgroup of some group G_i, $i \in I$, then $N_G(M)$ is infinite dihedral and $C_G(M) = \{1\}$;

13) if a cyclic subgroup $M = gp\{A\}$ of G is not conjugate in G to a subgroup of some group G_i, $i \in I$, and A is not a product of two involutions in G, then $N_G(M)$ is cyclic and $N_G(M) = C_G(M)$.

Now we have the following strengthenings of Theorems B and D [**6**].

Theorem B. *Let $\{G_i\}_{i\in I}$ be an arbitrary set of nontrivial groups containing either three groups or two groups of which one has order at least 3, H an arbitrary (for example, trivial) group, Ω^1 the free amalgam of the groups H and G_i, $i \in I$, and let f be an arbitrary generating mapping on $\Omega = \Omega^1 \setminus \{1\}$. Then the free amalgam Ω^1 can be embedded in an aspherical atoroidal group $G = gp\{\Omega\}$ such that*

1) the free amalgam of the groups G_i is embedded in a simple normal infinite subgroup L of G and G is the semidirect product of H and L;

2) every nontrivial subgroup of L is infinite cyclic or infinite dihedral (if one of the groups G_i, $i \in I$, or H has involutions), or conjugate in G to a subgroup $L_C = R_C \cap L$, where $R_C = gp\{C\}$, $C \in 2^{\Omega} \setminus 2^H$, and if $C \not\subseteq G_i$ for each $i \in I$, then L_C is simple and $L_C = gp\{cbab^{-1}c^{-1}; b, c \in f(C)\}$ for each $a \in f(C) \setminus H$;

3) if $C \not\subseteq G_i$ for each $i \in I$, then $\operatorname{Aut} L_C \cong R_C$ and $\operatorname{Out} L_C \cong R_C/L_C$ (in particular, $\operatorname{Aut} L \cong G$ and $\operatorname{Out} L \cong H$), and for each $g \in H \cap C$, g is a regular automorphism of L_C. .

Proof: Let $g_i : G_i \to H$ be the trivial homomorphism for each $i \in I$, $g_H : H \to H$ the natural isomorphism. Then the system $\{N_i\}_{i\in I}$ of nontrivial kernels of the homomorphisms g_H and g_i, $i \in I$, is the same as the set of the groups G_i, $i \in I$, and hence Theorem A applies to Ω^1 and f and yields the required G. □

For countable groups we have the following important result.

Theorem C. *Let $\{G_i\}_{i\in I}$ be a countable set of nontrivial countable groups containing either three groups or two groups of which one has order at least 3, H an arbitrary countable (for example, trivial) group. Then the free amalgam Ω^1 of the groups H and G_i, $i \in I$, can be embedded in a group $G = gp\{\Omega\}$, where $\Omega = \Omega^1 \setminus \{1\}$, with the following properties:*

1) the free amalgam of the groups G_i is embedded in a simple normal infinite subgroup $L = gp\{\Omega \setminus H\}$ of G and G is the semidirect product of H and L;

2) $\operatorname{Aut} L \cong G$ (and $\operatorname{Out} L \cong H$) and for each $g \in H \setminus \{1\}$, g is a regular automor-

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phism of L;

3) if $X, Y \in L$ with $X \in G_i \setminus \{1\}, Y \notin G_i$ for some $i \in I$, then either L is generated by the pair (X, Y) or X and Y are involutions, or X and XY are involutions in G;

4) every proper subgroup of L is either infinite cyclic or infinite dihedral (if one of the groups G_i, $i \in I$, or H has involutions), or contained in a subgroup conjugate in G to some G_i, $i \in I$.

PROOF: We define a generating mapping f on Ω in the following way: if $C \subseteq \Omega$ such that $C \not\subseteq G_i$ for each $i \in I$, $C \not\subseteq H$ and C is not dihedral (it follows from the statement of the theorem that such a subset C exists), then $f(C) = \Omega$. Then Theorem B applies to Ω^1 and this mapping f and yields the group G satisfying properties 1, 2 and 4 in the statement of the theorem. Assertion 3 of the theorem can be proved in the same way as in [**7**, Theorem 2]. ∎

The last application of Theorem A is devoted to construction of a "universal" uncountable group. It is easy to see that there is no countable groups containing every countable group, since any countable group has a countable set of finitely generated subgroups, but by [**2**], there exists a continuum of pairwise non-isomorphic finitely generated groups. On the other hand, Shelah [**9**] constructed an uncountable group with all proper subgroups countable, and the existence of such a group with some additional properties (such as Artinian and of finite exponent) follows immediately from [**3**, Corollary 5]. But there were no examples of uncountable groups G with all proper subgroups countable such that every countable group is contained in G. It is obvious that such groups can not be obtained without assuming CH (that is, $2^{\aleph_0} = \aleph_1$).

THEOREM D. *Let H be an arbitrary group with $1 \leqslant |H| \leqslant 2^{\aleph_0}$. Then assuming CH, there exists a simple uncountable group L in which all proper subgroups are countable such that L contains every countable group and* $\operatorname{Out} L \cong H$.

PROOF: Let $\{G_i\}_{i \in I}$ be the set of all pairwise non-isomorphic countable groups, $\Omega_1^1 = \{a_j; 1 \leqslant j < \omega_1\} \cup \{1\}$ the free amalgam of the groups G_i, $i \in I$, and also let $H = \{h_s; 1 \leqslant s < \chi\} \cup \{1\}$ for some ordinal number $\chi \leqslant \omega_1$ or $H = \{1\}$, where ω_1 is the first uncountable ordinal number. A generating mapping f on $\Omega = \left(\Omega_1^1 \cup H\right) \setminus \{1\}$ is defined in the following way: if C is a finite non-dihedral subset of Ω such that $C \not\subseteq G_i$ for each $i \in I$ and $C \not\subseteq H$, then let μ be the maximal ordinal number such that either a_μ or h_μ is contained in C, and we set

$$f(C) = f(\Omega(\mu) \cap H) \cup \bigcup_{i \in I} f(\Omega(\mu) \cap G_i),$$

where $\Omega(\mu) = \{a_j; 1 \leqslant j \leqslant \mu\} \cup \{h_s; 1 \leqslant s \leqslant \mu\}$ ($\Omega(\mu) = \{a_j; 1 \leqslant j \leqslant \mu\}$ in the case

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$H = \{1\}$). It is easy to see that this mapping f satisfies all conditions in the definition of a generating mapping on Ω.

Theorem B applied to $\Omega \cup \{1\}$ and f and yields a group G with a simple uncountable normal subgroup L such that every countable group is contained in L and $\operatorname{Out} L \cong H$. Let M be a non-cyclic and non-dihedral proper subgroup of L. Then by Theorem B, M is conjugate in G to a subgroup $L_C = R_C \cap L$, where $R_C = gp\{C\}$, $C \in 2^{\Omega} \setminus 2^H$. We may assume that $C \not\subseteq G_i$ for each $i \in I$, hence the set C is countable, since otherwise it follows from the definition of the mapping f that $f(C) = \Omega$ and $M = L$, and we arrive at a contradiction to the choice of M. Therefore, the subgroup M is countable, which completes the proof of the theorem. $\square$

The proof of Theorem A will be heavily based on the results from [**6**] and [**7**]. Unless otherwise stated, all definitions and notation may be found in [**7**] and [**8**].

2. Construction of the Group G

As in [**8**], we introduce the positive parameters

$$\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota,$$

where all the parameters are arranged according to "height", that is, each constant is chosen after its predecessor. Our proofs and some definitions are based on a system of inequalities involving these parameters. The value of the parameters can be chosen in such a way that all the inequalities hold. We then use the following notation:

$$\alpha' = 1/2 + \alpha,\ \beta' = 1 - \beta,\ \gamma' = 1 - \gamma,\ h = \delta^{-1},\ d = \eta^{-1},\ n = \iota^{-1}.$$

We may assume that n is an integer. We also use the notation introduced in Section 1.

We may assume that I is a well-ordered set. We also may assume that Ω^1 is a well-ordered set such that 1 is the maximal element of Ω^1 and if $a \in G_i \setminus \{1\}$ and $b \in G_j \setminus \{1\}$, where $i < j$, then $a < b$. On the set $\Omega_2 = \{ab \mid a \in \Omega^1,\ b \in \Omega$ and if $\{a, b\} \subseteq G_i$ for some $i \in I$, then $a = 1\}$ we introduce an order in the following way: $ab \leqslant cd$ if and only if either $b < d$ or $b = d$ and $a \leqslant c$ (with respect to the ordering of Ω^1).

By the statement of Theorem A, there is a homomorphism of the free product $G(1)$ of the groups G_i, $i \in I$, onto H such that its restriction to every group G_i is equal to g_i. Suppose that the kernel of this homomorphism is N.

Let $D_1 = \emptyset$, and suppose, by induction, that we have defined the set of relators $D_{i-1} \subseteq N$, $i \geqslant 2$, and set

$$G(i-1) = \langle G(1) \| R = 1; R \in D_{i-1} \rangle.$$

A word X is called *free* in rank $i-1$ if X is not conjugate in rank $i-1$ to an element of Ω^1, that is, to an image in $G(i-1)$ of an element of one of the free factors G_j. A non-empty word Y is said to be *simple* in rank $i-1$ if it is free in rank $i-1$, not conjugate in rank $i-1$ (that is, in $G(i-1)$) to a power of a shorter word and not conjugate in rank $i-1$ to a power of a period of rank $k<i$.

Now let P_i denote a maximal set of words of length i which are simple in rank $i-1$ with the property that $A, B \in P_i$ and $A \not\equiv B$ ("$\equiv$" means letter-for-letter equality of words of the same length) implies that A is not conjugate in rank $i-1$ to B or B^{-1}. The words in P_i are called *periods* of rank i. A special role in the construction of the group G will be played by the sets P_i' of all periods of rank i which are not equal in rank $i-1$ to a product of two involutions (of $G(i-1)$). (For short, a product of two involutions of a group will be called a *dihedral* element of a group.) We may assume (see Lemma 3.1 below) that if $a, c \in \Omega_1$, $b, e, g \in \Omega$ and $d, f \in \Omega^1$ such that a is of infinite order (if such an a exists), $\{a,b\} \not\subseteq G_i$, $\{c,e\} \not\subseteq G_j$ and $\{c,g\} \not\subseteq G_s$ for each $i,j,s \in I$, $fg, de \in \Omega_2$, $fg \neq de$ and $fge^{-1}d^{-1} \neq c$ in the case $c^2=1$, then the words $A_0 = a[a,b]^k a[a,b]^{2k} a[a,b]^{3k}$, $A_m = aA_0^m$, $B_0 = (cfg)^{-1}[[c,de]^n,[c,fg]](cfg)$, $B_m = (cfg)^{-1}[c,de]^n(cfg)B_0^m$ are non-dihedral periods of some ranks for each $k \geqslant 1$ and m, $|m| > n^6$.

For each period $A \in P_i' \cap N$, we fix a maximal subset Y_A such that:

1) if $T \in Y_A$, then $1 \leqslant |T| < d\,|A|$;

2) each double coset of the pair $gp\{A\}, gp\{A\}$ of subgroups of $G(i)$ contains at most one word in Y_A and this word is of minimal length among the words representing this double coset;

3) if $T \in Y_A$, then $T \in N$ and $F(\{T\}) \subseteq F(\{A\})$.

We may assume (see Lemma 3.1 below) that if a power F^t of a period F of some rank is conjugate to a word BC^m for some $m \geqslant n$, where C is a non-dihedral period of rank i not equal to A_0 or B_0, $|B| < \iota(m\,|C|)^{1/3}$ and $BCB^{-1} \neq C^{\pm 1}$ in $G(i-1)$, then $t=1$.

For each period $A \in P_i' \cap N$, we introduce the ordering of the set of natural numbers (or a finite segment of it) on the set Y_A such that the first element of the set Y_A belongs to Ω_1 (it follows from the statement of Theorem A that $Y_A \cap \Omega_1 \neq \emptyset$) and if $A = A_m$ or $A = B_m$, where $m = 0$ or $|m| > n^6$, then the first element of the set Y_A is a or $min(c,h)$ (with respect to the ordering of Ω^1), respectively, where $h = d$ if $d \in \Omega_1$, otherwise $h = 1$. We denote this order by $\leqslant_A$.

For each period $A \in P_i' \cap N$, $i \geqslant 7$, we now define some relations. If $A = A_m$, $|m| > n^6$, for some $a \in \Omega_1$ and $b \in \Omega$ such that $\{a,b\} \not\subseteq G_i$ for each $i \in I$ and a is of infinite order, then for each k, $5 \leqslant k \leqslant 15$, we introduce a relation

$$a^{-1}A^n a A^{n+k} a A^{n+30+k} \ldots a A^{n+30(h-2)+k} = 1. \tag{2.1}$$

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If $A = B_m$, $|m| > n^6$, for some $c \in \Omega_1$, $e, g \in \Omega$ and $d, f \in \Omega^1$ such that $\{c, e\} \not\subseteq G_i, \{c, g\} \not\subseteq G_j$ for each $i, j \in I$, $fg, de \in \Omega_2$, $fg \neq de$ and $fge^{-1}d^{-1} \neq c$ in the case $c^2 = 1$, then for each k, $16 \leqslant k \leqslant 25$, and $T = (cfg)^{-1}[c, de]^n(cfg)$, we consider the relation

$$T^{-1}A^nTA^{n+k}TA^{n+30+k}\ldots TA^{n+30(h-2)+k} = 1, \tag{2.2}$$

and if $b_1 = min(c, e)$, $b_2 = min(de, fg)$ (with respect to the ordering of Ω_2) and $T_i = (cfg)^{-1}[c, de]^i(cfg)$, $i \in \{1, 2\}$, then we set

$$(cfg)^{-1}b_icb_i^{-1}(cfg)A^nT_iA^{n+25}T_iA^{n+55}\ldots T_iA^{n+30(h-2)+25} = 1 \tag{2.3}$$

for each i, $1 \leqslant i \leqslant 2$. Let $T \in Y_A$ and $T \neq a$ in the case $A = A_m$, $|m| > n^6$. If a is the minimal element of the set Y_A and $T \neq a$, then we introduce the relation

$$aA^nTA^{n+10}TA^{n+40}\ldots TA^{n+30(h-2)+10} = 1, \tag{2.4}$$

and if $T = a$, then it follows from the definition of the set P_i that there exists $b \in F(\{A\})$ such that $\{a, b\} \not\subseteq G_i$ for each $i \in I$, and we consider the relation

$$bab^{-1}A^nTA^{n+10}TA^{n+40}\ldots TA^{n+30(h-2)+10} = 1. \tag{2.5}$$

If a is the first element of the set Y_A, $T \in Y_A \setminus \{a\}$ and $T \neq (cfg)^{-1}[c, de]^n(cfg)$ in the case $A = B_m$, $|m| > n^6$, then we introduce a relation

$$aA^nTA^{n+20}TA^{n+50}\ldots TA^{n+30(h-2)+20} = 1, \tag{2.6}$$

and if $T = a$, then, as above, we set

$$bab^{-1}A^nTA^{n+20}TA^{n+50}\ldots TA^{n+30(h-2)+20} = 1 \tag{2.7}$$

for some $b \in F(\{A\})$ such that $\{a, b\} \not\subseteq G_i$ for each $i \in I$. And if $T \in Y_A$, then let T_1 be the minimal element of the set $Y_A \setminus \{a^{\pm 1}\}$ such that $T <_A T_1$ (if such an element T_1 exists). Then we consider the relation

$$T_1A^nTA^{n+30}TA^{n+60}\ldots TA^{n+30(h-1)} = 1. \tag{2.8}$$

The left-hand sides of the relations (2.1)-(2.8) form the set S_i of *relators* of rank i. For each $i \geqslant 2$, we set $D_i = D_{i-1} \cup S_i$, and the group $G(i)$ is defined by its presentation:

$$G(i) = \langle G(1) || R = 1; R \in D_i \rangle. \tag{2.9}$$

Finally, we define

$$G = \langle G(1) || R = 1; R \in D = \bigcup_{i \geqslant 1} D_i \rangle.$$

By a *diagram of rank* i, where $i \geqslant 2$, we mean a diagram over the presentation (2.9). Contours of cells Π in the diagrams under considerations split to words of the form (2.1)-(2.8). Those sections of Π with labels $(A^{n+s})^{\pm 1}$ are called *long sections* while the others are called *short sections* of the contour.

3. Proof of Theorem A

We start our proof of the theorem with

Lemma 3.1. *The choice of the set of periods of the group G is correct.*

Proof: Let $a, c \in \Omega_1$, $b, e, g \in \Omega$ and $d, f \in \Omega^1$ such that a is of infinite order (if such an a exists), $\{a, b\} \not\subseteq G_i$, $\{c, e\} \not\subseteq G_j$ and $\{c, g\} \not\subseteq G_s$ for each $i, j, s \in I$, $fg, de \in \Omega_2$, $fg \neq de$ and $fge^{-1}d^{-1} \neq c$ in the case $c^2 = 1$. Also let $C = A_0 = a[a,b]^k a[a,b]^{2k} a[a,b]^{3k}$ for some $k \geqslant 1$ or $C = B_0 = (cfg)^{-1}[[c,de]^n, [c,fg]](cfg)$, and suppose that C is conjugate in some rank $i \geqslant 1$ to V, where V is either an element of Ω or a power of a period of rank $\leqslant i$, or a power of a simple word in rank i. Then it follows from [**8**, Lemma 26.5], [**8**, Corollary 22.2], [**8**, Lemma 21.7] and the definition of the relations in G that C and V are also conjugate in $G(1)$. Hence by the choice of C, we may assume that C is a period of some rank. Moreover, if C is a dihedral element in G, then there is $X \in G$ such that $XCX^{-1} = C^{-1}$ in some rank i. Then as above, we may assume that $i = 0$, and we arrive at a contradiction to the choice of C. Thus we may assume that A_0 and B_0 are non-dihedral periods.

If C_1 is a non-dihedral period of some rank i and B_1 is a word such that $B_1 C_1 B_1^{-1} \neq C_1^{\pm 1}$ in rank $i-1$ and $|B_1| < \iota(|m|\,|C_1|)^{1/3}$ for some m such that $|m| \geqslant n$, then by the proof of [**7**, Lemma 7] and [**8**, Lemma 34.7], a word $B_1 C_1^m$ is conjugate in G to a power F^t of a non-dihedral period F of some rank. Repeating the proof of [**8**, Lemma 27.3] with a reference to [**8**, Lemma 23.15] replaced by a reference to [**7**, Lemma 8], we obtain that $|t| = 1$. (In particular, it is true for A_m and B_m, where $|m| > n^6$.)

If F^{-t} is conjugate to a product $B_2 C_2^{m_2}$ of the same type, then there is a reduced annular diagram Δ with contours $z_1 p_1$ and $z_2 p_2$, where $\phi(z_i) \equiv B_i$ and $\phi(p_i) \equiv C_i^{m_i}$, $i = 1, 2$. Repeating the argument of [**7**, Lemma 8], we have that there exists a contiguity submap Γ of p_1 to p_2 such that the sum of lengths of its contiguity arcs is greater than $\beta'(|p_1| + |p_2|)$. Then by [**8**, Lemma 25.10] (with the correction from [**7**]), we have that $C_1 \equiv C_2$ and either $m_1 m_2 > 0$, p_1 and p_2 are C_1-anticompatible in Δ and C_1 is dihedral, which contradicts the choice of the words C_1 and C_2, or $m_1 m_2 < 0$ and p_1 and p_2 are C_1-compatible in Δ. In the second case, we have that $B_1 \in gp\{C_1\} B_2^{-1} gp\{C_1\}$, and in order to complete the proof of the lemma, it remains to consider the cases when either 1) $B_1 \equiv B_2 \equiv a$ and $C_1 \equiv C_2 \equiv A_0$ for some $a \in \Omega_1$ and $b \in \Omega$ such that $\{a, b\} \not\subseteq G_i$ for each $i \in I$ and a is of infinite order in G, or 2) $B_1 \equiv B_2 \equiv (cfg)^{-1}[c,de]^n(cfg)$ and $C_1 \equiv C_2 \equiv B_0$ for some $c \in \Omega_1$, $e, g \in \Omega$ and $d, f \in \Omega^1$ such that $\{c, e\} \not\subseteq G_j$, $\{c, g\} \not\subseteq G_s$ for each $j, s \in I$, $fg, de \in \Omega_2$, $fg \neq de$ and $fge^{-1}d^{-1} \neq c$ in the case $c^2 = 1$.

It follows from the proof of [**8**, Lemma 25.18] that in any case, either $B_1 \in gp\{C_1\}$,

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which is impossible, or $(C_1^s B_1)^2 = 1$ in G for some integer s. We note that $s \neq 0$, since B_1 is of infinite order in G. Then by [**8**, Lemma 34.7], $C_1^s B_1$ is conjugate in G to an element v of Ω, and it follows from [**8**, Corollary 22.2] and [**8**, Lemma 21.7] that $C_1^s B_1$ and v are conjugate in $G(1)$. We arrive at a contradiction to the choice of the words C_1 and B_1. Thus we may assume that $t = 1$, and the proof of the lemma is complete. ∎

Immediate verification shows that the presentations (2.9) of the groups $G(i)$ satisfy condition R (see [**8**, Sections 25 and 34]). So we can apply to the diagrams under considerations all the results from [**7**] and also [**6**, Lemmas 1–4] if in the definition (from [**6**]) of an I-diagram we demand that condition I3 holds for all contiguity submaps of $q_{i_1}^0$ to $q_{i_2}^0$, where $i_1, i_2 \in \{1,2\}$. We also need the following analogue of [**6**, Lemma 5].

Lemma 3.2. *Let C be a period of the group G, k an integer such that $|k| > 100\zeta^{-1}$, and also let W be a word which does not commute with C^k in G and whose length is minimal among all words in the double coset $gp\{C^k\}Wgp\{C^k\}$. Then $[C^k, W] = ZA^lZ^{-1}$ in G, where A is a period and Z is a minimal word in G, and*

$$F(\{A\}) = F(\{C, W\}),\ F(\{Z\}) \subseteq F(\{A\}). \tag{3.1}$$

Moreover, if $WC^kW^{-1} \neq C^{\pm k}$ in G, then also $BA^lB^{-1} \neq A^{\pm l}$ in G for $B = Z^{-1}C^kZ$.

Proof: By [**8**, Lemma 34.7], the word $[C^k, W]$ is conjugate in G to a word V, where either $|V| = 1$ or $V = A^l$ for some period A and an integer l. Let Δ be a reduced annular diagram with contours p and q such that $\phi(q) \equiv V^{-1}$, $p = p_1p_2p_3p_4$, $\phi(p_1) \equiv \phi(p_3^{-1}) \equiv C^k$, $\phi(p_2) \equiv \phi(p_4^{-1}) \equiv W$. By pasting together the paths p_2 and p_4^{-1}, we obtain a diagram Δ' on a sphere with three holes whose reduced form (that is, with j-pairs removed) is denoted by Δ_0. The cyclic sections p_1 and p_3 can be assumed smooth in Δ_0.

We note that there is no contiguity submap Γ of p_{i_1} to p_{i_2}, where $i_1, i_2 \in \{1,3\}$ and $i_1 \neq i_2$, such that $(p_{i_1}, \Gamma, p_{i_2}) \geqslant 1/100$, since otherwise it follows from [**8**, Lemma 25.10] that p_{i_1} and p_{i_2} are C-compatible in Δ_0, and using [**8**, Lemma 24.9], we arrive at a contradiction to the choice of the word W. Suppose now that for some $i \in \{1,3\}$, there is a contiguity submap Γ of p_i to p_i such that $(p_i, \Gamma, p_i) \geqslant 1/100$. Then by [**8**, Lemma 25.8] (with the correction from [**7**]), p_i is C-anticompatible, and for a compatible path t (see the definition of C-anticompatibility from [**7**]) and a word X we have that $C = X\phi(t)$ in G, where X and $\phi(t)$ are involutions in G. Hence there exists an annular subdiagram Δ_0' of Δ_0 with contours $p_i' t^\varepsilon$ and p_{4-i}, where p_i' is a subpath of p_i with label equal to a power of C and $|\varepsilon| = 1$. Therefore, a power of C is conjugate in G to an involution, which contradicts [**8**, Lemma 34.7]. Thus we obtain that Δ_0 satisfies conditions I1-I3 from the definition (from [**6**]) of an I-diagram.

If $|V| = 1$, then as in the proof of [**8**, Lemma 24.6], we obtain that there is a contiguity submap Γ of p_{i_1} to p_{i_2} for some $i_1, i_2 \in \{1,3\}$ such that $(p_{i_1}, \Gamma, p_{i_2}) > 1/10$, and we arrive at a contradiction to I3. Thus $V = A^l$ in G for some period A and an integer l.

The second assertion of the lemma can be proved in the same way as in [**6**, Lemma 5].

In order to prove the last assertion of the lemma, we note that the inequality $BA^lB^{-1} \neq A^{\pm l}$ is equivalent to the inequality $C^k[C^k, W]C^{-k} \neq [C^k, W]^{\pm 1}$ which is true, since otherwise we have that either $C^k(WC^{-k}W^{-1}) = (WC^{-k}W^{-1})C^k$ or $C^{2k}W = WC^{2k}$ in G, and it follows from [**8**, Lemma 34.9] and [**8**, Lemma 25.15] (with the correction from [**7**]) that $WC^kW^{-1} = C^{\pm k}$, which contradicts our assumption about W.

The proof of the lemma is complete. $\square$

By an *H-map* we understand a circular or annular B-map Δ with contours p (in the case of a circular map) or p and q (in the annular case), where $p = s_1t_1 \dots s_lt_l$, $l \leqslant 3$, $s_1, \dots, s_l$ are called *long sections of the first kind*, $t_1, \dots, t_l$ *short sections*, and q a *long section of the second kind*; all sections are assumed (cyclically) reduced and, for some $j > 1$, the following conditions hold.

H1. Every section of the first kind is a smooth section of rank j and $|s_1| \geqslant nj$.

H2. The section q of the second kind is either smooth or geodesic.

H3. The short sections are geodesic and the length of any short section is less than $max(dj, \iota|s_1|^{1/3})$.

Lemma 3.3. *The assertion of* [**8**, Lemma 23.15] *for C-maps is also true for H-maps.*

Proof: The lemma can be proved in the same way as [**8**, Lemma 23.15] taking into account the remark made in the proof of [**7**, Lemma 5]. $\square$

Lemma 3.4. *If $TA^kT^{-1} = A^{\varepsilon k}$ in G for some period A and an integer k, where $|\varepsilon| = 1$, then $TAT^{-1} = A^{\varepsilon}$ in G and either $\varepsilon = 1$ and $T \in gp\{A\}$ or $\varepsilon = -1$ and A is a dihedral element of G.*

Proof: If $\varepsilon = 1$, then by [**8**, Lemma 34.9], $T \in gp\{A\}$. Suppose now that $\varepsilon = -1$. Then $A^{km}T^{-1}A^{km}T = 1$ in G, where m is chosen in such a way that $m \geqslant n$ and $|T| < \iota(m|A|)^{1/3}$, and let Δ be a reduced circular diagram (of some rank) with contour $s_1t_1s_2t_2$, where $\phi(t_1) \equiv \phi(t_2^{-1}) \equiv T^{-1}$ and $\phi(s_1) \equiv \phi(s_2) \equiv A^{km}$. It follows from [**8**, Lemma 26.5] and the choice of m that Δ is an H-map. Then by Lemma 3.3, there exists a contiguity submap Γ of s_1 to s_2 such that $|\Gamma \wedge s_1| > |s_1|/2$, and it follows from [**8**, Lemma 25.8] (with the correction from [**7**]) that s_1 and s_2 are A-anticompatible in Δ and A is a dihedral element. Hence by [**7**, Lemma 3], we have that $TAT^{-1} = A^{-1}$, which completes the proof of the lemma. $\square$

LEMMA 3.5. *Let A be a period and T a word such that $TAT^{-1} \neq A^{\pm 1}$ in G and $F(\{T\}) \subseteq F(\{A\})$, and also let m be an integer with $|T| < \zeta\iota(|m|\,|A|)^{1/3}$ and $|m| > n$. Then $A^m T A^{2m} T A^{3m} T = Z C^l Z^{-1}$ and $T_1 C^l T_1^{-1} \neq C^{\pm l}$ in G, where C is a non-dihedral period, Z is a minimal word in G and $T_1 = Z^{-1} A^m Z$, and*

$$F(\{A\}) = F(\{C\}), F(\{Z\}) \subseteq F(\{A\}). \tag{3.2}$$

PROOF: By [7, Lemma 6], it remains to prove only (3.2) and the inequality $T_1 C^l T_1^{-1} \neq C^{\pm l}$.

It follows from [6, Lemma 1] and [6, Lemma 3] and the statement of the lemma that

$$F(\{Z, C\}) \subseteq F(\{T, A\}) = F(\{A\}).$$

So it is sufficient to show that $F(\{A\}) \subseteq F(\{C\})$.

Let Δ be a reduced annular diagram (of some rank) with contours p and q, where $\phi(p) \equiv A^m T A^{2m} T A^{3m} T$ and $\phi(q) \equiv C^{-l}$. It follows from the statement of the lemma and [8, Lemma 26.5] that Δ is an H-map. Among contiguity submaps $\Gamma_1, \Gamma_2, \ldots$ given by Lemma 3.3, the submaps in which at least one of the contiguity arcs has length greater than $\zeta^{-1}|A|$ are called *long* while the others are called *short*.

If Γ is a contiguity submap of a long section of the first kind to a distinct long section of the first kind, say of s_1 to s_2, such that its connecting line is homotopic in Δ to a subpath of p not containing s_3, then by [8, Lemma 26.5], [8, Lemma 21.1] and [8, Lemma 25.8], Γ is a short contiguity submap, since otherwise it follows from [8, Lemma 23.17] and [7, Lemma 3] that $TAT^{-1} = A^{-1}$ in rank $|A| - 1$, which contradicts the choice of T. But by the statement of the lemma, $|s_k| \geqslant (|s_1| + |s_2| + |s_3|)/6$ for each $k \in \{1,2,3\}$, then it follows from the definition of H-maps and Lemma 3.3 that there exists a contiguity submap Γ_i of a long section s_i to q for each $i \in \{1,2,3\}$ such that $|\Gamma_i \wedge s_i| > |s_i|/2$, and by [6, Lemma 2], $F(\{A\}) \subseteq F(\{C\})$.

The period C is non-dihedral, so if $T_1 C^l T_1^{-1} = C^{\varepsilon l}$, where $|\varepsilon| = 1$, then by Lemma 3.4, $\varepsilon = 1$ and we have that $A^m = \left(A^m T A^{2m} T A^{3m} T\right) A^m \left(A^m T A^{2m} T A^{3m} T\right)^{-1}$ in G. Again it follows from Lemma 3.4 that $A^{2m} T A^{3m} T A^t T = 1$ in G for some integer t. Let Δ be a reduced circular diagram (of some rank) with contour $s_1 t_1 s_2 t_2 s_3 t_3$, where $\phi(s_1) \equiv A^{2m}$, $\phi(s_2) \equiv A^{3m}$, $\phi(s_3) \equiv A^t$, $\phi(t_i) \equiv T$ for each $i \in \{1,2,3\}$. It follows from the statement of the lemma and [8, Lemma 26.5] that Δ is an H-map (even if $t = 0$). Then using [7, Lemma 3 and Lemma 3.3], we obtain that $TAT^{-1} = A^{-1}$ in G, which contradicts the choice of T.

The proof of the lemma is complete. $\square$

LEMMA 3.6. *Let A be a non-dihedral period and T a word such that $TAT^{-1} \neq A^{\pm 1}$ in G and $F(\{T\}) \subseteq F(\{A\})$, and also let m be an integer with $|T| < \iota(|m|\,|A|)^{1/3}$*

and $|m| > n$. *Then* $TA^m = ZC^{\varepsilon}Z^{-1}$, $|T_1|, |Z| < 3|C|$, $(\beta')^4 m|A| \leqslant |C|$ *and* $T_1CT_1^{-1} \neq C^{\pm 1}$ *in* G, *where* C *is a non-dihedral period of some rank,* $|\varepsilon| = 1$, Z *is a minimal word in* G *and* $T_1 = Z^{-1}TZ$, *and*

$$F(\{C\}) = F(\{A\}), F(\{Z\}) \subseteq F(\{C\}). \tag{3.3}$$

PROOF: The assertion of the lemma follows immediately from the proofs of [**7**, Lemma 9], [**6**, Lemma 6 and Lemma 3.1]. □

Now everything is ready to prove an anolog of [**6**, Lemma 6].

LEMMA 3.7. *Let* $R = gp\{C^k, W\}$, *where* C *is a period,* $C^k \in N \setminus \{1\}$ *and* W *is a minimal word in* G *such that* $WC^kW^{-1} \neq C^{\pm k}$ *in* G. *Then* R *contains a non-dihedral period* $C_1 \in N$ *such that* $F(\{C_1\}) = F(\{C, W\})$ *and* $n|C| < |C_1|$.

PROOF: By [**8**, Lemma 34.7], C is of infinite order in G, so there exists $p > 100\zeta^{-1}$ such that $C^p \in N \cap R$. By [**6**, Lemma 1], we may assume that W has the minimal length among all words in the double coset $gp\{C^p\}Wgp\{C^p\}$, and by [**6**, Lemma 3 and Lemma 3.2], $[C^p, W] = Z_1A^lZ_1^{-1}$, where A is a period, Z_1 is a minimal word in G, and for a word B which is minimal in G and equal in G to a word $Z_1^{-1}C^pZ_1$, we have that

$$F(\{B\}) = F(\{Z_1, C\}) \subseteq F(\{A\}),$$

$BA^lB^{-1} \neq A^{\pm l}$ and (3.1) holds. It is obvious that $A^l, B \in N$ and $BAB^{-1} \neq A^{\pm 1}$ in G.

Let t be an integer such that $|B| < \zeta\iota(lt|A|)^{1/3}$ and $lt > n$. Then by Lemma 3.5, $A^{lt}BA^{2lt}BA^{3lt}B = Z_2V^fZ_2^{-1}$ and $TV^fT^{-1} \neq V^{\pm f}$ in G, where V is a non-dihedral period, Z_2 and $T = Z_2^{-1}A^{lt}Z_2$ are minimal words in G, and (3.2) holds. It follows from (3.2) and [**6**, Lemma 3] that

$$F(\{T\}) = F(\{Z_2, A\}) = F(\{A\}) = F(\{V\}). \tag{3.4}$$

Moreover, $V^f, T \in N$, since $A^l, B \in N$ and N is a normal subgroup, and $TVT^{-1} \neq V^{\pm 1}$ in G.

Now we choose an integer m such that $|T| < \iota(mf|V|)^{1/3}$, $|Z_1|, |Z_2|, |C| < \iota^2 mf|V|$ and $mf > n$. By Lemma 3.6, $TV^{mf} = Z_3C_1^{\varepsilon}Z_3^{-1}$ and $|T_1|, |Z_3| < 3|C_1|$, where C_1 is a non-dihedral period, $|\varepsilon| = 1$, Z_3 and $T_1 = Z_3^{-1}TZ_3$ are minimal words in G, and (3.3) holds. Moreover, it follows from Lemma 3.6 that

$$|Z_1|, |Z_2|, |C| < \iota^2(\beta')^{-4}|C_1| < \iota|C_1|. \tag{3.5}$$

We also have that $C_1 \in N$, and by (3.1)-(3.3),

$$F(\{C, W\}) = F(\{A\}) = F(\{V\}) = F(\{C_1\}).$$

The words C_1 and T_1 are contained in $R_1 = Z^{-1}RZ$, where $Z = Z_1Z_2Z_3$. It follows from [**6**, Lemma 1], Lemma 3.6 and (3.1)-(3.5) that $F(\{T_1\}), F(\{Z\}) \subseteq F(\{C_1\})$ and $|Z| < 4|C_1|$. Hence there are $Z', T_1' \in Y_{C_1}$ such that $Z \in gp\{C_1\}Z'gp\{C_1\}$ and $T_1 \in gp\{C_1\}T_1'gp\{C_1\}$. By the definition of the relation (2.6) for C_1 and T_1' (or for C_1 and $(T_1')^2$ if $C_1 = B_m$, $|m| > n^6$, and $T_1' = (cfg)^{-1}[c, de]^n(cfg)$), the minimal element a of the set Y_{C_1} is contained in R_1. Now using the defining relation (2.8) for C_1 and a, we obtain that $a_1 \in R_1$, where a_1 is the minimal element of the set $Y_{C_1} \setminus \{a^{\pm 1}\}$, and so on. Thus we have that Z' is contained in R_1, hence $Z \in R_1$ and $R = R_1$, which completes the proof of the lemma. $\square$

Now we may obtain all the assertions of Theorem A, except assertion 5 about the automorphism groups of the groups L_C, where $C \not\subseteq G_i$ for each $i \in I$, if we repeat the proof of [**6**, Theorem A] replacing references to [**6**, Lemma 5 and Lemma 6], by references to Lemmas 3.2 and 3.7. We also need to make the following amendments.

1) In order to prove that the homomorphic image L of the subgroup N is an infinite subgroup of G, we can use the argument in the proof of [**8**, Theorem 26.1], with [**8**, Lemma 34.1] used in place of [**8**, Theorem 4.6].

2) Let M be an arbitrary non-cyclic subgroup of G containing a free element X. (An element X is called *free* in G if it is free in rank i for each $i \geqslant 1$.) By [**8**, Lemma 34.7], X is conjugate in G to a power of a period A. If $M \cap L = 1$, then the image A in H has infinite order, since by [**8**, Lemma 34.7], A is of infinite order in G. In the opposite case, as in the proof of [**6**, Theorem A], we obtain that M is conjugate in G to a subgroup $M_2 = gp\{C^l, \{W_j\}_{j\in J}\}$, where C is a period, $C^l \in L$ and for each $j \in J$, W_j is a minimal word in G such that W_j is not contained in $gp\{C\}$.

Now we assume that the group M (and therefore also M_2) is not infinite dihedral. Let Y be an arbitrary element of $L_K = R_K \cap L$, where $K = F(\{C\} \cup \{W_j\}_{j\in J})$. We note that if $A^t \in M_2$, where A is a period and $|t| \geqslant 1$, then there exists a word $Z \in M_2$ such that $ZA^tZ^{-1} \neq A^{\pm t}$, since otherwise it follows from [**8**, Lemma 34.9] that the group M_2 is either infinite cyclic or infinite dihedral.

By Lemma 3.7, a subgroup $gp\{C^l, Z\}$, where Z is an element of M_2 such that $ZC^lZ^{-1} \neq C^{\pm l}$, contains a non-dihedral period A such that $F(\{C\}) \subseteq F(\{A\})$. Then by the definition of a generating mapping on Ω, either $F(\{Y\}) \subseteq F(\{A\})$ or there are $W_{i_1}, \dots, W_{i_t}$, $t \geqslant 1$, such that $F(\{Y\}) \subseteq F(\{A, W_{i_1}, \dots, W_{i_t}\})$. In the second case, we may assume that $W_{i_s}AW_{i_s}^{-1} \neq A^{\pm 1}$ for each s, $1 \leqslant s \leqslant t$, since otherwise by Lemma 3.4, $W_{i_s} \in gp\{A\}$ and $F(\{W_{i_s}\}) = F(\{A\})$. Consider a subgroup $R_1 = gp\{A, W_{i_1}\}$. By Lemma 3.7, the group R_1 contains a period A_1 such that $F(\{A_1\}) = F(\{A, W_{i_1}\})$. Similarly, a group $R_2 = gp\{A_1, W_{i_2}\}$ contains a period A_2 such that $F(\{A_2\}) = F(\{A, W_{i_1}, W_{i_2}\})$, and so on.

As a result, we obtain a period $E \in L' = M_2 \cap L$ such that $F(\{Y\}) \subseteq F(\{E\})$.

Now if $|Y| > d|E|$, then by Lemma 3.7, a subgroup $gp\{E, Z\}$, where Z is an element of M_2 such that $ZEZ^{-1} \neq E^{\pm 1}$, contains a period E_1 such that $F(\{Y\}) \subseteq F(\{E_1\})$ and $n|E| < |E_1|$. Repeating the same trick several times, we have that L' contains a period B such that $F(\{Y\}) \subseteq F(\{B\})$ and $|Y| < d|B|$. Then, as in the proof of [**6**, Theorem A], we obtain that $Y \in L'$ and $L' = L_K$.

3) If $C \not\subseteq G_i$ for each $i \in I$, then by the statement of Theorem A, $f(C) \cap \Omega_1 \neq \emptyset$. Let $a \in f(C) \cap \Omega_1$ and $L'_C = gp\{cbab^{-1}c^{-1};\, b, c \in C\}$. It is obvious that $L'_C \leqslant L_C$. Now we prove that $L_C \leqslant L'_C$. For this purpose, we may repeat the proof of assertion 6 of [**6**, Theorem A] if we show that the group L'_C is not infinite dihedral. It follows from the statement of the theorem, the definition of the relations of G and [**8**, Lemma 34.11] that there is $b \in C$ such that $A = [a, b]$ is a free element. So it is sufficient to find a word $T \in L'_C$ such that $TAT^{-1} \neq A^{\pm 1}$.

By definition, L_C is a normal subgroup of a subgroup M and M is not cyclic or infinite dihedral. Moreover, it follows from assertion 4 of the theorem that $M \leqslant R_C$. Hence the group R_C is not infinite dihedral and there exists $c \in C$ such that $c \neq a$ and $c \neq b^{-1}$. Now we put $T = cbab^{-1}c^{-1}$ and assume that $TAT^{-1} = A^{\pm 1}$ in G. It follows from [**8**, Lemma 23.16] that this equation is also true in the group $G(1)$, and we arrive at a contradiction to the choice of c.

4) Let $C \in 2^{\Omega} \setminus \{\emptyset\}$ and $C \not\subseteq G_i$ for each $i \in I$. Then, as in 2), the group L_C contains two distinct non-dihedral periods A and B. By [**8**, Lemma 34.9], $C_G(A) = gp\{A\}$ and $C_G(B) = gp\{B\}$, and by [**8**, Lemma 34.7], $gp\{A\} \cap gp\{B\} = \{1\}$. Hence $C_G(L_C) = \{1\}$.

It is obvious that $N_G(L_C) \supseteq R_C$. Let $X \in N_G(L_C)$. Then $XAX^{-1} \in L_C$, where A is a non-dihedral period from L_C, and by [**6**, Lemma 3] and assertion 7 of Theorem A, $X \in R_C$. Thus $N_G(L_C) = R_C$.

5) In order to prove that a subgroup L_C is simple if $C \not\subseteq G_i$ for each $i \in I$, we repeat the argument in the proof of the simplicity of the subgroup L in [**6**, Theorem A] and consider the additional case when M is a normal subgroup of L_C, M is infinite dihedral and not contained in G_i for each $i \in I$. Then by the proofs of [**8**, Theorem 35.1] and assertion 4 of Theorem A, M contains a power A^t of a period A and it follows from Lemma 3.7 that $gp\{A^t\}$ is a normal subgroup of R_C, since otherwise M contains two infinite cyclic subgroups having the trivial intersection, which contradicts the choice of M. Hence it follows from [**8**, Lemma 34.9] that the group R_C is infinite dihedral, and we arrive at a contradiction to the fact that if B and E are distinct periods of some ranks such that $B, E \in R_C$, then by [**8**, Lemma 34.7], the groups $gp\{B\}$ and $gp\{E\}$ are infinite cyclic subgroups of R_C and $gp\{B\} \cap gp\{C\} = \{1\}$.

6) Assertion 11 of Theorem A follows immediately from [**8**, Lemma 34.10].

7) If a subgroup M of G is infinite dihedral and not conjugate in G to a subgroup

of some group G_i, $i \in I$, then we may assume that M contains a power A^t of a period A and A^t is a product in G of involutions X and Y. By [**8**, Lemma 34.9], $C_G(A^t) = gp\{A\}$ and it follows from [**8**, Lemma 34.10] that $C_G(M) = \{1\}$.

Let $Z \in N_G(M)$. Then by [**8**, Lemma 34.7], $ZA^tZ^{-1} = A^l$ for some integer l, since every element of M is either a power of A or an involution. It follows from [**8**, Lemma 25.17] (with the correction from [**7**]) that $t = \pm l$, and by Lemma 3.4, we have that either $Z \in gp\{A\}$ or $ZAZ^{-1} = A^{-1}$ in G. In the second case, it follows from [**8**, Lemma 34.9] that $ZX^{-1} \in gp\{A\}$, hence $N_G(M)$ is an infinite dihedral group.

8) If a cyclic subgroup $M = gp\{A\}$ of G is not conjugate in G to a subgroup of some group G_i, $i \in I$, and A is not a product of two involutions in G, then we may assume that A is a power B^t of a non-dihedral period B. Then by [**8**, Lemma 34.9 and Lemma 34.7], $N_G(M) = C_G(M) = gp\{B\}$.

It remains to prove assertion 5 of the theorem. Let ψ be an automorphism of a subgroup L_C, where $C \not\subseteq G_i$ for each $i \in I$.

LEMMA 3.8. *The element $\psi(a)$ is not free in G for each $a \in \Omega_1 \cap C$.*

PROOF: Assuming the contrary (and multiplying ψ by an inner automorphism of R_C), we have that $\psi(a) = A^t$ for some period A. Then by [**8**, Lemma 34.7], a is of infinite order in G. Let b be an arbitrary element of C such that $\{a, b\} \not\subseteq G_i$ for each $i \in I$, and also let $\psi(b) = T$. By raising a to a suitable power, we may assume that $t > 100\zeta^{-1}$. It follows from [**8**, Lemma 34.10] that $bab^{-1} \neq a^{\pm 1}$, hence $TAT^{-1} \neq A^{\pm 1}$. By Lemma 3.2 and [**6**, Lemma 3], we obtain (after multiplying ψ by an inner automorphism of R_C) that $\psi([a, b]) = S^l$ and $\psi(a) = B$, where S is a period, $BS^lB^{-1} \neq S^{\pm l}$ and $F(\{B\}) \subseteq F(\{S\})$.

There is $k \geqslant 1$ such that $k\,|l| > n$ and $|B| < \zeta\iota(k\,|l|\,|S|)^{1/3}$, and by Lemma 3.5 and [**6**, Lemma 3], we have (after multiplying ψ by an inner automorphism of R_C) that $\psi\big(a^{-1}A_0a = [a,b]^k a[a,b]^{2k} a[a,b]^{3k} a\big) = S_1^r$ and $\psi(a) = T_1$, where S_1 is a non-dihedral period, $T_1 S_1^r T_1^{-1} \neq S_1^{\pm r}$ and $F(\{T_1\}) \subseteq F(\{S_1\})$.

Now we may choose m such that $mr > 0$, $|m| > n^6$ and $|T_1| < \iota(mr\,|S_1|)^{1/3}$. Then it follows from Lemma 3.6, [**6**, Lemma 3] and the proof of Lemma 3.1 (after multiplying ψ by an inner automorphism of R_C) that $\psi\big(a^{-1}A_ma\big) = \psi\big(a\big(a^{-1}A_0a\big)^m\big) = E$ and $\psi(a) = T_2$, where E is a non-dihedral period, $E \in L_C$, $|T_2| < 3\,|E|$ and $F(\{T_2\}) \subseteq F(\{E\})$. It follows from [**6**, Lemma 1] and [**8**, Theorem 22.4] that (after multiplying ψ by an inner automorphism of R_C) there is $T_3 \in Y_E$ such that $T_2 = T_3E^p$, where $|p| \leqslant 4$. Applying the automorphism ψ to both sides of the defining relation (2.1) for A_m, a and $k = 10 - p$, we obtain that

$$T_2^{-1}E^nT_3E^{n+10}\dots T_3E^{n+30(h-2)+10} = 1,$$

and it follows from the definition of the relations (2.1), (2.4) and (2.5) for E and T_3

that T_2 is not a free element of G. But T_2 is conjugate in G to a power of A, and this contradiction completes the proof of the lemma. ∎

Lemma 3.9. *There exists $X \in R_C$ with the property that $\psi(G_i \cap L_C) = X(G_{k(i)} \cap L_C)X^{-1}$, $k(i) \in I$, for each $i \in I$ and if $c \in \Omega_1 \cap C$, $e \in C$ and $d \in C \cup \{1\}$ such that $\{c, e\} \not\subseteq G_i$ for each $i \in I$, then $\psi(c) = Xc_1X^{-1}$ and $\psi(dece^{-1}d^{-1}) = Xd_1e_1c_1e_1^{-1}d_1^{-1}X^{-1}$ for some $c_1 \in \Omega_1 \cap C$, $e_1 \in C$ and $d_1 \in C \cup \{1\}$.*

Proof: We may assume that in the statement of the lemma $de \in \Omega_2$ and $d \notin gp\{c\}$. It follows from assertion 4 of Theorem A that for each c, d and e from the statement of the lemma, there exist $f \in C \cup \{1\}$ and $g \in C$ such that $\{c, g\} \not\subseteq G_i$ for each $i \in I$, $fg \in \Omega_2$, $de \neq fg$ and $fge^{-1}d^{-1} \neq c$ in the case $c^2 = 1$. By Lemma 3.8, we have (after multiplying ψ by an inner automorphism of R_C) that $\psi([c, de]^n) = S^k$ and $\psi([c, fg]) = W$, where S is a period, $|k| > 100\zeta^{-1}$ and W is a minimal word in G such that $WS^kW^{-1} \neq S^{\pm k}$, since $[c, fg][c, de]^n[c, fg]^{-1} \neq [c, de]^{\pm n}$ in G. Then by Lemma 3.2 and [**6**, Lemma 3], we obtain (after multiplying ψ by an inner automorphism of R_C) that $\psi((cfg)B_0(cfg)^{-1}) = A^l$ and $\psi([c, de]^n) = B$, where A is a non-dihedral period, $BA^lB^{-1} \neq A^{\pm l}$ and $F(\{B\}) \subseteq F(\{A\})$, since B_0 is a non-dihedral period of G.

We choose an integer m such that $ml > 0$, $|m| > n^6$ and $|B| < \iota(ml\,|A|)^{1/3}$. Then it follows from Lemma 3.6, [**6**, Lemma 3] and the proof of Lemma 3.1 (after multiplying ψ by an inner automorphism of R_C) that either $\psi([c, de]^n) = [u, vy]^n$ in G for some $u \in \Omega_1 \cap C$, $y \in C$ and $v \in C \cup \{1\}$ such that $\{u, y\} \not\subseteq G_s$ for each $s \in I$ and $vy \in \Omega_2$, or $\psi((cfg)B_m(cfg)^{-1}) = E$ and $\psi([c, de]^n) = T$, where E is a non-dihedral period, $E \in L_C$, $|T| < 3|E|$ and $F(\{T\}) \subseteq F(\{E\})$. It follows from [**6**, Lemma 1] and [**8**, Theorem 22.4] that (after multiplying ψ by an inner automorphism of R_C) there is $T_1 \in Y_E$ such that $T = T_1E^p$, where $|p| \leqslant 4$. Applying the automorphism ψ to both sides of the defining relation (2.2) (conjugated by the element cfg) for $(cfg)B_m(cfg)^{-1}, [c, de]^n$ and $k = 10 - p$, we obtain that

$$T^{-1}E^nT_1E^{n+20}\dots T_1E^{n+30(h-2)+20} = 1,$$

and it follows from the definition of the relations (2.2), (2.6) and (2.7) for E and T_1 that again $\psi([c, de]^n) = [u, vy]^n$ for some $u \in \Omega_1 \cap C$, $y \in C$ and $v \in C \cup \{1\}$ such that $\{u, y\} \not\subseteq G_s$ for each $s \in I$ and $vy \in \Omega_2$.

By Lemma 3.8 [**8**, Lemma 34.7], we have that $\psi([c, de]) = [u, vy]$. It follows from Lemma 3.8 that (after multiplying ψ by an inner automorphism of R_C) $\psi(c) = c_1$ and $\psi(dece^{-1}d^{-1}) = Uc_1U^{-1}$ for some $c_1 \in \Omega_1 \cap C$ and $U \in R_C$. Then there is a reduced circular diagram Δ (of some rank) for the conjugacy of $[c_1, U]$ to $[u, vy]$. Pasting together the subpaths with labels U and U^{-1}, we arrive at a diagram Δ' on a sphere with three holes with contour labels equal to c_1, c_1^{-1} and $[u, vy]^{-1}$. The removal of

j-pairs from Δ' gives a reduced diagram Δ_0. By [8, Theorem 22.1], $r(\Delta_0) = 0$. Hence we obtain that $c_1 = u^\delta$ and $U = Z_1(vy)^\delta Z_2$, where $|\delta| = 1$, $Z_i \in G$ and $[c_1, Z_i] = 1$ for $i = 1,2$. Therefore, we have that $\psi(c) = c_1$ and $\psi\big(dece^{-1}d^{-1}\big) = Zd_1e_1c_1e_1^{-1}d_1^{-1}Z^{-1}$ for some $e_1 \in C$ and $d_1 \in C \cup \{1\}$ such that $d_1e_1 \in \Omega_2$, where $Z \in G$, $[c_1, Z] = 1$ and $Z = Z_{d,e}$ depends, in general, on the choice of e and d. Moreover, by [8, Lemma 34.10], $\{c_1, Z_{d,e}\} \subseteq G_s$ for some $s \in I$.

Suppose that $Z_{d,e} \neq Z_{f,g}$ for some $f \in C \cup \{1\}$ and $g \in C$ such that $fg \in \Omega_2$ and $f \notin gp\{c\}$. Then we may assume that $fge^{-1}d^{-1} \notin gp\{c\}$, and by the previous considerations, $\psi([c,de]^n[[c,de]^n,[c,fg]]^m)$, $|m| > n^6$, is conjugate in G to $[c_1,d_1e_1]^n[[c_1,d_1e_1]^n,[c_1,kl]]^{m_1}$, where $|m_1| > n^6$, $k \in C \cup \{1\}$, $l \in C$, $\{c_1,l\} \not\subseteq G_s$ for each $s \in I$ and $kl \in \Omega_2$. It follows from the proof of Lemma 3.1 and the choice of defining relations in G that $[Z_{d,e}[c_1,d_1e_1]^n Z_{d,e}^{-1}, Z_{f,g}[c_1,f_1g_1]Z_{f,g}^{-1}]$ is conjugate to $[[c_1,d_1e_1]^n,[c_1,kl]]$ in the group $G(1)$, which is impossible in our case. Thus we have that $Z_{d,e} = Z_c$ for each $d \in C\cup\{1\}$ and $e \in C$ such that $de \in \Omega_2$, where $\{Z_c, c_1\} \subseteq G_s$ for some $s \in I$ and $[c_1, Z_c] = 1$.

We note that if $a \in G_i \cap L_C$ and $\psi(a) \in G_{k(i)} \cap L_C$ for some $i, k(i) \in I$, then $\psi(G_i \cap L_C) = G_{k(i)} \cap L_C$, since by Lemma 3.8 and the proof of [8, Theorem 35.1] if $b \in G_i \cap L_C$ and $\psi(b) \notin G_{k(i)}$, then $\psi(ab)$ is not conjugate to an element of Ω, contradicting Lemma 3.8.

We may assume (after multiplying ψ by an inner automorphism of R_C) that $\psi(c) = c_1$ and $Z_c = 1$. It remains to prove that $\psi(e) \in \Omega_1 \cap C$ and $Z_e = 1$ for each $e \in \Omega_1 \cap C$. Let $e \in \Omega_1 \cap C$ such that $\{c,e\} \subseteq G_s$ for some $s \in I$ and $e \neq c$, and also let d be an arbitrary element of C with $\{e,d\} \not\subseteq G_i$ for each $i \in I$. By the previous considerations, $\psi(e) = e_1 \in \Omega_1 \cap C$. Then

$$\psi\big(dced^{-1}\big) = \psi\big(dcd^{-1}\big)\psi\big(ded^{-1}\big) = d_1c_1d_1^{-1}Z_ed_2e_1d_2^{-1}Z_e^{-1}$$

for some $d_1, d_2 \in C$, hence $Z_e = 1$, since otherwise $\psi\big(dced^{-1}\big)$ is a free element of G and we arrive at a contradiction to Lemma 3.8.

Now we consider the case when e is an arbitrary element of $\Omega_1 \cap C$ such that $\{c,e\} \not\subseteq G_i$ for each $i \in I$ (if such an e exists). Repeating the considerations from the beginning of the proof of the lemma for an element $[c,e]$, we obtain that $\psi(e) = e_1Z$ for some $e_1 \in \Omega_1 \cap C$ and $Z \in G$ such that $[c_1, Z] = 1$. By [8, Lemma 34.10], $\{Z, c_1\} \subseteq G_s$ for some $s \in I$, hence $Z = 1$, since otherwise $\psi(e)$ is a free element of G, which contradicts Lemma 3.8. Similary, we have that $\psi(c) = Z_ec_2Z = c_1$, where $\{Z_e, Z, e_1\} \subseteq G_j$ for some $j \in I$ and $c_2 \in C$, which is possible only if $Z_e = Z = 1$ and $c_1 = c_2$.

The proof of the lemma is complete. ∎

Lemma 3.10. *There exists $X \in R_C$ such that $\psi(a) = XaX^{-1}$ for each $a \in$*

$\Omega_1 \cap C$.

PROOF: By Lemma 3.9, we have (after multiplying ψ by an inner automorphism of R_C) that $\psi(G_i \cap L_C) = G_{k(i)} \cap L_C, k(i) \in I$, for each $i \in I$. Let s be the minimal element of I such that $G_i \cap L_C \neq \{1\}$ and $s \neq k(s)$. Hence $s < k(s)$ and there exists $p \in I$ such that $k(p) = s$ with $p > s$. Let c and e be arbitrary nontrivial elements of $G_s \cap L_C$ and $G_p \cap L_C$, respectively. By assertion 4 of Theorem A, the group L_C is not dihedral and there are $f \in C \cup \{1\}$ and $g \in C$ such that $\{c, g\} \not\subseteq G_j$ for each $j \in I$, $fg \in \Omega_2$, $e \neq fg$ and $fge^{-1} \neq c$ in the case $c^2 = 1$. Let $\psi(c) = c_1$, $\psi(e) = e_1$ and $\psi(fgcg^{-1}f^{-1}) = f_1g_1c_1g_1^{-1}f_1^{-1}$. Then $c < e$ and $c_1 > e_1$, and applying the automorphism ψ to both sides of the defining relation (2.3) (conjugated by the element cfg) for $(cfg)B_m(cfg)^{-1} = [c,e]^n[[c,e]^n,[c,fg]]^m$ and $T = [c,e]$, where $m > n^6$, we obtain that

$$c_1(B'_m)^n[c_1,e_1](B'_m)^{n+25}\dots[c_1,e_1](B'_m)^{n+30(h-2)+25} = 1,$$

where $B'_m = \psi((cfg)B_m(cfg)^{-1})$, and it follows from the definition of the relation (2.3) for $(c_1f_1g_1)^{-1}B'_m(c_1f_1g_1)$ and $T = (c_1f_1g_1)^{-1}[c_1,e_1](c_1f_1g_1)$ that $c_1 = e_1c_1e_1^{-1}$, which contradicts [**8**, Lemma 34.10]. Thus $\psi(G_i \cap L_C) = G_i \cap L_C$ for each $i \in I$.

Let c be the minimal element of $\Omega_1 \cap C$ such that $\psi(c) \neq c$. Hence $c < \psi(c) = c_1$, $c_1 \in \Omega_1 \cap C$, and there exists $d \in \Omega_1 \cap C$ such that $\psi(d) = c$ and $c < d$. By assertion 4 of Theorem A, there is $e \in C$ such that $\{c,e\} \not\subseteq G_i$ for each $i \in I$. Applying the automorphism ψ to both sides of the defining relation (2.4) (conjugated by the element ce) for $(ce)B_m(ce)^{-1} = [c,de]^n[[c,de]^n,[c,e]]^m$ and $T = c$, where $m > n^6$, we have that

$$(c_1e_1)c_1(c_1e_1)^{-1}(B'_m)^nc_1(B'_m)^{n+10}\dots c_1(B'_m)^{n+30(h-2)+10} = 1,$$

where $e_1c_1e_1^{-1} = \psi(ece^{-1})$ and $B'_m = \psi((ce)B_m(ce)^{-1})$. It follows from the definition of the relation (2.4) for $(c_1e_1)^{-1}B'_m(c_1e_1)$ and $T = (c_1e_1)^{-1}c_1(c_1e_1)$ that $c = c_1$, and this contradiction completes the proof of the lemma. ∎

LEMMA 3.11. *There exists $X \in R_C$ such that $\psi(dece^{-1}d^{-1}) = Xdece^{-1}d^{-1}X^{-1}$ for each $c \in \Omega_1 \cap C$, $e \in C$ with $\{c,e\} \not\subseteq G_i$ for each $i \in I$ and $d \in C \cup \{1\}$.*

PROOF: By Lemmas 3.9 and 3.10, we have (after multiplying ψ by an inner automorphism of R_C) that $\psi(dece^{-1}d^{-1}) = d_1e_1ce_1^{-1}d_1^{-1}$, where $e_1 \in C$, $\{e_1,c\} \not\subseteq G_i$ for each $i \in I$ and $d_1 \in C \cup \{1\}$. We may assume that $de, d_1e_1 \in \Omega_2$.

Let de be the minimal element of Ω_2 such that $dece^{-1}d^{-1} \in L_C$ and $\psi(dece^{-1}d^{-1}) = d_1e_1ce_1^{-1}d_1^{-1} \neq dece^{-1}d^{-1}$. Hence $de < d_1e_1$ and there exists $fg \in \Omega_2$ such that $fgcg^{-1}f^{-1} \in L_C$, $de < fg$ and $\psi(fgcg^{-1}f^{-1}) = dece^{-1}d^{-1}$. We have that $fge^{-1}d^{-1} \neq c$, since otherwise

$$dece^{-1}d^{-1} = \psi(fgcg^{-1}f^{-1}) = c\psi(dece^{-1}d^{-1})c^{-1} = cd_1e_1ce_1^{-1}d_1^{-1}c^{-1},$$

and by [**8**, Lemma 34.10], $\{e^{-1}d^{-1}cd_1e_1, c\} \subseteq G_s$ for some $s \in I$, which is impossible.

By applying the automorphism ψ to both sides of the defining relation (2.3) (conjugated by the element cfg) for $T = [c, de]^2$ and

$$(cfg)B_m(cfg)^{-1} = [c, de]^n[[c, de]^n, [c, fg]]^m,$$

where $m > n^6$, we obtain that

$$d_1e_1ce_1^{-1}d_1^{-1}(B'_m)^n[c, d_1e_1]^2(B'_m)^{n+25}\dots[c, d_1e_1]^2(B'_m)^{n+30(h-2)+25} = 1,$$

where $B'_m = \psi((cfg)B_m(cfg)^{-1})$, and it follows from the definition of the relation (2.3) for $(cde)^{-1}B'_m(cde)$ and $T = (cde)^{-1}[c, d_1e_1]^2(cde)$ that $d_1e_1ce_1^{-1}d_1^{-1} = dece^{-1}d^{-1}$, which contradicts our assumption.

The proof of the lemma is complete. ∎

Now assertion 5 of Theorem A about $\mathrm{Aut}\, L_C$ follows from Lemma 3.11 and assertion 6 of Theorem A. The assertion about regular automorphisms of L_C follows immediately from [**8**, Lemma 34.10].

The proof of Theorem A is complete. ∎

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