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ON LEVEL CROSSINGS FOR A GENERAL CLASS OF PIECEWISE-DETERMINISTIC MARKOV PROCESSES

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Abstract

We consider a piecewise-deterministic Markov process (X_t) governed by a jump intensity function, a rate function that determines the behaviour between jumps, and a stochastic kernel describing the conditional distribution of jump sizes. The paper deals with the point process N_+^b of upcrossings of some level b by (X_t). We prove a version of Rice's formula relating the stationary density of (X_t) to level crossing intensities and show that, for a wide class of processes (X_t), as $b \to \infty$, the scaled point process $(N_+^b(\nu_+(b)^{-1}t))$, where $\nu_+(b)$ denotes the intensity of upcrossings of b, converges weakly to a geometrically compound Poisson process.

Keywords: Level crossing; Rice's formula; compound Poisson limit theorem; piecewise-deterministic Markov process

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1. Introduction

We consider a real-valued piecewise-deterministic Markov process $(X_t)_{t\geq 0}$ whose distribution is determined by a drift coefficient $\mu\colon \mathbb{R} \to \mathbb{R}$, a jump intensity function $\lambda\colon \mathbb{R} \to [0, \infty)$, and a stochastic kernel $J(x, \mathrm{d}z)$ from $\mathbb{R}$ to $\mathbb{R}$. The process (X_t) is right continuous and jumps at (positive) epochs $T_1 < T_2 < \cdots$. Between the jumps, it moves along an integral curve determined by μ. We assume that μ is right continuous with left-hand limits and that $D_\mu := \{u\colon \mu(u) = 0\}$ is a locally finite set. The occurrence of jumps is governed by the stochastic jump intensity $\lambda(X_t)$. Given the nth jump epoch T_n, the conditional distribution of the size Z_n of the nth jump is $J(X_{T_n-}, \cdot)$, where X_{t-} is the value of the process just before $t > 0$. We will assume that the process is ergodic with invariant distribution π and refer to Appendix A for conditions guaranteeing ergodicity. It is then essentially well known [10], [31] that the stationary distribution π is absolutely continuous on $\mathbb{R} \setminus D_\mu$, and we let p denote its density. We note that π might have atoms in D_μ.

The process (X_t) is a generic model of applied probability. Special cases have been extensively studied in the literature. We just mention storage processes [14], [27], stress release models [7], [30], [31], queueing models [9], [27], and repairable systems [21]. It is mostly assumed that $J(x, \cdot)$ does not depend on $x \in \mathbb{R}$ and that the jumps are either only nonnegative or only nonpositive. An extensive discussion of several ergodicity properties for a constant

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(positive) μ and negative jumps is given in [19]. General properties of piecewise-deterministic Markov processes are studied in [11].

Now assume that X_0 has the distribution π. Then (X_t) is a *stationary process*, and the sequence (T_n) forms a *stationary point process*. We assume that the *intensity* of (T_n) (the expected number of points in an interval of unit length) is finite. Again, we refer to Appendix A for an explicit assumption that is sufficient for this finiteness.

We say that (X_t) has a (proper) *upcrossing* or *downcrossing* of level $u \in \mathbb{R}$ at time $s > 0$ if there is some $\delta > 0$ such that $X_t < u$ or, respectively, $X_t > u$ for $s - \delta \leq t < s$ and $X_t > u$ or, respectively, $X_t < u$ for $s < t \leq s + \delta$. If, in addition, $X_{s-} = X_s$ then we speak of a *continuous upcrossing* or, respectively, *downcrossing*. Otherwise, we speak of a *discontinuous upcrossing* or, respectively, *discontinuous downcrossing*. It is easy to see that the set of all continuous upcrossings and downcrossings forms a stationary point process N_c^u. Note that there are no continuous downcrossings of the level u in the case in which $\mu(u) > 0$ and no continuous upcrossings in the case in which $\mu(u) < 0$. The intensity of N_c^u is denoted by $\nu_c(u)$. As the intensity of (T_n) is assumed to be finite, it is easy to see that $\nu_c(u)$ is finite for any $u \in \mathbb{R}$.

Our first aim in this paper is to prove the following version of *Rice's formula*:

$$\nu_c(u) = |\mu(u)|p(u), \qquad u \notin D_\mu. \tag{1.1}$$

The simplicity of this result is striking. The formula can be explained by looking at the long-run proportion of time that (X_t) spends in an infinitesimal interval containing u. Equation (1.1) is a direct analog of the classical Rice formula [28], which holds for smooth processes and plays a rather important role in engineering. A rigorous treatment of Rice's formula is given in [22] and a more recent discussion is given in [24]. An analog of (1.1) for (discontinuous) Poisson shot noise processes has been studied in [4].

Let $\nu_{+,d}(b)$ and $\nu_{-,d}(b)$ respectively denote the intensities of discontinuous upcrossings and downcrossings of level b. Our proof of (1.1) uses the simple relation $\nu_c(u) = |\nu_{+,d}(u) - \nu_{-,d}(u)|$; see Lemma 3.1. In fact, (1.1) can be rewritten as

$$\nu_{-,d}(u) - \nu_{+,d}(u) = \mu(u)p(u), \qquad u \notin D_\mu. \tag{1.2}$$

Such equalities for level-crossing intensities are widely used in queueing theory. Here we refer the reader to the early reference [8] and the survey [12]. It is quite remarkable that the queueing literature does not take notice of the close relationship between (1.2) and the results in [28] (or [4]). Equation (1.2) is mostly derived for Poisson-driven models. In principle, the level-crossing method can also be applied in more general cases (see, e.g. [12]). There are, however, many implicit model assumptions that make a direct derivation of (1.1) nontrivial. So, to the best of the authors' knowledge, result (1.1) must be considered as new. Moreover, we will establish this formula under a minimal set of assumptions. In particular, the existence of the stationary density need not be assumed, but is a consequence of our model assumptions. Though assumed for convenience, even ergodicity is not needed.

Our second and main aim in this paper is to derive limit results for the point process N_+^b of all upcrossings of level $b \to \infty$. Whenever the intensity $\nu_+(b)$ of N_+^b is positive, we introduce the scaled point process $M^b(t) := N_+^b(\nu_+(b)^{-1}t)$, $t \geq 0$. It is stationary and has intensity 1. Under our assumptions (see the scenarios below), (1.1) will imply that the intensity $\nu_+(b)$ can be explicitly expressed as

$$\nu_+(b) = |\mu(b)|p(b) \tag{1.3}$$

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for all sufficiently large b. We refer the reader to Section 4 for more detail. We will study the limiting behaviour of M^b under the following three scenarios and some additional assumptions, see (4.4)–(4.6).

Scenario 1.1. We have $\mu(y) \to -\infty$ as $y \to \infty$, and there exists a $u_0 \in \mathbb{R}$ such that $J(x, (-\infty, 0)) = 0$ for $x \geq u_0$ (no negative jumps from states $x \geq u_0$).

Scenario 1.2. We have $\lambda(y) \to \infty$ as $y \to \infty$, $\mu(y)$ is positive for all sufficiently large y, and $J(x, (0, \infty)) = 0$ for all $x \in \mathbb{R}$ (no positive jumps).

Scenario 1.3. As $y \to \infty$, we have $\mu(y) \to \mu(\infty) \in \mathbb{R} \setminus \{0\}$ and $\lambda(y) \to \lambda(\infty) \in [0, \infty)$. In the case in which $\mu(\infty) < 0$, there exists a $u_0 \in \mathbb{R}$ such that $J(x, (-\infty, 0)) = 0$ for $x \geq u_0$, and in the case in which $\mu(\infty) > 0$ we have $J(x, (0, \infty)) = 0$ for all $x \in \mathbb{R}$. Moreover, $J(y, \cdot)$ converges weakly, as $y \to \infty$, to a probability measure $J(\infty, \cdot)$ on $\mathbb{R}$.

In the first two scenarios the point process M^b will converge, as $b \to \infty$, in distribution to a Poisson process. The explanation of this phenomenon is quite simple. Fixing a level $u > u_0$, the trajectory of the process (X_t) can be split into independent and identically distributed (i.i.d.) cycles between the successive continuous crossings of this level. Then hitting a high level b during a particular cycle will be a 'rare event'. In both scenarios, with a probability arbitrarily close to 1 for large enough b, given that the level b was exceeded during a cycle, there is exactly one upcrossing of that level during this cycle.

In the third scenario the limiting behaviour of M^b is slightly more complicated. The crossing of a high level b is still a rare event. However, given that the level b was exceeded during a cycle, the conditional distribution of the number of continuous crossings of that level during this cycle will be geometric with a parameter that converges, as $b \to \infty$, to some number $\rho \in (0, 1)$. Therefore, the limit is a *geometrically compound Poisson process* Π_ρ, which is defined as follows. Each point of a homogeneous Poisson process of intensity $(1 - \rho)$ gets (independently of the other points) a mass $k \in \{1, 2, \dots\}$ with probability $(1 - \rho)\rho^{k-1}$. The resulting stationary point process Π_ρ has *independent increments* and geometrically distributed multiplicities. As the above geometric distribution has mean $1/(1 - \rho)$, the intensity of Π_ρ is 1.

For Gaussian processes with smooth trajectories, it is well known that the point process of the times of crossing a high level are asymptotically Poisson; see, e.g. [18], [25], and the references therein. To the best of the authors' knowledge, the present paper is the first to establish such a limit theorem for jump processes. Compound Poisson limits for exceedances of a high level by a stationary Markov chain were obtained in [29]; such limits for exceedances and also for upcrossings of sequences are summarised in [13]. A general compound Poisson limit theorem for stationary, strongly mixing random measures was derived in [23]. We use an approach based on the abovementioned cycles and a modified version of the argument employed in [29] to analyse exceedances in general Markov processes. An immediate consequence of our main theorem is that the first time of crossing a high level is asymptotically exponentially distributed. A discussion of this well-known phenomenon can be found, for instance, in [1] and Section VI.4 of [2]. However, in the present framework the result seems to be new.

This paper is organised as follows. Section 2 contains the detailed definition of the process as well as some of its fundamental properties. Section 3 provides the proof of Rice's formula. The Poisson limit theorem is the topic of Section 4.

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2. Definition and basic properties of the process

We consider a right-continuous function $\mu\colon \mathbb{R} \to \mathbb{R}$ with left-hand limits such that the set D_μ of 0s of μ is locally finite. We assume that $\mu(u-)$ has the same sign as $\mu(u)$ whenever $\mu(u) \neq 0$. We also assume that, for any $x \in \mathbb{R}$, there exists a unique continuous function $q(x, \cdot)\colon [0, \infty) \to \mathbb{R}$ satisfying the integral equation

$$q(x,t) = x + \int_0^t \mu(q(x,s))\, \mathrm{d}s, \qquad t \geq 0. \tag{2.1}$$

The jump intensity λ is assumed to be measurable, locally bounded, and such that

$$\int_0^\infty \lambda(q(x,s))\, \mathrm{d}s = \infty, \qquad x \in \mathbb{R}. \tag{2.2}$$

For the jump distribution, we assume that $J(x, \{0\}) = 0$ for all $x \in \mathbb{R}$ (see also Remark 2.1, below).

Formally, our process (X_t) is defined as follows. We consider a measurable space $(\Omega, \mathcal{F})$ that is rich enough to carry a *marked point process* $\Phi = ((T_n, Z_n))_{n\geq 1}$ on $[0, \infty)$ with real-valued random variables (marks) Z_n and a real-valued random variable X_0. Between the jumps, the process is defined by $X_t := q(X_0, t)$ on $[0, T_1)$ and $X_t = q(X_{T_n}, t - T_n)$ on $[T_n, T_{n+1})$, $n \geq 1$. At the jump epochs T_n, we have $X_{T_n} := X_{T_n-} + Z_n$, where we note that $X_{T_n-} = q(X_{T_{n-1}}, T_n - T_{n-1})$. Finally, we define $X(t) := \Delta$ for $t \geq T_\infty$, where Δ is a point external to $\mathbb{R}$ and $T_\infty := \lim_{n\to\infty} T_n$.

For any probability measure σ on $\mathbb{R}$, we consider a probability measure P_σ on $(\Omega, \mathcal{F})$ such that $\mathrm{P}_\sigma(X_0 \in \cdot) = \sigma(\cdot)$ and the following properties hold. The conditional distribution of T_1 given X_0 is specified by

$$\mathrm{P}_\sigma(T_1 \leq t \mid X_0) = 1 - \exp\biggl[-\int_0^t \lambda(q(X_0,s))\, \mathrm{d}s\biggr] \quad \mathrm{P}_\sigma\text{-almost surely (a.s.).} \tag{2.3}$$

Similarly, we assume, for $n \geq 1$, that, P_σ-a.s.,

$$\mathrm{P}_\sigma(T_{n+1} - T_n \leq t \mid X_0, T_1, Z_1, \dots, T_n, Z_n) = 1 - \exp\biggl[-\int_0^t \lambda(q(X_{T_n},s))\, \mathrm{d}s\biggr]. \tag{2.4}$$

By (2.2), the jump epochs T_n are indeed all finite a.s. The conditional distributions of the jump sizes are given by

$$\mathrm{P}_\sigma(Z_{n+1} \in \cdot \mid X_0, T_1, Z_1, \dots, T_n, Z_n, T_{n+1}) = J(X_{T_{n+1}-}, \cdot) \quad \mathrm{P}_\sigma\text{-a.s.},\ n \geq 0. \tag{2.5}$$

Since $J(x, \{0\}) = 0,\ x \in \mathbb{R}$, we can assume that $Z_n(\omega) \neq 0$ for all $n \geq 1$ and $\omega \in \Omega$.

The conditional distribution of Φ given X_0 is now completely specified. Our assumptions imply that (X_t) is a homogeneous Markov process with respect to the family $\{\mathrm{P}_x : x \in \mathbb{R}\}$, where $\mathrm{P}_x := \mathrm{P}_{\delta_x}$ is the measure corresponding to the initial distribution supported by x. The expectations with respect to P_σ and P_x are denoted by E_σ and E_x, respectively. Actually, (X_t) is a piecewise-deterministic Markov process in the terminology of [11].

Remark 2.1. As $Z_n \neq 0$ for all $n \geq 1$, there is a one-to-one correspondence between Φ and (X_t). The former condition can be easily dispensed with by suitably augmenting the process (X_t).

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Remark 2.2. In many applications (queueing and dam models, and repairable systems) the process (X_t) is nonnegative, in the sense that $X_t \geq 0$ for all $t \geq 0$ whenever $X_0 \geq 0$. Such a situation can be accommodated by choosing the characteristics so that $(-\infty, 0)$ becomes *transient* for the process. A possible choice is $\mu(x) = 1$ and $\lambda(x) = 0$ for $x < 0$. Any stationary distribution of (X_t) is then concentrated on $[0, \infty)$.

Remark 2.3. We assumed that the solution $q(x, t)$ of (2.1) is defined for all $t \geq 0$. This could be generalised as follows. Suppose that, for any $x \in \mathbb{R}$, there is a $t_\infty(x) \in (0, \infty]$ such that $q(x, \cdot)$ is the unique continuous function on $[0, t_\infty(x))$ satisfying (2.1) for all $t \in [0, t_\infty(x))$. Assuming, instead of (2.2), that $\int_0^{t_\infty(x)} \lambda(q(x, s))\, \mathrm{d}s = \infty$, $x \in \mathbb{R}$, we can still use (2.3), (2.4), and (2.5) to define a marked point process Φ such that, a.s., $T_1 < t_\infty(X_0)$ and $T_{n+1} - T_n < t_\infty(X_{T_n})$, $n \geq 1$. Hence, we can define the Markov process (X_t) as before. All the results of this paper remain valid in this more general framework.

The next result provides the (generalised) *infinitesimal generator* of (X_t). Set

$$\tau_m := \inf\{t \geq 0 : |X(t)| \geq m\}, \qquad m \in \mathbb{N}.$$

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous and let its Radon–Nikodym derivative f', as well as the function $x \mapsto \lambda(x) \int (f(x+z) - f(x)) J(x, \mathrm{d}z)$, be locally bounded. Then, for any probability measure σ on $\mathbb{R}$,*

$$\begin{aligned} \mathrm{E}_\sigma\, f(X_{t\wedge\tau_m}) = \mathrm{E}_\sigma\, f(X_0) + \mathrm{E}_\sigma \int_0^{t\wedge\tau_m} f'(X_s)\mu(X_s)\, \mathrm{d}s \\ + \mathrm{E}_\sigma \int_0^{t\wedge\tau_m} \int_{\mathbb{R}} (f(X_s + z) - f(X_s))\lambda(X_s) J(X_s, \mathrm{d}z)\, \mathrm{d}s. \end{aligned} \tag{2.6}$$

Proof. Denote by $(\mathcal{F}_t)$ the filtration generated by X_0 and the restriction of Φ to $[0, t] \times \mathbb{R}$. Using basic results on marked point processes (see, e.g. Chapter 4 of [20]), we obtain, from (2.3), (2.4), and (2.5),

$$\mathrm{E}_\sigma \sum_{n=1}^{\infty} h(T_n, Z_n) = \mathrm{E}_\sigma \int_0^\infty \int_{\mathbb{R}} h(t, z)\lambda(X_t) J(X_t, \mathrm{d}z)\, \mathrm{d}t \tag{2.7}$$

for all *predictable* $h \colon \Omega \times [0, \infty) \times [0, \infty) \to [0, \infty)$. We can now proceed as in Section 8 of [19] to obtain the result.

We have to make two basic assumptions on the process. They will be discussed in Appendix A.

Assumption 2.1. *The process (X_t) is Harris ergodic. Its invariant distribution is denoted by π.*

In view of Remark 2.1, the marked point process Φ is stationary under P_π; see [3] for more detail on this stationarity. In particular, the distribution of $(N(t+s) - N(s))_{t\geq 0}$ does not depend on $s \geq 0$, where $N(t) := \mathrm{card}\{n \geq 1 \colon T_n \leq t\}$ is the number of jumps in the time interval $[0, t]$. The (stationary) *intensity* of N is defined by $\lambda_\pi := \mathrm{E}_\pi\, N(1)$.

Assumption 2.2. *We have $\lambda_\pi < \infty$.*

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Let $g\colon [0,\infty)\times\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be measurable. Applying (2.7) with $\sigma=\pi$ and $h(t,z):=g(t,X_{t-},z)$, and using Fubini's theorem, we obtain

$$\mathrm{E}_\pi\sum_{n=1}^{\infty} g(T_n, X_{T_n-}, Z_n)=\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} g(s,x,z)\lambda(x)J(x,\,\mathrm{d}z)\pi(\,\mathrm{d}x)\,\mathrm{d}s. \tag{2.8}$$

Choosing $g(s,x,z)=\mathbf{1}\{0\le s\le 1\}$, we obtain the equality in

$$\lambda_\pi=\int\lambda(x)\pi(\,\mathrm{d}x)<\infty. \tag{2.9}$$

A quick consequence of Proposition 2.1 is the following (basically well-known) integral equation for π.

Proposition 2.2. *Let $f\colon\mathbb{R}\to\mathbb{R}$ be bounded and differentiable with a continuous derivative f' that has a compact support. Then*

$$\int f'(x)\mu(x)\pi(\,\mathrm{d}x)=\iint\lambda(x)(f(x)-f(x+z))J(x,\,\mathrm{d}z)\pi(\,\mathrm{d}x).$$

Proof. The assumptions on f allow us to use (2.6). Because of Assumption 2.1, the process (X_t) is real valued and locally bounded. Hence, we have, P_π-a.s., $\tau_m\to\infty$ as $m\to\infty$. As f is bounded, the left-hand side of (2.6) converges to $\mathrm{E}_\pi f(X_t)=\mathrm{E}_\pi f(X_0)$. As f' has a compact support and μ is locally bounded, the second term on the right-hand side of (2.6) converges as well. For the third term, we can use (2.9) and bounded convergence to conclude that

$$0=\mathrm{E}_\pi\int_0^t f'(X_s)\mu(X_s)\,\mathrm{d}s+\mathrm{E}_\pi\int_0^t\int_{\mathbb{R}}\lambda(X_s)(f(X_s+z)-f(X_s))J(X_s,\,\mathrm{d}z)\,\mathrm{d}s.$$

Using Fubini's theorem and stationarity again, we obtain the assertion.

Some relationships between π and the stationary distribution of the imbedded process (X_{T_n}) can be found in [10].

3. Rice's formula

In this section we will prove the following assertion, establishing Rice's formula, (1.1).

Theorem 3.1. *Under Assumptions 2.1 and 2.2, the stationary distribution π has a density p on $\mathbb{R}\setminus D_\mu$ satisfying $\nu_c(u)=|\mu(u)|p(u)$ for all $u\notin D_\mu$.*

We prepare the proof with an auxiliary result. The point processes of discontinuous upcrossings and downcrossings are denoted by $N^u_{+,d}$ and $N^u_{-,d}$, respectively. In this section we take P_π to be the underlying probability measure. Then Φ is a stationary marked point process, and $N^u_{+,d}$ and $N^u_{-,d}$ are (jointly) stationary point processes. Their intensities are denoted by $\nu_{+,d}(u)$ and $\nu_{-,d}(u)$, respectively.

Lemma 3.1. *For any $u\in\mathbb{R}\setminus D_\mu$, we have $\nu_c(u)=\nu_{+,d}(u)-\nu_{-,d}(u)$ in the case in which $\mu(u)<0$ and $\nu_c(u)=\nu_{-,d}(u)-\nu_{+,d}(u)$ in the case in which $\mu(u)>0$.*

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Proof. Assume that $\mu(u) < 0$. (The argument for the other case is the same.) As the solution of (2.1) is unique and μ is right continuous, there are no continuous upcrossings of level u. Therefore, between any two successive downcrossings there must be exactly one discontinuous upcrossing of u. The point process of all continuous downcrossings is just given by N_c^u. Hence, we have, for any $t \geq 0$,

$$N_{+,d}^u(t) - 1 \leq N_{-,d}^u(t) + N_c^u(t) \leq N_{+,d}^u(t) + 1.$$

Taking expectations gives

$$\nu_{+,d}(u)t - 1 \leq \nu_{-,d}(u)t + \nu_c(u)t \leq \nu_{+,d}(u)t + 1.$$

Dividing by t and letting $t \to \infty$ yields the assertion.

Proof of Theorem 3.1. Let $u \in \mathbb{R}$ be such that $\mu(u) < 0$. Since $\mu(u-) < 0$ by our assumption on μ, we have

$$N_{+,d}^u = \{s > 0 \colon X_{s-} < u < X_s\}, \qquad N_{-,d}^u = \{s > 0 \colon X_{s-} \geq u \geq X_s,\ X_{s-} > X_s\}.$$

Applying (2.8) yields

$$\nu_{+,d}(u) = \iint \mathbf{1}\{x < u < x + z\}\lambda(x) J(x,\, \mathrm{d}z)\pi(\mathrm{d}x), \tag{3.1}$$

$$\nu_{-,d}(u) = \iint \mathbf{1}\{x \geq u \geq x + z\}\lambda(x) J(x,\, \mathrm{d}z)\pi(\mathrm{d}x). \tag{3.2}$$

Let f be a function satisfying the assumptions of Proposition 2.2. By (3.1) and (3.2) (and the corresponding formulae for $\mu > 0$), we have (giving $\nu_{-,d}(u) - \nu_{+,d}(u)$ some fixed value when $\mu(u) = 0$)

$$\begin{aligned}
&\int f'(u)(\nu_{-,d}(u) - \nu_{+,d}(u))\, \mathrm{d}u \\
&\quad = \iiint f'(u)\, \mathbf{1}\{x > u \geq x + z\}\, \mathrm{d}u \lambda(x) J(x,\, \mathrm{d}z)\pi(\mathrm{d}x) \\
&\qquad - \iiint f'(u)\, \mathbf{1}\{x + z > u \geq x\}\, \mathrm{d}u \lambda(x) J(x,\, \mathrm{d}z)\pi(\mathrm{d}x) \\
&\quad = \iint \mathbf{1}\{z < 0\}(f(x) - f(x + z))\lambda(x) J(x,\, \mathrm{d}z)\pi(\mathrm{d}x) \\
&\qquad - \iint \mathbf{1}\{z > 0\}(f(x + z) - f(x))\lambda(x) J(x,\, \mathrm{d}z)\pi(\mathrm{d}x) \\
&\quad = \iint (f(x) - f(x + z))\lambda(x) J(x,\, \mathrm{d}z)\pi(\mathrm{d}x).
\end{aligned}$$

Therefore, we obtain, from Proposition 2.2,

$$\int f'(u)(\nu_{-,d}(u) - \nu_{+,d}(u))\, \mathrm{d}u = \int f'(u)\mu(u)\pi(\mathrm{d}u).$$

The class of functions f' that are allowed in the above formula is rich enough to conclude first that π is absolutely continuous on $\mathbb{R} \setminus D_\mu$ and second that the density p satisfies

$$\mu(u)p(u) = \nu_{-,d}(u) - \nu_{+,d}(u)$$

for almost all $u \notin D_\mu$. Lemma 3.1 implies the assertion.

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Remark 3.1. The above proof shows that the density p satisfying Rice's formula, (1.1), has limits from the left and the right. The proof also shows how to construct a right-continuous version of this density. Such a version satisfies (1.1) with the exception of at most countably many $u \notin D_\mu$.

4. Asymptotics of level crossings

In this section we write $\mathrm{P} := \mathrm{P}_\pi$. Consider the point process N_+^b of all upcrossings of some level $b \in \mathbb{R}$ and let $\nu_+(b)$ denote its intensity (under P). It is given by

$$\nu_+(b) = \mathbf{1}\{\mu(b) > 0\}\nu_c(b) + \nu_{+,d}(b), \qquad b \notin D_\mu,$$

where we refer to the introduction for the definition of the intensities $\nu_c(b)$ and $\nu_{+,d}(b)$. From Lemma 3.1, (3.1), and (3.2), we obtain $\nu_+(b) \to 0$ as $b \to \infty$. If $\mu(x) < 0$ and $J(x, (-\infty, 0)) = 0$ for all $x \geq u_0$ (no negative jumps from levels above u_0), we have $\nu_{-,d}(b) = 0$ and $\nu_+(b) = \nu_{+,d}(b) = \nu_c(b)$ for $b \geq u_0$. If $J(x, [(u_0 - x)^+, \infty)) = 0$ for all $x \in \mathbb{R}$ (no positive jumps to levels above u_0) and $\mu(x) > 0$ for all $x \geq u_0$, we have $\nu_{+,d}(b) = 0$ and $\nu_+(b) = \nu_{-,d}(b) = \nu_c(b)$ for $b \geq u_0$. (Here $a^+ := \max\{a, 0\}$ denotes the positive part of a.) In either case, Theorem 3.1 implies that (1.3) holds for $b \geq u_0$. Whenever $\nu_+(b) > 0$, we introduce the following scaled point process M^b (on $[0, \infty)$):

$$M^b(t) := N_+^b(\nu_+(b)^{-1}t), \qquad t \geq 0.$$

In each of Scenarios 1.1–1.3 described in the introduction we will prove (under additional technical assumptions) the convergence

$$M^b \xrightarrow{\mathrm{w}} \Pi_\rho \quad \text{as } b \to \infty, \tag{4.1}$$

where '$\xrightarrow{\mathrm{w}}$' denotes weak convergence of point processes (see, e.g. [15, p. 316]), under the probability measure P, where $\rho \in [0, 1)$ is explicitly determined by the characteristics of (X_t) (see Theorem 4.1, below) and the geometrically compound Poisson process Π_ρ was defined in the introduction. If $\rho = 0$ then Π_ρ is a unit-rate Poisson process. Actually, we will prove the weak convergence of $\mathrm{P}_\sigma(M^b \in \cdot)$ for an arbitrary initial distribution σ.

In Scenarios 1.1 and 1.2 we assume that the jumps (from high enough levels) in the respective processes are dominated in distribution. This means that there exists a $u_0 \in \mathbb{R}$ and a family of nonincreasing (right-continuous) functions $(\bar{H}(u, \cdot))_{u \geq u_0}$ such that $\sup_{x \geq u} J(x, (z, \infty)) \leq \bar{H}(u, z)$ for all $z \in \mathbb{R}$ and $u \geq u_0$. In other words, denoting by $\xi(x)$ a generic random variable (RV) with the distribution $J(x, \cdot)$, this means that there exist RVs $\overline{\xi}(u)$ such that

$$\xi(x) \overset{\mathrm{D}}{\leq} \overline{\xi}(u), \qquad x \geq u \geq u_0, \tag{4.2}$$

where '$\overset{\mathrm{D}}{\leq}$' denotes the usual stochastic order. We assume that $\mathrm{E}\,\overline{\xi}(u) < \infty$. In Scenario 3 we will assume in addition that there exist RVs $\underline{\xi}(u)$ such that

$$\underline{\xi}(u) \overset{\mathrm{D}}{\leq} \xi(x), \qquad x \geq u \geq u_0. \tag{4.3}$$

Furthermore, set

$$\overline{\mu}(u) := \sup_{x \geq u} \mu(x), \qquad \overline{\lambda}(u) := \sup_{x \geq u} \lambda(x), \qquad \underline{\lambda}(u) := \inf_{x \geq u} \lambda(x).$$

Next we will make Scenarios 1.1–1.3 more precise.

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Scenario 4.1. We have $\mu(y) \to -\infty$ as $y \to \infty$, and there exists a $u_0 \in \mathbb{R}$ such that (4.2) holds and $J(x, (-\infty, 0]) = 0$ for all $x \geq u_0$. Moreover,

$$\mathrm{E}\,\overline{\xi}(u_0) + \frac{\overline{\mu}(u_0)}{\overline{\lambda}(u_0)} < 0. \tag{4.4}$$

Scenario 4.2. We have $\lambda(y) \to \infty$ as $y \to \infty$. Furthermore, there is a $u_0 \in \mathbb{R}$ such that (4.2) holds, $\mu(y) > 0$ for all $y \geq u_0$, $J(x, [(u_0 - x)^+, \infty)) = 0$ for all $x \in \mathbb{R}$, and

$$\mathrm{E}\,\overline{\xi}(u_0) + \frac{\overline{\mu}(u_0)}{\underline{\lambda}(u_0)} < 0, \tag{4.5}$$

where we can assume, without loss of generality, that $\overline{\xi}(u_0) < 0$.

Scenario 4.3. As $y \to \infty$, we have $\mu(y) \to \mu(\infty) \in \mathbb{R} \setminus \{0\}$ and $\lambda(y) \to \lambda(\infty) \in [0, \infty)$. There is some $u_0 \in \mathbb{R}$ such that (4.2) and (4.3) hold, $J(x, (-\infty, 0)) = 0$ for all $x \geq u_0$ in the case in which $\mu(\infty) < 0$, and $J(x, [(u_0 - x)^+, \infty)) = 0$ for all $x \in \mathbb{R}$ in the case in which $\mu(\infty) > 0$. Furthermore, we have, for $y \to \infty$, $\overline{\xi}(y), \underline{\xi}(y) \xrightarrow{\mathrm{w}} \xi(\infty)$, where $\xi(\infty)$ is an integrable RV satisfying

$$\mathrm{E}\,\xi(\infty) + \frac{\mu(\infty)}{\lambda(\infty)} < 0. \tag{4.6}$$

Remark 4.1. Each of the inequalities (4.4)–(4.6) implies the second ergodicity condition (A.2), below. In case of (4.4) and (4.5) this is due to the monotonicity properties of $\overline{\xi}(u)$, $\overline{\mu}$, $\overline{\lambda}$, and $\underline{\lambda}$.

Example 4.1. As an example of Scenario 4.1, consider a process (X_t) governed by the equation

$$\mathrm{d}X_t = -\beta X_t \,\mathrm{d}t + \mathrm{d}Y_t,$$

where $\beta > 0$ is constant and (Y_t) is a compound Poisson process (with intensity λ and i.i.d. jumps $Z_j > 0$). Such processes appear in a number of important applications and are known under several names (in particular, as storage processes with linear release functions [14] and as outputs of shot-noise excited filters with exponential memory; see, e.g. [16]). As is well known, (X_t) is ergodic if and only if $\mathrm{E}\ln(1 \vee Z_1) < \infty$, and if this is the case, there is a simple closed-form expression for the Laplace transform of its stationary density (in particular, if the Z_js are exponentially distributed with parameter α then the stationary distribution π is gamma with parameters λ and α); see, e.g. [6].

Also, observe that, for a somewhat more general process of the form

$$X_t = \sum_{n\colon t \leq T_n} h_n(t - T_n),$$

where (T_n, h_n) is a stationary marked point process (with rate λ for the underlying point process N and marks h_n taking values in the space of nonincreasing, nonnegative functions), the intensity of level upcrossings was given in [17]. It has a closed form in terms of the Palm probability measure P^0 of P with respect to N and can equivalently be written as

$$\nu_+(u) = \lambda\,\mathrm{P}^0(u - h_0(0) < X_{0-} \leq u), \qquad u > 0.$$

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Example 4.2. Scenario 4.2 can be illustrated by the so-called stress release processes used in statistical seismology (see, e.g. [7], [30], and [31]). The model assumes that $\mu(y) \equiv \mu$ and $\lambda(y) = \exp[\beta(y - y_0)]$, where $\mu, \beta > 0$ and $y_0 \in \mathbb{R}$ are all constants, and that $Z_n < 0$ are i.i.d. RVs (usually assumed to follow a truncated Pareto distribution). Thus, the constructed stress process (X_t) is always ergodic, the characteristic function of its stationary density having a closed-form infinite product representation [7].

Example 4.3. Processes covered by Scenario 4.3 appear in queueing theory and also as storage processes. For instance, we can consider a work-modulated single-server queue, (X_t), being the workload process (for a formal definition and detailed treatment of such a process, we refer the reader to the discussion of 'Model 2' in [9]; see also [27]). Then we will have $\mu(y) \leq 0$, $y > 0$, and $\mu(0) = 0$. The asymptotic homogeneity conditions from the scenario mean that, under heavy load, the dependence of the arrival and service processes on the load asymptotically vanishes.

Theorem 4.1. *Let the assumptions of either Scenario 4.1, 4.2, or 4.3 be satisfied, and assume that $\nu_c(b) > 0$ for all sufficiently large b. Then $\mathrm{P}_\sigma(M^b \in \cdot)$ converges for any distribution σ on $\mathbb{R}$ weakly to $\mathrm{P}(\Pi_\rho \in \cdot)$ as $b \to \infty$. The number ρ is given by $\rho = 0$ in Scenarios 4.1 and 4.2, and in Scenario 4.3 by*

$$\rho = \begin{cases} -\dfrac{\lambda(\infty)}{\mu(\infty)}\,\mathrm{E}\,\xi(\infty) & \text{if } \mu(\infty) < 0, \\ 1 - \dfrac{w\mu(\infty)}{\lambda(\infty)} & \text{if } \mu(\infty) > 0, \end{cases} \tag{4.7}$$

where w is the only positive solution of the equation

$$\mathrm{E}\,\mathrm{e}^{w\xi(\infty)} = 1 - \frac{w\mu(\infty)}{\lambda(\infty)}.$$

Remark 4.2. The ergodicity assumption in Theorem 4.1 (see Assumption 2.1) has been made for technical convenience. It would have been enough to assume the existence of a stationary distribution π such that the positivity assumption made in the theorem is satisfied. The theorem then still holds for all distributions σ whose support is 'attracted' by the support of π.

Remark 4.3. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing, continuously differentiable function such that $g(x) \to \infty$ as $x \to \infty$. Then $(X_t^g) := (g(X_t))$ is again a piecewise-deterministic Markov process, as defined in Section 2. The characteristics of (X_t^g) are given by $\mu^g(y) = g'(g^{-1}(y))\mu(g^{-1}(y))$, $\lambda^g(y) = \lambda(g^{-1}(y))$, and $J^g(y, \cdot) = J(g^{-1}(y), g^{-1}(y + \cdot) - g^{-1}(y))$. If the point processes of upcrossings defined in terms of (X_t^g) satisfy a compound limit theorem, as in Theorem 4.1, then so do the corresponding processes defined in terms of (X_t). Therefore, the assertion of the theorem remains true in the more general case, when the assumptions of either Scenario 4.1, 4.2, or 4.3 holds for the transformed process (X_t^g).

As a corollary, we find that the first crossing time

$$T(b) := \inf\{t > 0\colon X_t > b\} \tag{4.8}$$

is asymptotically exponentially distributed.

Corollary 4.1. *Under the assumptions of Theorem 4.1, we have, for any $s \geq 0$,*

$$\mathrm{P}_\sigma((1 - \rho)\nu_c(b)T(b) > s) \to \mathrm{e}^{-s} \quad \text{as } b \to \infty.$$

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Proof. For any $s \geq 0$ and $b \geq u_0$, we have $\nu_+(b) = \nu_c(b)$ and

$$\begin{aligned} \mathrm{P}_\sigma((1-\rho)\nu_c(b)T(b) > s) &= \mathrm{P}_\sigma(X_0 < b,\ M^b((1-\rho)^{-1}s) = 0) \\ &= \mathrm{P}_\sigma(M^b((1-\rho)^{-1}s) = 0) - \mathrm{P}_\sigma(X_0 \geq b,\ M^b((1-\rho)^{-1}s) = 0). \end{aligned} \tag{4.9}$$

The second term on the right-hand side of (4.9) converges to 0 as $b \to \infty$. As any fixed finite number of points (in our case 0 and $(1-\rho)^{-1}s$) are almost surely not contained in Π_ρ, we find, from (4.1) and a standard property of weak convergence of point processes (see [15, Theorem 16.16]), that the first term in (4.9) converges to $\mathrm{P}(\Pi_\rho((1-\rho)^{-1}s) = 0) = \mathrm{e}^{-s}$. This completes the proof.

Remark 4.4. Define $T_1(b) := T(b)$ and, inductively, $T_{n+1}(b) := \inf A_n$, $n \geq 1$, where A_n is the set of all $t > T_n(b)$ such that $X_t > b$ and $X_s < b$ for some $s \in (T_n(b), t)$. Under the assumptions of Theorem 4.1, we obtain, for any $n \geq 1$ and $s \geq 0$ as above,

$$\mathrm{P}_\sigma((1-\rho)\nu_c(b)T_n(b) > s) \to \mathrm{P}(\Pi_\rho((1-\rho)^{-1}s) \leq n-1) \quad \text{as } b \to \infty.$$

An easy calculation shows that, for instance,

$$\begin{aligned} \mathrm{P}(\Pi_\rho((1-\rho)^{-1}s) \leq 1) &= \mathrm{e}^{-s}(1 + (1-\rho)s), \\ \mathrm{P}(\Pi_\rho((1-\rho)^{-1}s) \leq 2) &= \mathrm{e}^{-s}\biggl(1 + (1-\rho^2)s + \frac{(1-\rho)^2 s^2}{2}\biggr). \end{aligned}$$

Corollary 4.2. *Let the assumptions of Theorem 4.1 be satisfied, and let $B \subset [0, \infty)$ be a bounded Borel set whose boundary has Lebesgue measure* 0. *Then $M^b(B) \xrightarrow{\mathrm{W}} \zeta_B$ as $b \to \infty$, where ζ_B is a nonnegative integer-valued RV with the Laplace transform*

$$\mathrm{E}\exp[-z\zeta_B] = \exp\biggl[-|B|(1-\rho)\biggl(1 - \frac{1-\rho}{\mathrm{e}^z - \rho}\biggr)\biggr], \qquad z \geq 0. \tag{4.10}$$

Here $|B|$ denotes the Lebesgue measure of B.

Proof. The right-hand side of (4.10) is just the Laplace transform of $\Pi_\rho(B)$. Hence, the result is a direct consequence of Theorem 4.1 and Theorem 16.16 of [15].

Remark 4.5. The random variable $\zeta_B \stackrel{\mathrm{D}}{=} \Pi_\rho(B)$ (where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution) is infinitely divisible with a Lévy measure having mass $|B|(1-\rho)^2\rho^{k-1}$ at $k \geq 1$.

4.1. Proof of Theorem 4.1

A possible way of proving the theorem is to apply standard point process techniques on weak convergence employing generating functions. However, a simpler and shorter alternative is to use a regenerative process approach to analysing exceedances of stationary Markov chains from [29]. The natural choice of i.i.d. cycles in our case is the segments of the trajectory of (X_t) between consecutive continuous crossing of a fixed level. For a fixed $u \in \mathbb{R}$, we define an increasing sequence $\tau_n(u)$, $n \geq 0$, of stopping times inductively by $\tau_0(u) := 0$ and $\tau_{n+1}(u) := \inf\{t > \tau_n(u) \colon N_c^u(t) \geq n+1\}$, where $\inf \varnothing := \infty$. Hence, $N_c^u(t)$ is the cardinality of $\{n \geq 1 \colon \tau_n(u) \leq t\}$. Let $N_c^u(\infty) := \lim_{t\to\infty} N_c^u(t)$ and $\tau(u) := \tau_1(u)$.

Lemma 4.1. *Assume that $u \in \mathbb{R}$ satisfies $\nu_c(u) > 0$. Then $\mathrm{P}(\tau(u) < \infty) = 1$ and $\mathrm{P}(N_c^u(\infty) = \infty) = \mathrm{P}_u(N_c^u(\infty) = \infty) = 1$. Moreover, $\nu_c(u) = (\mathrm{E}_u\, \tau(u))^{-1}$.*

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Proof. Take a $u \in \mathbb{R}$. The strong Markov property implies that $(X_t)_{t \le \tau(u)}$ and $(\mathbf{1}\{\tau(u) < \infty\} X_{\tau(u)+t})_{t \ge 0}$ are independent for any initial distribution. This fact will often be used in the sequel. In particular, N_c^u is a renewal process (with a possibly defective distribution of interpoint distances). Under P, N_c^u is also a stationary point process. If $\nu_c(u) > 0$, this clearly implies that $\mathrm{P}(\tau(u) < \infty) = 1$. The equality $\nu_c(u) = (\mathrm{E}_u\, \tau(u))^{-1}$ is then a consequence of the elementary renewal theorem. In particular, $\mathrm{E}_u\, \tau(u) < \infty$, so that the relations $\mathrm{P}(N_c^u(\infty) = \infty) = \mathrm{P}_u(N_c^u(\infty) = \infty) = 1$ are obvious. This completes the proof of the lemma.

Lemma 4.1 clearly implies that if $\nu_c(u) > 0$ then $\mathrm{P}(\tau_n(u) < \infty) = \mathrm{P}_u(\tau_n(u) < \infty) = 1$ for all $n \ge 1$.

Lemma 4.2. *Assume that $u \in \mathbb{R}$ satisfies $\nu_c(u) > 0$, and let $b \in \mathbb{R}$. Then $\nu_c(b) > 0$ if and only if $\mathrm{P}_u(\tau(b) < \infty) > 0$. In this case $\mathrm{P}_u(\tau(b) < \infty) = 1$.*

Proof. Assume that $\nu_c(b) > 0$. Since $\mathrm{P}(\tau(u) < \infty) = 1$ by Lemma 4.1, we must have $\mathrm{P}_u(\tau(b) < \tau(u)) > 0$. Since $\mathrm{P}_u(N_c^u(\infty) = \infty) = 1$, we can use a geometrical trial argument to obtain $\mathrm{P}_u(\tau(b) < \infty) = 1$. Assume, conversely, that $\mathrm{P}_u(\tau(b) < \infty) > 0$. Then we must have $\mathrm{P}_u(\tau(b) < \tau(u)) > 0$ and, hence, $\mathrm{P}_u(\tau(b) < \infty) = 1$. Lemma 4.1 implies that $\mathrm{P}(\tau(b) < \infty) = 1$. Hence, N_c^b is a nonempty, stationary point process under P and must, therefore, have a positive intensity $\nu_c(b)$. This completes the proof of the lemma.

In the remainder of the proof of Theorem 4.1 we will assume that u_0 is chosen according to the assumptions of either Scenario 4.1, 4.2, or 4.3. It is then no loss of generality to assume that $\mu(u) \ne 0$ and $\nu_c(u) > 0$ for all $u \ge u_0$. We now fix $u \ge u_0$. Denote by

$$\zeta_n := \sup\{X_t \colon \tau_n(u) \le t < \tau_{n+1}(u)\}$$

the maximum of the process (X_t) over the nth cycle. The above lemma and the assumptions of Theorem 4.1 imply that the distribution of ζ_1 has an unbounded support and, hence,

$$R(b) := \frac{1-\rho}{\nu_c(u)\, \mathrm{P}(\zeta_1 > b)} \equiv \frac{1-\rho}{\nu_c(u)\, \mathrm{P}_u(\tau(u) > \tau(b))} < \infty, \qquad b \in \mathbb{R},$$

and $R(b) \to \infty$ as $b \to \infty$. The proof of Theorem 3.1 of [29] shows that the first crossing times $T(b)$ (see (4.8)) satisfy

$$\lim_{b \to \infty} \mathrm{P}_x(T(b) > R(b)) = \mathrm{e}^{-(1-\rho)}, \qquad x \in \mathbb{R}.$$

Here we have also used the fact that ergodicity of (X_t) and Lemma 4.1 together imply that

$$\mathrm{P}_x(\tau(u) < \infty) = 1, \qquad x \in \mathbb{R}.$$

Next we introduce the scaled process $L^b(t) := N_+^b(R(b)t)$ and choose a function $r(b),\ b \in \mathbb{R}$, such that, as $b \to \infty$,

$$r(b) \to \infty, \qquad r(b) = o(R(b)), \qquad R(b)\, \mathrm{P}_u(\tau(u) > r(b)) \to 0$$

(which is always possible since $\mathrm{E}_u\, \tau(u) = 1/\nu_c(u) < \infty$ by Lemma 4.1). Denote by L_0^b the point process of the points $(T_n'(b))$, obtained as follows: $T_1'(b) := T(b)$ and $T_{n+1}'(b) := \inf A_n'$, A_n' being the set of all $t > T_n'(b) + r(b)$ such that $X_t > b$ and $X_s < b$ for some $s \in (T_n'(b) + r(b), t)$. This is an analogue of the process N_T' of [29, p. 376]: a process of the

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locations of 'clusters' of upcrossings of level b. Since having an uprocssing of b in the nth cycle is equivalent to having $\zeta_n > b$, we can easily verify that the argument proving Theorem 3.2 of [29] also demonstrates that our process L_0^b converges in distribution to a Poisson process Π'_ρ with intensity $1 - \rho$.

To complete the proof of Theorem 4.1, it remains to determine the asymptotic behaviour of the distribution of the size of the clusters of upcrossings as $b \to \infty$ and then use Theorem 3.3 of [29]. We will begin by introducing the probability of having a nontrivial cluster of upcrossings:

$$\gamma(u, b) := \mathrm{P}_b(\tau(u) > \tau(b)), \qquad u < b,$$

where we note that

$$\mathrm{P}_u(N_+^b(\tau(u)) = N_c^b(\tau(u))) = 1, \qquad b > u \geq u_0. \tag{4.11}$$

In the case in which $\mu < 0$ on (u_0, ∞), this is due to the absence of negative jumps from a level above b. In the case in which $\mu > 0$ on (u_0, ∞) we even have $N_+^b = N_c^b$ because there are no positive jumps to a level above b. From (4.11) and the strong Markov property, we obtain

$$\mathrm{P}_u(N_+^b(\tau(u)) = k \mid N_+^b(\tau(u)) > 0) = (1 - \gamma(u, b))\gamma^{k-1}(u, b), \qquad k \geq 1. \tag{4.12}$$

Now we will analyse the limiting behaviour of $\gamma(u, b)$. The assumptions of Theorem 4.1 are used in the following key lemma.

Lemma 4.3. *Under either Scenario 4.1, 4.2, or 4.3, for all sufficiently large u,*

$$\lim_{b \to \infty} \gamma(u, b) = \rho,$$

where ρ is defined in Theorem 4.1.

Proof. Without loss of generality, we can assume that $\bar{H}(\cdot, z)$ is nonincreasing for any $z \in \mathbb{R}$ (or, equivalently, that $\overline{\xi}(u) \overset{\mathrm{D}}{\leq} \overline{\xi}(v)$ for $u > v$) and that $\underline{H}(\cdot, z)$ is nondecreasing. Recall that u_0 is chosen according to the assumptions of either Scenario 4.1, 4.2, or 4.3. We always take $u, b \in \mathbb{R}$ such that $b > u \geq u_0$. Note that, in Scenarios 4.1 and 4.2, the drift conditions (4.4) and (4.5), respectively, hold with u in place of u_0. In Scenario 4.1 or 4.2 the argument will run roughly as follows. Owing to the imposed conditions, for a large enough initial value b, the trajectory of (X_t) will very quickly drop by a given large quantity C. Since in the part of the state space above the level u the process can be shown to be dominated (at its jump points) by a random walk with i.i.d. jumps and a negative trend, we can choose C large enough to ensure that the process will not climb back by C prior to dropping below the level u. If (X_t) does not drop quickly enough from a high level then it is likely there will be several crossings of that level before the process returns to the range of its 'normal values'. This case requires the more restrictive conditions formulated in Scenario 4.3.

First consider Scenario 4.1 and fix an arbitrary $\varepsilon > 0$. Since the process has a negative drift coefficient in the half-line $[u, \infty)$, it can only exceed the level $b > u$ by a jump, so we can restrict ourselves to considering the values $X_{T_1}, X_{T_2}, \dots$:

$$\mathrm{P}_x(\tau(b) < \tau(u)) \leq \mathrm{P}_x(\sup\{X_{T_k} : T_k \leq \tau(u)\} \geq b), \qquad x \geq u. \tag{4.13}$$

Furthermore, for x and t such that $q(x, t) > u$, we have

$$\mathrm{P}_x(T_1 > t) = \exp\biggl[-\int_0^t \lambda(q(x, s))\,\mathrm{d}s\biggr] \geq \exp[-\overline{\lambda}(u)t] = \mathrm{P}(\chi(\overline{\lambda}(u)) > t), \tag{4.14}$$

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where $\chi(w)$ is a RV following the exponential distribution with parameter w. Therefore, we can easily see that the right-hand side of (4.13) does not exceed $\mathrm{P}(S \geq b-x)$, where $S := \sup_{k\geq 1} S_k$ is the global supremum of a random walk

$$S_k = \zeta_1 + \cdots + \zeta_k, \qquad k \geq 1, \tag{4.15}$$

with i.i.d. jumps $\zeta_k \stackrel{\mathrm{D}}{=} \overline{\xi}(u) + \overline{\mu}(u)\chi(\overline{\lambda}(u))$, where $\overline{\xi}(u)$ and $\chi(\overline{\lambda}(u))$ are independent of each other. Since $\mathrm{E}\,\zeta_k < 0$ by (4.4), S is a proper RV, and we can choose C so large that $\mathrm{P}(S \geq C) < \varepsilon$.

Next we assume that $b \geq u + C$. For any $t \geq 0$, the equation $q(b,t) = b - C$ has a unique solution $t = t(b,C)$. Since $\mu(y) \to -\infty$ as $y \to \infty$, we have $t(b,C) \to 0$ as $b \to \infty$. In particular, we obtain, from (4.14), $\mathrm{P}_b(T_1 \leq t(b,C)) < \varepsilon$ for all large enough b. Then we have $\mathrm{P}_b(X_{t(b,C)} = b - C) > 1 - \varepsilon$, and finally, due to (4.13) and our choice of C, $\mathrm{P}_b(\tau(b) < \tau(u)) < 2\varepsilon$. Since ε was arbitrary small, this completes the proof of the lemma in the case of Scenario 4.1.

Now consider Scenario 4.2. In this case, jumps from levels $x \geq u$ are negative, and we can concentrate on the values $X_{T_1-}, X_{T_2-}, \ldots$. For a given $\varepsilon > 0$, choose a $C < \infty$ such that $\mathrm{P}(S \geq C) < \varepsilon$ for the random walk (4.15) with i.i.d. jumps $\zeta_k \stackrel{\mathrm{D}}{=} \overline{\xi}(u) + \overline{\mu}(u)\chi(\underline{\lambda}(u))$, where $\overline{\xi}(u)$ and $\chi(\overline{\lambda}(u))$ are independent of each other. Consider a stopping time T with values in $\{T_1, T_2, \ldots\}$. Then, as we can easily see, given that $X_{T-} < b - C$, the probability of the process exceeding b on the time interval $[T, \infty)$ prior to dropping below u will again be less than ε.

Since the deterministic drift is now positive on (u_0, ∞),

$$\mathrm{P}_x(T_1 > t) = \exp\biggl[-\int_0^t \lambda(q(x,s))\,\mathrm{d}s\biggr] \leq \exp[-\underline{\lambda}(x)t] = \mathrm{P}(\chi(\underline{\lambda}(x)) > t), \qquad x \geq u.$$

Therefore, given $X_0 = b$, we have $X_{T_1-} \stackrel{\mathrm{D}}{\leq} b + \overline{\mu}(u)\chi(\underline{\lambda}(b))$, and it is not difficult to see that, for $m \geq 1$, $\bar{X}_m := \sup_{T_1 \leq t \leq T_m} X_t$ and $\underline{X}_m := \inf_{T_1 \leq t \leq T_m} X_t$,

$$\begin{aligned}
&\mathrm{P}_b(\{\bar{X}_m \geq b\} \cup \{\underline{X}_m \geq b - C\}) \\
&\qquad \leq \mathrm{P}\Bigl(\max_{1\leq k\leq m} S_k \geq -\overline{\mu}(u)\chi(\underline{\lambda}(b-C))\Bigr) + \mathrm{P}(S_m \geq -C),
\end{aligned} \tag{4.16}$$

where (S_k) is a random walk given by (4.15) with i.i.d. jumps $\zeta_k \stackrel{\mathrm{D}}{=} \overline{\xi}(u) + \overline{\mu}(u)\chi(\underline{\lambda}(b-C))$, and (S_k) and $\chi(\underline{\lambda}(b-C))$ which appear in (4.16) are independent of each other. Now choose m so large that $\mathrm{P}(S_m \geq -C) < \varepsilon$ (this is possible due to (4.5)) and then choose b so large that the first term on the right-hand side of (4.16) is also less than ε. The latter is possible due to the following observation. Setting

$$S_k' := \xi_1' + \cdots + \xi_k', \qquad S_k'' := \xi_1'' + \cdots + \xi_k'', \qquad k \geq 1,$$

where (ξ_k') and (ξ_k'') are independent sequences of i.i.d. RVs with $\xi_k' \stackrel{\mathrm{D}}{=} \overline{\xi}(u)$ and $\xi_k'' \stackrel{\mathrm{D}}{=} \chi(\underline{\lambda}(b - C))$, the event in that term implies that

$$\max_{1\leq k\leq m} S_k' \geq -\overline{\mu}(u) \max_{1\leq k\leq m} S_k'' - \overline{\mu}(u)\xi_{m+1}''.$$

The RV on the left-hand side is almost surely negative, with a distribution independent of b. Because it is assumed that $\lambda(y) \to \infty$ as $y \to \infty$, the distribution of the right-hand side converges to δ_0 as $b \to \infty$.

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Thus, on the event complementary to the one on the left-hand side of (4.16), the process (X_t) will drop at one of the times $T_1, \dots, T_m$ below the level $b - C$ (denote this epoch by T^*), without having continuously crossed the level b prior to that time. Also, due to our choice of C and to the strong Markov property, the process will reach the level b on the time interval $[T^*, \tau(u)]$ with probability less than ε. This means that $\mathrm{P}_b(\tau(b) < \tau(u)) < 3\varepsilon$. This completes the proof of the lemma in the case of Scenario 4.2.

Finally, in Scenario 4.3 we start by considering $b > v > u$ and noting that $\{\tau(v) > \tau(b)\} \subset \{\tau(u) > \tau(b)\}$ holds P_b-a.s. Hence, we have, as $b \to \infty$,

$$\begin{aligned}\gamma(u, b) &= \mathrm{P}_b(\tau(v) > \tau(b)) + \mathrm{P}_b(\tau(u) > \tau(b) > \tau(v)) \\ &= \gamma(v, b) + \mathrm{P}_b(\tau(b) > \tau(v))\,\mathrm{P}_v(\tau(u) > \tau(b)) \\ &= \gamma(v, b) + o(1),\end{aligned}$$

where we made use of the strong Markov property and also of the obvious relation $\mathrm{P}_v(\tau(u) > \tau(b)) \to 0$ due to ergodicity. Therefore, the quantities

$$\limsup_{b\to\infty} \gamma(u, b) = \limsup_{b\to\infty} \gamma(v, b) =: \gamma_+, \tag{4.17}$$

$$\liminf_{b\to\infty} \gamma(u, b) = \liminf_{b\to\infty} \gamma(v, b) =: \gamma_-, \tag{4.18}$$

do not depend on v.

Now we assume that $\mu(\infty) < 0$. Then crossing level b can only occur due to a jump, and since to get from level $x > b$ down to level u will require a continuous downcrossing of b, we obtain

$$\mathrm{P}_b(\tau(b) < \tau(u)) = \mathrm{P}_b(\sup\{X_{T_k} - b \colon T_k < \tau(u)\} > 0). \tag{4.19}$$

Next we observe that, for the segment of the process on the time interval $[0, \tau(u)]$, we have $\underline{S}_k \overset{\mathrm{D}}{\leq} X_{T_k} - b \overset{\mathrm{D}}{\leq} \bar{S}_k$, where $(\bar{S}_k)_{k\geq 1}$ and $(\underline{S}_k)_{k\geq 1}$ are random walks with i.i.d. jumps

$$\overline{\xi}_k \overset{\mathrm{D}}{=} \overline{\xi}(u) + \overline{\mu}(u)\chi(\overline{\lambda}(u)) \quad \text{and} \quad \underline{\xi}_k \overset{\mathrm{D}}{=} \underline{\xi}(u) + \underline{\mu}(u)\chi(\underline{\lambda}(u)),$$

respectively, where we again make the usual independence assumptions. Owing to (4.6) and the uniform integrability of $(\overline{\xi}(u))$, we obtain $\mathrm{E}\,\underline{\xi}_k \leq \mathrm{E}\,\overline{\xi}_k < 0$ for all large enough u, so that then

$$\underline{S}^u := \sup_{k\geq 1} \underline{S}_k \overset{\mathrm{D}}{\leq} \bar{S}^u := \sup_{k\geq 1} \bar{S}_k < \infty \quad \text{a.s.}$$

It is not difficult to see that, for $b > 2u$,

$$\mathrm{P}(\underline{S}^u > 0) + V(u, b) \leq \mathrm{P}_b(\sup\{X_{T_k} - b \colon T_k < \tau(u)\} > 0) \leq \mathrm{P}(\bar{S}^u > 0),$$

where

$$V(u, b) := \mathrm{P}(\sup\{\underline{S}_k : k \leq \eta\} > 0) - \mathrm{P}(\underline{S}^u > 0) - \mathrm{P}\Big(\underline{\mu}(u)\chi(\underline{\lambda}(u)) > \frac{b}{2} - u\Big),$$

and $\eta := \inf\{k > 0 \colon \underline{S}_k < -b/2\}$. Since, clearly, $\eta \to \infty$ a.s. as $b \to \infty$, we obtain $\lim_{b\to\infty} V(u, b) = 0$, and so, owing to (4.17), (4.18), and (4.19), we have

$$\mathrm{P}(\underline{S}^u > 0) \leq \gamma_- \leq \gamma_+ \leq \mathrm{P}(\bar{S}^u > 0).$$

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Now, by virtue of Theorem 6 and Condition B of [5, p. 114], we obtain

$$\lim_{u\to\infty} \mathrm{P}(\underline{S}^u > 0) = \lim_{u\to\infty} \mathrm{P}(\bar{S}^u > 0) = \mathrm{P}(S > 0),$$

where $S = \sup_{k\geq 1} S_k$ for a random walk (S_k) with i.i.d. jumps $\zeta_k \stackrel{\mathrm{D}}{=} \xi(\infty) + \mu(\infty)\chi(\lambda(\infty))$ with negative mean. This clearly implies that $\gamma_+ = \gamma_- = \lim_{b\to\infty}\gamma(u, b) = \mathrm{P}(S > 0)$, and the value of the last probability is well known to be given by the first expression on the right-hand side of (4.7); see, e.g. Theorem VIII.5.7 and Corollary III.6.5 of [2].

The argument in the case in which $\mu(\infty) > 0$ is very similar, the main difference being the value of $\mathrm{P}(S > 0)$. But again it is well known that this value is given by the second expression on the right-hand side of (4.7); see, e.g. Theorem X.5.1 of [2]. This completes the proof of the lemma.

The lemma implies that, under the conditions of the theorem, the conditional distribution of the number of upcrossings of b during a cycle, given the level was reached during the cycle, will converge to the geometric law with probabilities $(1-\rho)\rho^{k-1}$, $k = 1, 2, \dots$. This, together with the argument used to prove Theorem 3.3 of [29], shows that L^b converges to Π_ρ in distribution. To complete the demonstration of Theorem 4.1, it remains to note that

$$R(b) \equiv \frac{1-\rho}{\nu_c(u)\,\mathrm{P}(\zeta_1 > b)} \sim \frac{1}{\nu_c(b)} = \frac{1}{\nu_+(b)}. \tag{4.20}$$

The last equality is obvious from the assumptions of the theorem (cf. the beginning of this section). To prove the asymptotic equivalence, observe that

$$\mathrm{E}_u\, N_c^b(\tau(u)) = \mathrm{E}_u(N_c^b(\tau(u)) \mid N_c^b(\tau(u)) > 0)\,\mathrm{P}_u(N_c^b(\tau(u)) > 0) = \frac{\mathrm{P}(\zeta_1 > b)}{1-\gamma(u, b)}$$

(using (4.12)), so that from Lemma 4.3 we have

$$R(b) \sim \frac{1}{\nu_c(u)\,\mathrm{E}_u\, N_c^b(\tau(u))}.$$

Now (4.20) follows from the equilibrium equation, $\nu_c(u)\,\mathrm{E}_u\, N_c^b(\tau(u)) = \nu_c(b)$, which holds owing to the following simple observation: its right-hand side gives the intensity of continuous crossings of level b, whereas its left-hand side is the mean number of cycles (formed by continuous crossings of u) per time unit multiplied by the mean number of crossings of b per cycle. This completes the proof of the theorem.

Observe that relation (4.20) together with Theorem 3.1 relates the tail asymptotics of the distribution of the cycle maximum ζ_1 to that of the stationary density p of the process. Namely, we have the following corollary.

Corollary 4.3. *Under the assumptions of Theorems 3.1 and 4.1, for any large enough u (the level used to construct the cycles),*

$$\mathrm{P}(\zeta_1 > b) \sim \frac{1-\rho}{\nu_c(u)}|\mu(b)|p(b) \quad \textit{as } b\to\infty.$$

Remark 4.6. Theorem 4.1 assumes that $\nu_c(b) = \nu_+(b) > 0$ for all sufficiently large b. This can be checked with the help of Lemma 4.2. To indicate how this can be done, we fix a $u \in \mathbb{R}$ satisfying $\nu_c(u) > 0$. Consider first the case in which $q(u, t) \to \infty$ as $t \to \infty$. Since λ is

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locally bounded, we then have $\mathrm{P}_u(N_+^b(\infty) > 0) > 0$, implying that $\mathrm{P}_u(N_+^b(\infty) > 0) = 1$. A second case is that there are $\varepsilon, \delta > 0$ such that $\lambda(x)J(x, [\varepsilon, \infty)) > 0$ for all $x \geq u - \delta$. Owing to the possibility of many positive jumps in a small period of time, we again have $\mathrm{P}_u(N_+^b(\infty) > 0) > 0$.

Appendix A

First we formulate some conditions under which Assumption 2.1 will hold true. Introduce the mean values

$$m^-(x) := -\int_{-\infty}^0 zJ(x, \mathrm{d}z), \qquad m^+(x) := \int_0^\infty zJ(x, \mathrm{d}z), \qquad x \in \mathbb{R},$$

and

$$m(x) := m^+(x) - m^-(x) = \int zJ(x, \mathrm{d}z), \qquad x \in \mathbb{R}.$$

Assumption A.1. *For all $x \in \mathbb{R}$, $m^-(x) + m^+(x) < \infty$ and $\lambda(x)(m^-(x) + m^+(x))$ is a locally bounded function on $\mathbb{R}$.*

In the next assumption we use the convention $0/0 := 0$.

Assumption A.2. *We have*

$$\lim_{x \to -\infty} \frac{1}{m^+(x)} \int_{-x}^{\infty} (x+z)J(x, \mathrm{d}z) = \lim_{x \to \infty} \frac{1}{m^-(x)} \int_{-\infty}^{-x} (x+z)J(x, \mathrm{d}z) = 0. \tag{A.1}$$

Next we formulate a basic ergodicity assumption.

Assumption A.3. *There is an $\varepsilon > 0$ such that*

$$\begin{aligned} &\liminf_{x \to -\infty} (\mu(x) + \lambda(x)m^+(x)(1-\varepsilon) - \lambda(x)m^-(x)) > 0, \\ &\limsup_{x \to \infty} (\mu(x) + \lambda(x)m^+(x) - \lambda(x)m^-(x)(1-\varepsilon)) < 0. \end{aligned} \tag{A.2}$$

Remark A.1. Assume that two of the limits

$$\lim_{x \to -\infty} \mu(x), \qquad \lim_{x \to -\infty} \lambda(x)m^-(x), \qquad \lim_{x \to -\infty} \lambda(x)m^+(x)$$

exist and are finite, and make a similar assumption on the corresponding limits as $x \to \infty$. Then Assumption A.3 is equivalent to

$$\liminf_{x \to -\infty} (\mu(x) + \lambda(x)m(x)) > 0 > \limsup_{x \to \infty} (\mu(x) + \lambda(x)m(x)).$$

For a constant (positive) μ and negative jumps, this is the well-known ergodicity condition for the stress release model (see [19], [30], and [31]).

The next assumption states that all bounded sets are *small* for the process (see [27]). Previous studies (see, e.g. [19], [27], and [31]) show that this is a rather weak though sometimes tedious to check assumption. We will not discuss it any further.

Assumption A.4. *For any bounded interval $I \subset \mathbb{R}$, there is a $t_0 > 0$ and a nontrivial measure $\mathbb{Q}$ on $\mathbb{R}$ such that*

$$\mathrm{P}_x(X_{t_0} \in \cdot) \geq \mathbb{Q}(\cdot), \qquad x \in I.$$

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Theorem A.1. *If Assumptions A.1, A.2, A.3, and A.4 are satisfied then* $\mathrm{P}_x(T_\infty = \infty) = 1$ *for all* $x \in \mathbb{R}$ *and* (X_t) *has a unique invariant distribution* π.

Proof. We proceed similarly to [19]. For any $m \geq 1$, the process $(X_{t\wedge\tau_m})$ is again Markov. By (2.6), its generalised generator $\mathcal{A}_m$ (cf. [27]) is given by

$$\mathcal{A}_m f(x) = \mu(x) f'(x) + \lambda(x) \int (f(x+z) - f(x)) J(x,\, \mathrm{d}z), \qquad |x| < m,$$

where f satisfies the assumptions of Proposition 2.1. By Assumption A.1 we can take $f(x) := |x|$ to obtain, for $|x| < m$,

$$\begin{aligned}\mathcal{A}_m f(x) &= \operatorname{sgn}(x)\mu(x) + \lambda(x) \int (|x+z| - |x|) J(x,\, \mathrm{d}z) \\ &= \operatorname{sgn}(x)\mu(x) + \operatorname{sgn}(x)\lambda(x)m^+(x) - \operatorname{sgn}(x)\lambda(x)m^-(x) \\ &\quad + 2\bigg(\mathbf{1}\{x < 0\}\lambda(x) \int_{-x}^{\infty} (x+z) J(x, \mathrm{d}z) - \mathbf{1}\{x \geq 0\}\lambda(x) \int_{-\infty}^{-x} (x+z) J(x, \mathrm{d}z)\bigg),\end{aligned} \tag{A.3}$$

where $\operatorname{sgn}(x) \in \{-1, 1\}$ is the sign of $x \in \mathbb{R}$, defined in a right-continuous way, and where the second equality comes from

$$|x+z| - |x| = 2(\mathbf{1}\{-z < x < 0\} - \mathbf{1}\{z < -x \leq 0\})(x+z) + \operatorname{sgn}(x) z, \qquad z \neq 0.$$

We define

$$\varepsilon(x) := \mathbf{1}\{x < 0\} \frac{1}{2m^+(x)} \int_{-x}^{\infty} (x+z) J(x,\, \mathrm{d}z) - \mathbf{1}\{x \geq 0\} \frac{1}{2m^-(x)} \int_{-\infty}^{-x} (x+z) J(x,\, \mathrm{d}z).$$

Then $\varepsilon(x) \geq 0$ and from (A.1) we have $\varepsilon(x) \to 0$ as $|x| \to \infty$. We can now rewrite (A.3) as

$$\begin{aligned}\mathcal{A}_m f(x) = \operatorname{sgn}(x)\mu(x) &+ \operatorname{sgn}(x)\lambda(x)m^+(x)(1 - \mathbf{1}\{x < 0\}\varepsilon(x)) \\ &- \operatorname{sgn}(x)\lambda(x)m^-(x)(1 - \mathbf{1}\{x \geq 0\}\varepsilon(x)).\end{aligned} \tag{A.4}$$

Using our assumptions in (A.4), we easily get numbers $\varepsilon > 0$, $x_0 > 0$, and $d \geq 0$ such that

$$\mathcal{A}_m f(x) \leq -\varepsilon + \mathbf{1}\{|x| \leq x_0\} d, \qquad |x| < m,\ m \in \mathbb{N}. \tag{A.5}$$

In particular, we can apply Theorem 2.1 of [27] to conclude that, for any $x \in \mathbb{R}$, $\tau_m \to \infty$ P_x-a.s. as $m \to \infty$. This proves the first assertion. We are then in a position to apply Theorem 4.2 of [27] to complete the proof of the theorem.

Remark A.2. Under the conditions of Theorem A.1, the process (X_t) is even *positive Harris recurrent*; see [27]. Only a weak additional assumption is needed to obtain *Harris ergodicity*, i.e. the total variation convergence of $\mathrm{P}_x(X_t \in \cdot)$ to π for any $x \in \mathbb{R}$. By Theorem 6.1 of [26], one such assumption is *irreducibility* of one *skeleton chain*.

Next we discuss Assumption 2.2. If λ is a bounded function then this assumption is trivially satisfied. If not then we can impose the following slightly stronger version of Assumption A.3 and a weak positivity assumption on $m^-(x) + m^+(x)$.

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Assumption A.5. *There is an $\varepsilon > 0$ such that*

$$\liminf_{x\to-\infty}(\mu(x) + \lambda(x)m^+(x)(1-\varepsilon) - \lambda(x)m^-(x)(1+\varepsilon)) > 0,$$
$$\limsup_{x\to\infty}(\mu(x) + \lambda(x)m^+(x)(1+\varepsilon) - \lambda(x)m^-(x)(1-\varepsilon)) < 0.$$

Theorem A.2. *If Assumptions A.1, A.2, A.4, and A.5 are satisfied and, moreover,*

$$\liminf_{|x|\to\infty}(m^-(x) + m^+(x)) > 0, \tag{A.6}$$

then $\mathrm{P}_x(T_\infty = \infty) = 1$ *for all* $x \in \mathbb{R}$ *and* (X_t) *has a unique invariant distribution* π *satisfying* $\int \lambda(x)\pi(\mathrm{d}x) < \infty$.

Proof. Using the assumptions in (A.4), we can easily strengthen (A.5) to

$$\mathcal{A}_m f(x) \le -\max\{\varepsilon, \lambda(x)\} + \mathbf{1}\{|x| \le x_0\}d, \qquad |x| < m,\ m \in \mathbb{N}.$$

Hence, we can apply Theorem 4.2 of [27] to obtain $\int \lambda(x)\pi(\mathrm{d}x) < \infty$.

Remark A.3. In the framework described in Remark 2.2, Assumption A.3 can be reduced to (A.2). A similar remark applies to Assumptions A.2 and A.5, and to (A.6).

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