PERTURBATION OF THE CONTINUOUS SPECTRUM OF EVEN ORDER DIFFERENTIAL OPERATORS

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1. Introduction. Let L_0 be a differential operator of even order $n = 2\nu$ on the half open interval $0 \leqslant t < \infty$ which is formally self adjoint and satisfies the conditions of Kodaira **(5**, p. 503**)**. We consider a perturbed operator of the form $L_\epsilon = L_0 + \epsilon q$ where $q(t)$ is a real-valued bounded function and ϵ is a real parameter. The object of this paper is to set up conditions on the operator L_0 and the function $q(t)$ such that L_ϵ determines a self-adjoint operator H_ϵ and such that the spectral resolution operator $E^\epsilon(\Delta)$ corresponding to H_ϵ is analytic in a neighbourhood of $\epsilon = 0$, where Δ is a closed bounded interval.

Our conditions are a natural generalization of conditions considered by Moser for the case $n = 2$**(6)**. Moser has given a number of examples showing that when his conditions do not hold $E^\epsilon(\Delta)$ need not be analytic. However, Moser's conditions are not necessary. Brownell has demonstrated analyticity of $E^\epsilon(\Delta)$ for second order differential operators (in E_n) under conditions different from Moser's **(2)**.

Our main result is Theorem 4 which gives sufficient conditions that $E^\epsilon(\Delta)$ be analytic. Theorem 4 is an easy consequence of Theorem 3. The proof of Theorem 3 hinges upon the Neumann expansion for the resolvent kernel of the perturbed operator H_ϵ and on the behaviour of the resolvent kernel of the unperturbed operator H_0 under change of boundary conditions at $t = 0$. We discuss the former of these topics in § 4 and the latter in § 3. Section 2 is devoted to definitions and needed facts. The restrictions that we impose on L_0, q are stated at the end of § 2.

The assumption that $q(t)$ is bounded can be removed. In § 6 we indicate briefly how this may be done.

The significance of analyticity of the spectral measure $E^\epsilon(\Delta')$ for $\Delta' \subset \Delta$, Δ a fixed bounded interval, is that it implies that points in the spectrum of H_ϵ which lie inside Δ remain fixed under the perturbation **(6**; **7)**. Our assumptions imply that Δ contains only points of the continuous spectrum of H_0 (cf. assumption (ii)). Therefore, our results may be interpreted as sufficient conditions that the continuous spectrum remain fixed under perturbation.

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2. Basic definitions and assumptions. We shall use the standard notation from the theory of ordinary differential operators **(3; 5)**. The notation (u, v) will mean the inner product of two functions in $L_2(0, \infty)$. The norm of u is $||u|| = (u, u)^{\frac{1}{2}}$. Let $[u, v](t)$ be the bilinear form associated with the differential operator L_0 such that

$$\int_0^t (L_0 u\bar{v} - u\,\overline{L_0 v})\,dt = [u, v](t) - [u, v](0). \tag{2.1}$$

Since $t = 0$ is a regular point there exists a complete canonical set of boundary functions $\psi_{0j}(t)$ and regular solutions $s_j(t, \lambda)$ of $L_0 u = \lambda u$, $j = 1, \ldots, n$ such that

$$[\psi_{0j}, \psi_{0k}](0) = [\psi_{0j}, s_k](0) = [s_j, s_k](0) = \epsilon_{jk} \tag{2.2}$$

and $\epsilon_{jk} = +1$, $k = j + \nu$, $\epsilon_{jk} = -1$, $k = j - \nu$, $\epsilon_{jk} = 0$ otherwise **(4; 5**, p. 505**)**. We shall suppose the differential problem

$$L_0 u = \lambda u, \; [\psi_{0j}, u](0) = 0, \quad j = 1, \ldots, \nu \tag{2.3}$$

is self adjoint **(5**, p. 521**)**. In the case $n = 2$ this reduces to the limit point case at $t = \infty$.

Repeated indices will mean summation unless the contrary is explicitly stated. Latin indices are to be summed over $1, \ldots, n$ and Greek over $1, \ldots, \nu$.

Let $\mathscr{D}$ be the set of functions in $L_2(0, \infty)$ such that for $u \in \mathscr{D}$ we have $u^{(i)}(t) \in \mathscr{C}^i[0, \infty)$, $i = 1, \ldots, n-1$, $u^{(n-1)}(t)$ is absolutely continuous in every closed subinterval of $[0, \infty)$, and $L_0 u \in L_2(0, \infty)$. Let $\mathscr{D}_\infty$ be the set of functions in $\mathscr{D}$ which vanish outside some closed bounded interval. The operator L_0 determines a self-adjoint operator H_0 as follows: We define $\mathscr{D}_{H_0}$ to be the set of functions

$$\mathscr{D}_{H_0} = \{u | u \in \mathscr{D} \quad \text{and} \quad [\psi_{0j}, u](0) = 0, j = 1, \ldots, \nu\}$$

and define $H_0 u = L_0 u$ for $u \in \mathscr{D}_{H_0}$ **(5**, p. 521**)**. Since we are assuming $q(t)$ bounded it follows at once that $L_\epsilon = L_0 + \epsilon q$ determines a self-adjoint operator H_ϵ with

$$\mathscr{D}_{H_\epsilon} = \mathscr{D}_{H_0}$$

and

$$H_\epsilon u = L_\epsilon u, \, u \in \mathscr{D}_{H_0}.$$

The assumption that the boundary value problem (2.3) is self-adjoint implies the following facts (which are all derived from **(5)**): There exist ν vectors $f_\beta(\lambda) = (f_\beta^1, \ldots, f_\beta^n)$, $\beta = \nu + 1, \ldots, n$ such that $w_\beta(t, \lambda) = f_\beta{}^j s_j$ are the eigenfunctions of $L_0 u = \lambda u$, $\mathscr{I}(\lambda) \neq 0$, $w_B(t, \lambda) \in L_2(0, \infty)$. Corresponding to the boundary conditions $[\psi_{0j}, u](0) = 0$ we may choose vectors $f_\alpha = (\delta_\alpha{}^1, \ldots, \delta_\alpha{}^n)$, $\alpha = 1, \ldots, \nu$. Then $w_\alpha = f_\alpha{}^j s_j$ satisfy $[\psi_{0j}, w_\alpha](0) = 0$, $j = 1, \ldots, \nu$, $\alpha = 1, \ldots, \nu$ by (2.2).

The Green's function corresponding to H_0 may be constructed as follows. Define the characteristic matrix M^{ij} by

$$M_{ij} = \sum_{\alpha,\beta} F_{\alpha\beta} f_\beta{}^j f_\alpha{}^i$$

where $\alpha = 1, \ldots, \nu$, $\beta = \nu + 1, \ldots, n$ and $F_{\alpha\beta}$ is the inverse matrix of $[w_\alpha, w_\beta](t)$. The Green's function is by (**5**, p. 511)

$$G^0(t, \tau, \lambda) = M^{jk}(\lambda) s_j(t, \lambda) s_k(\tau, \lambda), \qquad t \geqslant \tau. \tag{2.4}$$

The spectral resolution operator $E^0(\Delta)$ corresponding to H^0 is defined in terms of the Green's function* by

$$E^0(\Delta)u = \frac{1}{2\pi i} \lim_{\delta\to 0+} \mathscr{I}\left\{\int_{\Gamma(\delta)} (G^0(t, \cdot, \lambda), \bar{u}) d\lambda\right\}, \; u \in \mathscr{D}_\infty \tag{2.5}$$

where $\Gamma(\delta)$ is the polygonal path connecting the points $\alpha + i\delta$, $\alpha + 2i\delta$, $\beta + 2i\delta$, $\beta + i\delta$, $\Delta = \{l | \alpha \leqslant l \leqslant \beta\}$. Formula (2.5) may be written (**5**, p. 528)

$$E^{(0)}(\Delta)u = \int_\Delta s_j(t, l)(s_k, \bar{u}) d\rho^{jk}(l) \qquad u \in \mathscr{D}_\infty \tag{2.6}$$

where

$$\rho^{jk}(\Delta) = \frac{1}{2\pi i} \lim_{\delta\to 0+} \mathscr{I}\left\{\int_{\Gamma(\delta)} M^{jk}(\lambda) d\lambda\right\}. \tag{2.7}$$

For two arbitrary l-measurable vector functions $\phi_i(l)$, $\psi_i(l)$, $i = 1, \ldots, n$ we have the inequality

$$\left|\int_{-\infty}^{\infty} \phi_j(l) \overline{\psi_k(l)} d\rho^{jk}(l)\right|^2 \leqslant \int_{-\infty}^{\infty} \phi_j \bar{\phi}_k d\rho^{jk}(l) \int_{-\infty}^{\infty} \psi_j \bar{\psi}_k d\rho^{jk}(l). \tag{2.8}$$

If $u \in \mathscr{D}_\infty$, $\phi_j = (s_j, u)$ then by (**5**, p. 537)

$$||u||^2 = \int_{-\infty}^{\infty} |\phi_j(l)|^2 d\rho^{jk}(l). \tag{2.9}$$

The following assumptions are basic for the theorems to be given below. We shall require† that L_0 and q are such that, for l in a fixed finite interval Δ,

(i) $\int_0^\infty \Phi^2(t) |q(t)| dt \leqslant \gamma < \infty$

where $\Phi(t) = \sup |s_j(t, \lambda)|$, $j = 1, \ldots, n$, $l \in \Delta$, $0 < \delta < \delta_0$, $\lambda = l + i\delta$.

(ii) $\lim_{\delta\to 0+} |M^{jk}(l + i\delta)| \leqslant K$

for $l \in \Delta$, $0 < \delta < \delta_0$, $j, k = 1, \ldots, n$.

*We assume the end points of Δ are not in the point spectrum.

†This assumption is weakened in § 6.

(iii) for all vector functions $\phi^i(l)$ defined on Δ,*

$$\phi^j(l)\overline{\phi^k(l)}\rho^{jk}(\Delta') - \phi^j(l)\overline{\phi^j(l)}\rho^{jj}(\Delta') \geqslant 0 \tag{2.10}$$

for $l \in \Delta' \subset \Delta$.

(iv) if s_{j+p}' are permutations of the regular solutions s_j according to the rules $s_{j+p}' = s_{j+p}$ for $j + p \leqslant n$ and $s_{j+p}' = s_{j+p-n}$ for $j + p > n$, then for $p = 1, \ldots, n$

$$\int_\Delta s_{j+p}' s_{k+p}' d\rho^{jk}(l) \tag{2.11}$$

is the kernel of a bounded operator with bound P^2.

The assumptions (i) and (ii) reduce to ones considered by Moser for the case $n = 2$ (**6**, pp. 367, 388). Assumption (i) asserts roughly that the operator q. is relatively bounded with respect to L_0. Assumption (ii) implies that M^{jk} does not have any poles in Δ so that Δ contains only continuous spectrum. Assumptions (iii) and (iv) are unnecessary in the case $n = 2$ as they are automatically satisfied. Assumption (iii) is a definiteness condition on the bilinear form associated with the matrix $\rho^{jk}(\Delta')$. This condition is trivially satisfied if $\rho^{jk}(\Delta')$ is diagonal and for that reason holds when $n = 2$. Assumption (iv) is the key assumption upon which our proof of Theorem 4 depends. The fact that (iv) holds when $n = 2$ is also used by Moser in his paper (**6**, p. 382). In § 3 we shall discuss the meaning of assumption (iv) and show that it is valid for a broad class of operators L_0.

3. Changes in boundary conditions at $t = 0$. In this section we shall study kernels

$$\int_\Delta s_j(t, l) s_k(\tau, l) d\tilde{\rho}^{jk}(l)$$

corresponding to self-adjoint boundary value problems of the form

$$L_0 u = \lambda u, \; [\tilde{\psi}_{0j}, u](0) = 0 \qquad j = 1, \ldots, \nu \tag{3.1}$$

where the functions $\tilde{\psi}_{0j}$ are linear combinations of ψ_{0j}. The object of this section is to show that, under certain restrictions on L_0, and by appropriate choice of the boundary functions $\tilde{\psi}_{0i}$, that the kernels (2.11) of assumption (iv) may be written in terms of the kernel

$$\int_\Delta s_j(t, l) s_k(\tau, l) d\tilde{\rho}^{jk}(l).$$

Therefore we will have a means of testing when assumption (iv) holds. The theorem is the following:

*Ibid.

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THEOREM 1. *If L_0 is a differential operator satisfying assumption* (ii) *and if the functions $f_\beta{}^j(\lambda)$ corresponding to L_0 satisfy the property that for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$, the determinants of the* $(\nu \times \nu)$ *minors of the matrix*

$$\begin{pmatrix} f^1_{\nu+1}(\lambda) \dots f^n_{\nu+1}(\lambda) \\ \dots \\ f^1_{\nu+\nu}(\lambda) \dots f^n_{\nu+\nu}(\lambda) \end{pmatrix} \tag{3.2}$$

have moduli greater than k_1 and less than k_2, $0 < k_1 < k_2$, and the difference of the arguments α of any two $(\nu \times \nu)$ minors lies in a sector such that $0 < \theta \leqslant \alpha \leqslant \pi - \theta < \pi$, $\sin\theta > k_1$, *then for some function $a_{jk}(l)$*

$$\int_\Delta s'_{j+p}(t, l) s'_{k+p}(\tau, l) d\rho^{jk}(l) = \int_\Delta s_j(t, l) s_k(\tau, l) a_{ij}(l) d\tilde{\rho}^{jk}(l) \tag{3.3}$$

where $a_{jk}(l)$ are uniformly bounded and $\tilde{\rho}^{jk}(\Delta)$ is the spectral density matrix corresponding to a self-adjoint problem $L_0 u = \lambda u$, $[\tilde{\psi}_{0j}, u](0) = 0, j = 1, \dots, \nu$.

Proof. First we introduce the notation j_p, j_p', j_p'' for permutations of $j = 1, \dots, n$ defined by:

$$\begin{cases} j_p = j + p, j + p \leqslant n, j_p = j + p - n, j + p > n \\ j_p' = j - p, j \geqslant p + 1, j_p' = n + j - p, j \leqslant p \\ j_p'' = j + p + \nu, j + p \leqslant \nu, j_p'' = j + p - \nu, j + p > \nu. \end{cases}$$

Define

$$\tilde{\psi}_{0j} = \delta^k_{j_p} \psi_{0k}, \quad j = 1, \dots, \nu.$$

Using (2.2) we get

$$[\tilde{\psi}_{0j}, \tilde{\psi}_{0k}](0) = 0, \qquad j, k = 1, \dots, \nu. \tag{3.4}$$

Formula (3.4) shows that the problem (3.1) is self adjoint when $\tilde{\psi}_{0j} = \delta_{j_p}{}^k \psi_{0k}$. Let $\tilde{M}^{jk}(\lambda)$ be the characteristic matrix corresponding to (3.1). Then $\tilde{M}^{jk}(\lambda)$ can be explicitly constructed (cf. § 2) as follows:

$$\tilde{M}^{jk}(\lambda) = \sum_{\alpha,\beta} \tilde{F}_{\alpha\beta} \tilde{f}_\beta{}^j \tilde{f}_\alpha{}^k, \quad \alpha = 1, \dots, \nu, \beta = \nu + 1, \dots, n \tag{3.5}$$

where

$$\tilde{f}_\alpha{}^j = \delta^j_{\alpha_p}, \quad \alpha = 1, \dots, \nu$$

and $\tilde{f}_\beta{}^j = f_\beta{}^j$, $\beta = \nu + 1, \dots, n$, $\tilde{F}_{\alpha\beta}$ is the inverse of $[\tilde{w}_\alpha, \tilde{w}_\beta](t)$, $\tilde{w}_\alpha = \tilde{f}_\alpha{}^j s_j(t, \lambda)$, $\tilde{w}_\beta = \tilde{f}_\beta{}^j s_j(t, \lambda)$. Using (2.2) we have

$$[\tilde{\omega}_\alpha, \tilde{\omega}_\beta] = \sum_{j=1}^{\nu} \delta^j_{\alpha_p} f_\beta^{j+\nu} - \delta^{j+\nu}_{\alpha_p} f^j_\beta = \begin{cases} f_\beta^{\nu+\alpha_p}, & \alpha_p \leqslant \nu \\ -f_\beta^{\alpha_p - \nu}, & \alpha_p > \nu \end{cases} \tag{3.6}$$

By (3.5), (3.6) $\tilde{M}^{jk}(\lambda)$ may be written*

*The sign is positive if $k \leqslant \nu$ and negative if $k \geqslant \nu$.

(3.7) $$\tilde{M}^{jk}(\lambda) = (\pm)\det \tilde{A}(j,k)/\det \tilde{B}, \qquad k = 1_p, 2_p, \ldots, \nu_p$$

where $\tilde{B}$ is the matrix

(3.8) $$\tilde{B} = \begin{pmatrix} f_{\nu+1}^{1_p''}(\lambda) \ldots f_{\nu+1}^{\nu_p''}(\lambda) \\ \ldots \\ f_{\nu+\nu}^{1_p''}(\lambda) \ldots f_{\nu+\nu}^{\nu_p''}(\lambda) \end{pmatrix}$$

and $\tilde{A}(j,k)$ is the matrix obtained from $\tilde{B}$ by replacing the elements of the k_p'th column with the terms $f_{\nu+1}{}^j, f_{\nu+2}{}^j, \ldots, f_{\nu+\nu}{}^j$. The hypothesis of the theorem implies that for $j, k = 1_p, 2_p, \ldots, \nu_p$, $K_1 \leqslant \det|\tilde{A}(j,k)| \leqslant k_2$.

Now that $\tilde{M}^{jk}$ has been constructed the remainder of the proof consists in demonstrating that (3.3) holds for some $a_{jk}(l)$. By the definition of j_p' we may write

(3.10) $$s'_{j+p}(t,l)s'_{k+p}(\tau,l)\mathscr{I}\{M^{jk}(\lambda)\} = s_j(t,l)s_k(\tau,l)\mathscr{I}\{M^{j'pk'p}(\lambda)\}.$$

(Note that

$$\mathscr{I}\{M^{j'pk'p}(\lambda)\} = 0, \quad j,k \neq 1_p, 2_p, \ldots, \nu_p.)$$

Now define $a_{jk}(\lambda)$ by the equation

(3.11) $$a_{jk}(\lambda) = \begin{cases} \mathscr{I}\{M^{j'pk'p}(\lambda)\}/\{\mathscr{I} M^{jk}(\lambda)\}, & j,k = 1_p, 2_p, \ldots, \nu_p \\ 0, \text{otherwise.} \end{cases}$$

Since $\tilde{M}^{jk} = \pm\det\tilde{A}(j,k)/\det\tilde{B}$ we have by (ii), (3.9)

(3.12) $$|a_{jk}(\lambda)| \leqslant Kk_2/k_1 \sin\theta < Kk_2/k_1{}^2$$

so that $a_{jk}(\lambda)$ are uniformly bounded, $l \in \Delta$, $0 < \delta < \delta_0$. By using (2.7), (3.11) and the theorem of Helly-Bray (**8**, p. 164) it follows that for $\Delta' \subset \Delta$

(3.13) $$\rho^{j'pk'p}(\Delta') = \lim_{\delta\to 0+}\frac{1}{\pi}\int_{\Delta'}\mathscr{I}\{M^{j'pk'p}(\lambda)\}\,dl$$
$$= \lim_{\delta\to 0+}\frac{1}{\pi}\int_{\Delta'} a_{jk}(\lambda)\mathscr{I}\{M^{jk}(\lambda)\}\,dl.$$

From (3.12), (3.13) we have

(3.14) $$|\rho^{j'pk'p}(\Delta')| \leqslant K\,k_2/k_1{}^2 \quad (\text{variation} \quad \rho^{jk}(\Delta')),\ \Delta' \subset \Delta.$$

The inequality (3.14) implies that functions $a_{ij}(l)$ exist (**8**, p. 215) such that

(3.15) $$\rho^{j'pk'p}(\Delta') = \int_{\Delta'} a_{ij}(l)d\rho^{jk}(l),\ \Delta' \subset \Delta.$$

By (3.10), (3.15) we obtain (3.3).

Theorem 1 leads to a sufficient condition that assumption (iv) should hold; if the hypothesis of Theorem 1 is satisfied and if

$$\int_\Delta s_j(t,l)s_k(\tau,l)a_{jk}(l)d\tilde{\rho}^{jk}(l)$$

is the kernel of a bounded operator then assumption (iv) holds. One can easily show by direct calculation that in the case $n = 2$ the hypothesis of Theorem 1 is satisfied and

$$\int_{\Delta} s_j(t, l)s_k(\tau, l)a_{jk}(l)d\tilde{\rho}^{jk}(l)$$

is the kernel of a bounded operator. Therefore, assumption (iv) holds automatically when $n = 2$ (**6**, p. 382).

4. Neumann series for the resolvent. Following (**1**, p. 560) we define functions $G^{(\nu)}(t, \tau, \lambda)$ by setting $G^{(0)} = G^{(0)}$ and

$$G^{(\nu)} = [+ G^{(\nu-1)}q]\cdot[G^{(0)}] = [+ G^0q]^\nu\cdot[G^0], \quad \nu = 1, 2, \ldots, \tag{4.1}$$

where the brackets indicate integration as follows

$$[G^0q]\cdot[G^0] = \int_0^\infty G^0(t, \xi, \lambda)q(\xi)G^0(\xi, \tau, \lambda)\, d\xi.$$

The object of this section is to show that $G^\epsilon = \sum(-\epsilon)^\nu G^{(\nu)}$ is the kernel of the resolvent of the operator H_ϵ.

LEMMA 1. *If* $G^{(\nu)}$ *is defined by* (4.1) *and assumptions* (i) *and* (ii) *hold, then for* $|\epsilon| < (\gamma K n^2)^{-1}$, $l \in \Delta$, $0 < \delta < \delta_0$ *the series* $G^\epsilon = \sum(-\epsilon)^\nu G^{(\nu)}$ *converges uniformly and absolutely and*

$$|G^{(\nu)}| \leqslant \Phi(t)\Phi(t)\gamma^\nu(Kn^2)^{\nu H}, \; \nu = 0, 1, 2 \ldots. \tag{4.2}$$

Proof. The inequality (4.2) holds for $\nu = 0$ by assumption (ii) and (2.4). Suppose (4.2) true for $(\nu - 1)$. Then by (4.1)

$$G^{(\nu)} = + \int_0^\infty G^{(0)}(t, \xi, \lambda)q(\xi)G^{(\nu-1)}(\xi, \tau, \lambda)\, d\xi. \tag{4.3}$$

Using assumptions (i), (ii), and (2.4) we get

$$\begin{aligned} |G^{(\nu)}| &\leqslant \left| \sum M^{jk}(\lambda)\left\{ s_j(t, \lambda) \int_0^t s_k(\xi, \lambda)q(\xi)G^{(\nu-1)}(\xi, \tau, \lambda)\, d\xi \right.\right. \\ &\qquad \left.\left. + s_k(t, \lambda) \int_t^\infty s_j(\xi, \lambda)q(\xi)G^{(\nu-1)}(\xi, \tau, \lambda)\, d\xi \right\}\right| \\ &\leqslant Kn^2\Phi(t) \int_0^\infty \Phi(\xi)|q(\xi)|\gamma^{\nu-1}\Phi(\xi)\Phi(\tau)(Kn^2)^\nu\, d\xi \\ &\leqslant (Kn^2)^{\nu+1}\gamma^{\nu-1}\Phi(t)\Phi(\tau) \int_0^\infty \Phi^2(\xi)|q(\xi)|\, d\xi \\ &\leqslant (Kn^2)^{\nu+1}\gamma^\nu\Phi(t)\Phi(\tau). \end{aligned} \tag{4.4}$$

This proves (4.2). The absolute convergence of the series for G^ϵ follows from (4.2). We also need the following lemma:

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LEMMA 2. *If*

$$\mathscr{G}^{(\nu)}(\lambda)u = \int_0^\infty G^{(\nu)} u \, d\tau$$

where $G^{(\nu)}$ *is defined by* (4.1) *and if assumptions* (i) *and* (ii) *hold, then* $\mathscr{G}^{(\nu)}(\lambda)$ *is a bounded operator and*

$$(4.5) \qquad || \, |q|^{\frac{1}{2}} \mathscr{G}^{(\nu)} u|| \leqslant (\gamma K n^2)^\nu \frac{\max |q|^{\frac{1}{2}}}{\mathscr{I}(\lambda)} ||u|| \qquad \nu = 0, 1, 2, \ldots,$$

Proof. For $\nu = 0$

$$|| \, |q|^{\frac{1}{2}} \mathscr{G}^{(0)}(\lambda) u|| \leqslant \max |q|^{\frac{1}{2}} ||| \mathscr{G}^{(0)}(\lambda)||| \, ||u|| \leqslant \max |q|^{\frac{1}{2}} \frac{1}{\mathscr{I}(\lambda)} ||u||.$$

Suppose (4.5) true for $(v - 1)$. Then using (2.4) and assumptions (i) and (ii),

$$(4.6) \quad | \, |q|^{\frac{1}{2}} \mathscr{G}^{(\nu)}(\lambda) u| \leqslant \sum |M^{jk}| \Big\{ |q(t)|^{\frac{1}{2}} |s_j(t, \lambda)| \int_0^t |s_k(\xi, \lambda) q(\xi) \mathscr{G}^{(\nu-1)} u| d\xi$$

$$+ |q(t)|^{\frac{1}{2}} |s_k(t, \lambda)| \int_t^\infty |s_j(\xi, \lambda) q(\xi) \mathscr{G}^{(\nu-1)} u| d\xi \Big\}$$

$$\leqslant (K n^2) |q(t)|^{\frac{1}{2}} \Phi(t) \int_0^\infty \Phi(\xi) |q(\xi) \mathscr{G}^{(\nu-1)} u| d\xi.$$

From (4.6) it follows

$$(4.7) \quad || \, | \cdot q|^{\frac{1}{2}} \mathscr{G}^{(\nu)} u||^2 = \int_0^\infty | \, |q|^{\frac{1}{2}} \mathscr{G}^{(\nu)} u|^2 dt$$

$$\leqslant (K n^2)^2 \int_0^\infty \Phi^2(t) |q(t)| dt \int_0^\infty \Phi^2(\xi) |q(\xi)| d\xi \int_0^\infty |q(\xi)| |\mathscr{G}^{(\nu-1)} u|^2 d\xi$$

$$\leqslant (K n^2)^2 \gamma^2 (\gamma K n^2)^{2\nu - 2} \left\{ \frac{\max |q|^{\frac{1}{2}}}{\mathscr{I}(\lambda)} \right\}^2 ||u||^2$$

$$\leqslant (\gamma K n^2)^{2\nu} \left\{ \frac{\max |q|^{\frac{1}{2}}}{\mathscr{I}(\lambda)} \right\}^2 ||u||^2.$$

Lemma 1 and Lemma 2 imply:

THEOREM 2. *If* $G^{(\nu)}$ *is defined by* (4.1) *and assumptions* (i) *and* (ii) *hold, then for* $|\epsilon| < (\gamma K n^2)^{-1}$, $l \in \Delta$, $0 < \delta < \delta_0$ *the series* $G^\epsilon = \sum (-\epsilon)^\nu G^{(\nu)}$ *represents the kernel of the resolvent* $R^\epsilon(\lambda) = (H_\epsilon - \lambda 1)^{-1}$ *of the operator* H_ϵ.

Proof. Let

$$\mathscr{B}^\epsilon(\lambda) = 1 + (+q) \sum_{\nu=0}^\infty (-\epsilon)^{\nu+1} \mathscr{G}^{(\nu)}(\lambda).$$

By Lemma 2 the series for $\mathscr{B}^\epsilon(\lambda)$ converges uniformly in norm for $|\epsilon| < (\gamma K n^2)^{-1}$ and defines a bounded operator. Since $\mathscr{G}^\epsilon(\lambda) = \mathscr{G}^{(0)}(\lambda) \mathscr{B}^\epsilon(\lambda)$ and both $\mathscr{G}^{(0)}(\lambda)$,

$\mathscr{B}^{\epsilon}(\lambda)$ are bounded operators it follows $\mathscr{G}^{\epsilon}(\lambda)$ is a bounded operator. In order to show that $\mathscr{G}^{\epsilon}(\lambda)$ is the resolvent it is sufficient to show the range of $\mathscr{G}^{\epsilon}(\lambda)$ is in $\mathscr{D}_{H_0}$ and

$$(4.8) \qquad (L_\epsilon - \lambda 1)\mathscr{G}^{\epsilon}(\lambda)u = u, \qquad u \in L_2(0,\infty)$$

$$(4.9) \qquad \mathscr{G}^{(\epsilon)}(\lambda)(L_\epsilon - \lambda 1)u = u, \qquad u \in \mathscr{D}_{H_0}.$$

Since the range of $\mathscr{G}^{0}(\lambda)$ is $\mathscr{D}_{H_0}$ and since $\mathscr{B}^{\epsilon}(\lambda)$ is bounded it follows the range $\mathscr{G}^{\epsilon}(\lambda)$ is contained in $\mathscr{D}_{H_0}$. Formula (4.8) can be proved by direct calculation using the definition of $G^{(\nu)}$ and Lemmas 1 and 2 (we shall omit the computation as it is standard (**1**, p. 562)). To prove (4′9) set

$$w = u - \mathscr{G}^{\epsilon}(\lambda)(L_\epsilon - \lambda 1)u, \quad u \in \mathscr{D}_{H_0}.$$

Since w is the difference of two elements of $\mathscr{D}_{H_0}$ it follows

$$w \in \mathscr{D}_{H_0}.$$

Then

$$(H_\epsilon - \lambda 1)w = (L_\epsilon - \lambda 1)w = (L_\epsilon - \lambda 1)u - (L_\epsilon - \lambda 1)\mathscr{G}^{\epsilon}(\lambda)(L_\epsilon - \lambda 1)u = 0.$$

This implies $w = (H_\epsilon - \lambda 1)^{-1}\, 0 = 0$.

For later use define the modified resolvent kernels $\bar{G}^{(\nu)}(t, \tau, \lambda)$ by

$$(4.5) \qquad \bar{G}^{(0)} = M^{jk}(l + i\delta)s_j(t, l)\, s_k(\tau, l), \qquad t \geqslant \tau$$

$$(4.6) \qquad \bar{G}^{(\nu)} = [\bar{G}^{(0)}q]^{\nu} \,.\, [\bar{G}^{0}] \qquad \nu = 1, 2 \ldots .$$

Since $s_j(t, \lambda)$ are entire in λ the functions $\bar{G}^{(\nu)}$ have the same type of singularities along the real axis as $G^{(\nu)}$. Also $\bar{G}^{(\nu)}$ satisfy Lemmas 1 and 2.

5. Analyticity of $E^{\epsilon}(\Delta)$. In this section we show that the spectral measure $E^{\epsilon}(\Delta)$ corresponding to H_ϵ is an analytic operator in a neighbourhood of $\epsilon = 0$. Define the function $\mathscr{E}^{(\nu)}(t, \tau)$ by

$$(5.1) \qquad \mathscr{E}^{(\nu)} = \lim_{\delta \to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} G^{(\nu)} d\lambda\right\}.$$

We shall show that $\mathscr{E}^{(\nu)}$ are kernels of bounded operators $E^{(\nu)}(\Delta)$ and that $E^{(\epsilon)}(\Delta) = \sum \epsilon^{\nu} E^{(\nu)}(\Delta)$ for sufficiently small ϵ. To do this first consider the approximate kernel $\hat{\mathscr{E}}^{(\nu)}$ defined by

$$(5.2) \qquad \hat{\mathscr{E}}^{(\nu)} = \lim_{\delta \to 0+} \frac{1}{\pi} \int_{\Delta} \mathscr{I}(\bar{G}^{(\nu)})\, dl$$

where $\bar{G}^{(\nu)}$ is defined by (4.6). We shall first prove that $\hat{\mathscr{E}}^{(\nu)} = \mathscr{E}^{(\nu)*}$:

LEMMA 3. *If $\mathscr{E}^{(\nu)}(\Delta)$, $\hat{\mathscr{E}}^{(\nu)}(\Delta)$ are defined by* (5.1) *and* (5.2) *and if assumptions* (i), (ii), *and* (iii) *hold, then* $\mathscr{E}^{(\nu)}(\Delta) = \hat{\mathscr{E}}^{(\nu)}(\Delta)$.

*The existence of $\hat{\mathscr{E}}^{(\nu)}$ is insured by (4.5), (4.6), and (ii) cf. (**9**, p. 346, 22.23).

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Proof. By a routine calculation which will be omitted one can show using (ii), (2.4), (4.1), and (4.6) that for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$,

$$|G^{(\nu)}(t, \tau, \lambda) - \bar{G}^{(\nu)}(t, \tau, \lambda)| \leqslant M_1\delta, \tag{5.3}$$

where M_1 depends on (t, τ) but is independent of λ. Using (5.3) we have

$$\mathscr{E}^{(\nu)}(\Delta) = \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} G^{(\nu)} d\lambda\right\} = \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} \bar{G}^{(\nu)} d\lambda\right\}. \tag{5.4}$$

Next (4.2) implies

$$\lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} \bar{G}^{(\nu)} d\lambda\right\} = \lim_{\delta\to 0+} \frac{1}{\pi} \int_{\Delta} \mathscr{I}(\bar{G}^{(\nu)}) dl = \hat{\mathscr{E}}^{(\nu)}. \tag{5.5}$$

By (5.4) and (5.5) $\mathscr{E}^{(\nu)}(\Delta) = \hat{\mathscr{E}}^{(\nu)}(\Delta)$.

THEOREM 3. *If* $\hat{\mathscr{E}}^{(\nu)}(\Delta)$ *is defined by* (5.2) *and if assumptions* (i), (ii), (iii), *and* (iv) *hold then* $\hat{\mathscr{E}}^{(\nu)}(\Delta)$ *is the kernel of a bounded operator* $E^{(\nu)}(\Delta)$ *and*

$$|(E^{(\nu)}(\Delta)u, v)| \leqslant p^2(4\nu)(\gamma K n^2)^{\nu} n^3 ||u||\, ||v|| \qquad u, v \in L_2(0, \infty). \tag{5.6}$$

Proof. From the definition of $\bar{G}^{(\nu)}$ one can show by induction that

$$\mathscr{I}(\bar{G}^{(\nu)}) = \sum_{\mu+\chi=\nu} [\overline{\bar{G}^{(\nu)}} q]^{\mu} \cdot \mathscr{I}(\bar{G}^0) \cdot [q\bar{G}^{(0)}]^{\chi}. \tag{5.7}$$

Next by (2.4) and (4.5) $\mathscr{I}(\bar{G}^0) = \mathscr{I}(M^{jk}) s_j(t, l) s_k(\tau, l)$, $t \geqslant \tau$ and (5.7) may be written

$$\mathscr{I}(\bar{G}^{(\nu)}) = \sum_{\mu+\chi=\nu} \bar{H}_j^{(\mu)}(t) H_k^{(\chi)}(\tau) \mathscr{I}(M^{jk}) \tag{5.8}$$

where

$$\begin{aligned}(5.9)\quad H_j^{(\mu)}(t) = \sum_{p,m} M^{pm} s_p(t, l) \int_0^t d\xi_1 \int_0^\infty & s_m(\xi_1, l) q(\xi_1) \\ & \bar{G}^{(\mu-2)}(\xi_1, \xi_2, \lambda) q(\xi_2) s_j(\xi_2, l) d\xi_2 \\ + M^{mp} s_p(t, l) \int_t^\infty d\xi_1 \int_0^\infty & s_m(\xi_1, l) q(\xi_1) \bar{G}^{(\mu-2)}(\xi_1, \xi_2, \lambda) q(\xi_2) s_j(\xi_2, l) d\xi_2 \\ = \sum_p s_p(t, l) \left\{\int_0^t \eta_{j,p}^{(\mu)}(\xi, \lambda) d\xi \right. & \left. + \int_t^\infty \zeta_j^{(\mu)}(\xi, \lambda) d\xi\right\}.\end{aligned}$$

The integrals in (5.9) converge absolutely and may be estimated using (4.2). Define for fixed values of j, p, μ (no summation)

$$_1Q_{p,\mu}^j(t, \lambda) = s_p(t, l) \int_0^t \eta_{j,p}^{(\mu)}(\xi, \lambda) d\xi \tag{5.10}$$

$$_2Q_{p,\mu}^j(t, \lambda) = s_p(t, l) \int_t^\infty \zeta_{j,p}^{(\mu)}(\xi, \lambda) d\xi. \tag{5.11}$$

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Using (ii) and (4.2) it is easily seen that

$$(5.12)\qquad \left|\int_0^t \eta_{j,p}^{(\mu)}(\xi,\lambda)d\xi\right| \leqslant (\gamma K n^2)^\mu$$

$$\left|\int_t^\infty \zeta_{j,p}^{(\mu)}(\xi,\lambda)d\xi\right| \leqslant (\gamma K n^2)^\mu.$$

With the notation j_p introduced in Theorem 1 equation (5.8) becomes

$$(5.13)\qquad \mathscr{I}(\bar{G}^{(\nu)}) = \sum_{\mu+\chi=\nu}\ \sum_{i_1,i_2=1} ({}_{i_1}\bar{Q}^j_{p,\mu})({}_{i_2}Q^k_{r,\chi})\mathscr{I}(M^{jk})$$

$$= \sum_{\mu+\chi=\nu}\ \sum_{i_1 i_2=1}^{2} ({}_{i_1}\bar{Q}^j_{j_p,\mu})({}_{i_2}Q^k_{k_r,\chi})\mathscr{I}(M^{jk}).$$

When (5.13) is inserted in (5.2) and operations of limit and integration are interchanged we get for $u, v \in \mathscr{D}_\infty$

$$(5.14)\qquad (E^{(\nu)}(\Delta)u, v) = \sum_{\mu+\chi=\nu}\ \sum_{i_1,i_2=1}^{2} \lim_{\delta\to 0+}\frac{1}{\pi}\int_\Delta ({}_{i_1}\bar{Q}^j_{jp\mu}, v)({}_{i_2}Q^k_{kr,\chi}, \bar{u})\,\mathscr{I}(M^{jk}(\lambda)dl.$$

The interchange of limit operations in (5.14) is justified since the integrand is less than an absolutely integrable function (the integrand is less than $\Phi(t)|v(t)|\Phi(\tau)|u(\tau)|(\gamma K n^2)^\nu 2K$ by (5.12) and (ii) and this function is integrable for $u, v \in \mathscr{D}_\infty$). The remainder of the proof consists in estimating the term of (5.14). For p, r, μ, i_1, i_2 fixed (no summation) we have by the Schwarz inequality

$$(5.15)\qquad \left|\int_\Delta ({}_{i_1}\bar{Q}^j_{jp,\mu}, v)({}_{i_q}Q^k_{kr,\chi}, \bar{u})\mathscr{I}(M^{jk}(\lambda)dl\right|^2$$

$$\leqslant \int_\Delta ({}_{i_1}\bar{Q}^j_{jp\mu}, v)({}_{i_1}Q^k_{kp\mu}, \bar{v})\mathscr{I}(M^{jk}(\lambda))dl$$

$$\times \int_\Delta ({}_{i_2}Q^j_{jr,\chi}, \bar{u})({}_{i_2}\bar{Q}^k_{kr,\chi}, u)\mathscr{I}(M^{jk}(\lambda))dl$$

since $\mathscr{I}(M^{jk}(\lambda))$ is a non-negative matrix (of (**5**, p. 534)).

Again since $\mathscr{I}(M^{jk})$ is a non-negative matrix $|\mathscr{I}(M^{jk}(\lambda))| \leqslant (\mathscr{I}(M^{jj}))^{\frac{1}{2}}(\mathscr{I}(M^{kk}))^{\frac{1}{2}}$, and we have

$$(5.16)\qquad \left|\int_\Delta ({}_{i_1}Q^j_{jp,\mu}, \bar{v})({}_{i_1}\bar{Q}^k_{kp\mu}, v)\mathscr{I}(M^{jk}(\lambda))dl\right|$$

$$\leqslant \int_\Delta |({}_{i_1}Q^j_{jp,\mu}, \bar{v})||({}_{i_1}\bar{Q}^k_{kp,\mu}, v)|(\mathscr{I}(M^{ij}))^{\frac{1}{2}}(\mathscr{I}(M^{kk}))^{\frac{1}{2}}dl$$

$$\leqslant \frac{n}{2}\left(\int_\Delta |({}_{i_1}Q^j_{jp,\mu}, \bar{v})|^2\mathscr{I}(M^{jj})dl + \int_\Delta |({}_{i_1}\bar{Q}^k_{kp,\mu}, v)|^2\mathscr{I}(M^{kk})dl\right)$$

$$\leqslant n\int_\Delta |{}_{i_1}Q^j_{jp,\mu}, \bar{v})|^2\mathscr{I}(M^{jj}(\lambda))dl.$$

By (5.10) and the Schwarz inequality

$$(5.17)\quad |({}_1Q^j_{jp,\mu}, v)|^2 = \left|\int_0^\infty \eta_j^{(\mu)}(t, \lambda)dt \int_t^\infty s_{jp}(\xi, l)v(\xi)d\xi\right|^2$$

$$\leqslant \int_0^\infty |\eta^{(\mu)}(t)|dt \int_0^\infty |\eta^{(\mu)}(t)| \left|\int_t^\infty s_{j_p}(\xi, l)v(\xi)d\xi\right|^2 dt$$

where $|\eta^{(\mu)}(t)| = \sup_{j\lambda p} |\eta^\mu_{j,p}|$. Now apply assumptions (iii) and (iv), and (5.12), (5.16), and (5.17) to obtain

$$(5.18)\quad \lim_{\delta\to 0+} \frac{1}{\pi}\left|\int_\Delta ({}_1Q^j_{jp,\mu}, \bar{v})({}_1\bar{Q}^k_{kp,\mu}, v)\mathscr{I}(M^{jk}(\lambda))dl\right|$$

$$\leqslant (\gamma K n^2)^\mu \int_0^\infty |\eta^{(\mu)}(t)| \lim_{\delta\to 0+}\frac{1}{\pi}\int_\Delta \left|\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right|^2 \mathscr{I}(M^{jj}(\lambda)dl\, dt$$

$$= (\gamma K n^2)^\mu \int_0^\infty |\eta^{(\mu)}(t)| \int_\Delta \left|\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right|^2 d\rho^{jj}(l)dt$$

$$\leqslant (\gamma K n^2)^\mu \int_0^\infty |\eta^{(\mu)}(t)| \int_\Delta \left(\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right)\left(\int_t^\infty s_{pk}(\xi, l)v(\xi)\xi\right) d\rho^{jk}(l)dt$$

$$\leqslant n(\gamma K n^2)^{2\mu}p^2||v||^2, \qquad v \in \mathscr{D}_\infty.$$

The identity

$$\lim_{\delta\to 0+}\frac{1}{\pi}\int_\Delta \left|\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right|^2 \mathscr{I}(M^{jk}(\lambda))dl = \int_\Delta \left|\int_t^\infty s_{pj}(\xi, l)v(\xi)d\xi\right|^2 d\rho^{jj}(l)$$

is by the theorem of Helly-Bray (**8**, pp. 163, 209). In exactly the same manner as (5.18) was obtained we get

$$(5.19)\qquad \lim_{\delta\to 0+}\int_\Delta ({}_iQ^j_{j,\mu}, \bar{v})({}_i\bar{Q}^j_{jp\mu}, v)\mathscr{I}(M^{jk})dl \leqslant (\gamma K n^2)^{2\mu}p^2||v||^2 n$$

$$(5.20)\qquad \lim_{\delta\to 0+\Delta} ({}_iQ^k_{kr,\chi}, u)({}_i\bar{Q}^k_{kr,\chi}, \bar{u})\mathscr{I}(M^{jk})dl \leqslant (\gamma K n^2)^{2\chi}p^2||u||^2 n$$

$$i = 1, 2.$$

Using (5.14), (5.14), (5.19), and (5.20) we have

$$(5.21)\qquad |(\mathrm{E}^{(\nu)}(\Delta)u, v)| \leqslant \nu P^2(\gamma K n^2)^\nu(4n^3)||u||\,||v||, \qquad u, v \in \mathscr{D}_\infty.$$

The inequality (5.20) must hold for all u, v in $L_2(0, \infty)$ and $E^{(\nu)}(\Delta)$ determines a bounded operator by a theorem by Frechet (**6**, p. 385).

Now we shall state our main theorem:

THEOREM 3. *If $L_\epsilon = L_0 + \epsilon q$ is a differential operator such that the problem $L_0 u = \lambda u$, $[\psi_{0j}, u](0) = 0$, $j = 1, \ldots, \nu$ is self adjoint and satisfies conditions* (i), (ii), (iii), *and* (iv) *then for $|\epsilon| < (\gamma K n^2)^{-1}L_\epsilon$ determines a self-adjoint operator H_ϵ and the spectral measure $E^\epsilon(\Delta)$ corresponding to H_ϵ is an analytic operator.*

Proof. For $|\epsilon| < (\gamma K n^2)^{-1}$ we have the equalities

$$(5.22) \qquad \sum \epsilon^\nu (E^{(\nu)}(\Delta)u, v) = \sum (-\epsilon)^\nu \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} (\mathscr{G}^{(\nu)}u, v)d\lambda\right\}$$
$$= \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} \sum (-\epsilon)^\nu (\mathscr{G}^{(\nu)}u, v)d\lambda\right\}$$
$$= \lim_{\delta\to 0+} \frac{1}{\pi} \mathscr{I}\left\{\int_{\Gamma(\delta)} (\mathscr{G}^{\epsilon}u, v)d\lambda\right\}$$
$$= (E^\epsilon u, v), \qquad u, v \in \mathscr{D}_\infty.$$

The first two equalities in (5.21) follow from (5.1) and the fact that the function $G^{(\nu)}(t, \tau, \lambda)u(\tau)v(t)$ is less than an integrable function for $u, v \in D_\infty$. (By Lemma 1

$$|G^{(\nu)}(t, \tau, \lambda)u(\tau)\overline{v(t)}| \leqslant \gamma^\nu (Kn^2)^{\nu+1}\Phi(t)\Phi(\tau)|u(\tau)||\overline{v(t)}|$$

and $\Phi(t)\Phi(\tau)|u(\tau)||v(t)|$ is integrable when $u, v \in D_\infty$.) The third equality in (5.22) is by Theorem 2 and the fourth equality in (5.21) is by (2.5). From (5.6) and (5.22) it follows that $E^\epsilon(\Delta)$ is a bounded analytic operator by a theorem of Frechet (**6**, p. 385).

6. Weakened assumptions. The restrictions placed on q in preceding sections may be weakened. In fact Theorems 3 and 4 remain valid when assumption (i) is replaced by

$$\text{(i)}' \qquad \int_0^\infty \Phi_1^2(t)|q(t)|^\nu dt \leqslant \gamma_1 < \infty \qquad \nu = 1, 2, \ldots,$$

where $\Phi_1(t) = \sup |s_j(t, l)|, j = 1, \ldots, n, l \in \Delta$. It is not necessary to assume q bounded. We shall omit giving the details of the proof of how Theorems 3 and 4 follow from (i)′ but simply outline the necessary steps in the argument: First of all one observes, by reviewing the proof of Theorems 3 and 4, that the series $\hat{E}^\epsilon(\Delta) = \sum \epsilon^\nu E^\nu(\Delta)$ represents a bounded operator for $|\epsilon| < (\gamma_1 K n^2)^{-1}$. It remains to redefine H_ϵ, show it self adjoint with domain $\mathscr{D}_{H_0}$, and show that $\hat{E}^\epsilon(\Delta)$ is the spectral measure of H_ϵ. To define H_ϵ one shows that $\mathscr{G}^\epsilon(\lambda)$, defined in Theorem 2, is a bounded operator for $|\epsilon| < (\gamma_1 K n^2)^{-1}$, $\mathscr{I}(\lambda) > 4$ using (i)′. Then $H_\epsilon - \lambda 1$ is defined to be the inverse of $\mathscr{G}^\epsilon(\lambda)$. Using properties of $\mathscr{G}^\epsilon(\lambda)$ one shows H_ϵ is self adjoint,

$$\mathscr{D}_{H_\epsilon} = \mathscr{D}_{H_0}, \quad L_\epsilon u = H_\epsilon u, \quad u \in \mathscr{D}_{H_0}.$$

Finally to show that $\hat{E}^\epsilon(\Delta)$ is the spectral measure corresponding to H_ϵ we use a limiting argument. Define operators $L_\epsilon(a, b) = L_0 + \epsilon q(a, b, t)$ where

$$(6.1) \qquad q(a, b, t) = \begin{cases} q(t)\Phi_1^2(t)/\Phi_b^2(t), & t \leqslant a \\ 0, & t > a \end{cases}$$

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and $\Phi_b(t) = \sup |s_j(t, \lambda)|$, $j = 1, \ldots, n$, $l \in \Delta$, $0 < \delta < b$. The operators $L_\epsilon(a, b)$ satisfy assumption (i) so that Theorems 2, 3, and 4 hold for $L_\epsilon(a, b)$, $|\epsilon| < (\gamma_1 K n^2)^{-1}$. Now the resolvent $\mathscr{G}^\epsilon(\lambda, a, b)$ of $H_\epsilon(a, b)$ converges strongly to the resolvent $\mathscr{G}^\epsilon(\lambda)$ of H_ϵ, $a \to \infty$, $b \to 0$. By a well-known theorem of Rellich the spectral measure $E^\epsilon(\Delta, a, b)$ converges strongly to $E^\epsilon(\Delta)$, $a \to \infty$, $b \to 0$. On the other hand, $E^\epsilon(\Delta, a, b)$ converges strongly to $\hat{E}^\epsilon(\Delta)$ so $E^\epsilon(\Delta) = \hat{E}^\epsilon(\Delta)$.

Note that the results of § 5 hold if L_0 has a singular point at $t = 0$ since the boundary conditions there are given in the abstract form **(6)**.

It is important to consider weakening assumption (iii). An alternative assumption is the following:

(iii)′ There exists a unimodular matrix $V_j^k(\lambda)$ which is analytic in λ, $l \in \Delta$, $-\delta_0 < \delta < \delta_0$, $\bar{V}_j^k(\lambda) = V_j^k(\bar{\lambda})$ such that the spectral density matrix $\tilde{\rho}^{jk}(l)$ defined by

$$(6.2) \qquad \tilde{\rho}^{jk}(l) = \int_\alpha^l V_r^j(l) V_s^k(l) d\rho^{rs}(l) \qquad \Delta = [\alpha, \beta]$$

is a diagonal matrix. We may derive Theorem 3 using (iii)′ in place of (iii) simply by using $\tilde{\rho}^{jk}$ in place of ρ^{jk} and also $\tilde{s}_j = U_j^k s_k$, $\tilde{M}^{jk} = V_r^j V_s^k M^{rs}$ in place of s_j, M^{jk}. (U_j^k means the inverse of V_k^j **(5**, p. 536**)**.

An alternative to assumption (iv) is the following set of three conditions:
(iv)′ $M^{jk} = 0, j = r + 1, \ldots, n$.
(iv)″ if s_{j+p}' are permutations of the regular solutions s_j, $j = 1, \ldots, r$ according to the rules $s_{j+p}' = s_{j+p}$, $j + p \leqslant r$, $s_{j+p}' = s_{j+p-r}$, $j + p > r$ then for $p = 1, \ldots, r$

$$\int_\Delta s'_{j+p}(t, l) s'_{k+p}(\tau, l) d\rho^{jk}(l)$$

are kernels of bounded operators with bound P^2.

(iv)‴ for $k = r + 1, \ldots, n$.

$$\int_0^\infty \left(|M^{jk}| |s_j|^2 \left(\int_0^t |s_k|^2 dt \right) \right) |q| dt < P^2, \qquad l \in \Delta.$$

We may derive Theorem 3 with (iv)′, (iv)″, and (iv)‴ in place of assumption (iv) by minor modifications of the argument. Formulas (5.19) and (5.20) must be re-proved using (iv)″ when $i = 1$ and (iv)‴ when $i = 2$.

For the case $n = 4$ and $L_0 = d^4/dt^4$, $\psi_{01} = t$, $\psi_{02} = t^3/3!$ assumptions (ii), (iii)′, (iv)′, (iv)″, and (iv)‴ are satisfied with $r = 3$ provided $\Delta = [\alpha, \beta]$ is any interval of the form $0 < \alpha \leqslant t \leqslant \beta < \infty$. The expansion theorem for this case has been obtained by Windau **(10)**. Using Windau's results one may easily verify that assumptions (ii), (iii)′, (iv)′, (iv)″, (iv)‴ hold.

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