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RESEARCH ARTICLE

Some remarks on Riesz transforms on exterior Lipschitz domains

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Abstract
Let $n \geq 2$ and $\mathcal{L} = -\mathrm{div}(A\nabla\cdot)$ be an elliptic operator on $\mathbb{R}^n$. Given an exterior Lipschitz domain Ω, let $\mathcal{L}_D$ be the elliptic operator $\mathcal{L}$ on Ω subject to the Dirichlet boundary condition. Previously, it was known that the Riesz transform $\nabla\mathcal{L}_D^{-1/2}$ is not bounded for $p > 2$ and $p \geq n$, even if $\mathcal{L} = \Delta$ is the Laplace operator and Ω is a domain outside a ball. Suppose that A are CMO coefficients or VMO coefficients satisfying certain perturbation property, and $\partial\Omega$ is C^1. We prove that for $p > 2$ and $p \in [n, \infty)$, it holds that

$$\inf_{\phi\in\mathcal{K}_p(\mathcal{L}_D^{1/2})} \|\nabla(f-\phi)\|_{L^p(\Omega)} \sim \left\|\mathcal{L}_D^{1/2} f\right\|_{L^p(\Omega)}$$

for $f \in \mathring{W}^{1,p}(\Omega)$. Here, $\mathcal{K}_p(\mathcal{L}_D^{1/2})$ is the kernel of $\mathcal{L}_D^{1/2}$ in $\mathring{W}^{1,p}(\Omega)$, which coincides with $\tilde{\mathcal{A}}_0^p(\Omega) := \{f \in \mathring{W}^{1,p}(\Omega) : \mathcal{L}_D f = 0\}$ and is a one-dimensional subspace. As an application, we provide a substitution of L^p-boundedness of $\sqrt{t}\nabla e^{-t\mathcal{L}_D}$ which is uniform in t for $p \geq n$ and $p > 2$.

Contents

1. Introduction and main results

In this paper, motivated by the recent works [10, 16, 20] on the Riesz transform on exterior Lipschitz domains, we continue to study the boundedness of the Riesz transform, associated with second-order divergence form elliptic operators on the exterior Lipschitz domain Ω having the Dirichlet boundary condition, on $L^p(\Omega)$ with $p \in (2, \infty)$.

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain; that is, $\mathbb{R}^n \setminus \overline{\Omega}$ is a bounded Lipschitz domain of $\mathbb{R}^n$, where $\overline{\Omega}$ denotes the closure of Ω in $\mathbb{R}^n$. Recall that a bounded domain O is Lipschitz provided for each point x in the boundary ∂O, there is $r > 0$, such that $B(x, r) \cap \partial O$ is a rotated graph of Lipschitz function. Furthermore, assume that $A \in L^\infty(\mathbb{R}^n)$ is a real-valued and symmetric matrix that satisfies the uniformly elliptic condition; that is, there exists a constant $\mu_0 \in (0, 1]$ such that, for any $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$\mu_0|\xi|^2 \le (A(x)\xi, \xi) \le \mu_0^{-1}|\xi|^2,$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$.

Denote by $\mathcal{L}$ the operator $-\mathrm{div}(A\nabla\cdot)$ on $\mathbb{R}^n$, and by $\mathcal{L}_D$ the operator $-\mathrm{div}(A\nabla\cdot)$ on Ω subject to the Dirichlet boundary condition (see, for instance, [24, Section 4.1] for the detailed definitions of $\mathcal{L}$, $\mathcal{L}_D$). When $A := I_{n\times n}$ (the unit matrix), we simply denote these operators respectively by Δ and Δ_D. Moreover, let $O \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Denote by $\mathcal{L}_{D,O}$ the operator $-\mathrm{div}(A\nabla\cdot)$ on O subject to the Dirichlet boundary condition.

Let U be a domain in $\mathbb{R}^n$ or $U = \mathbb{R}^n$. Denote by $\mathcal{D}(U)$ the space of all infinitely differentiable functions with compact support in U endowed with the inductive topology, and by $\mathcal{D}'(U)$ the topological dual of $\mathcal{D}(U)$ with the weak-$*$ topology which is called the space of distributions on U. Let $p \in (1, \infty)$. For any $x \in \mathbb{R}^n$, let $\rho(|x|) := (1 + |x|^2)^{1/2}$ and $\lg(|x|) := \ln(2 + |x|^2)$.

We define the weighted Sobolev space $W^{1,p}(\mathbb{R}^n)$ by

$$W^{1,p}(\mathbb{R}^n) := \left\{u \in \mathcal{D}'(\mathbb{R}^n) : \|u\|_{W^{1,p}(\mathbb{R}^n)} := \left\|\frac{u}{\rho(|x|)}\right\|_{L^p(\mathbb{R}^n)} + \|\,|\nabla u|\,\|_{L^p(\mathbb{R}^n)} < \infty\right\}$$

when $p \ne n$, and

$$W^{1,n}(\mathbb{R}^n) := \left\{u \in \mathcal{D}'(\mathbb{R}^n) : \|u\|_{W^{1,n}(\mathbb{R}^n)} := \left\|\frac{u}{\rho(|x|)\lg(|x|)}\right\|_{L^p(\mathbb{R}^n)} + \|\,|\nabla u|\,\|_{L^p(\mathbb{R}^n)} < \infty\right\},$$

where ∇u denotes the *distributional gradient* of u; see [1, 2], for instance. Moreover, for the *exterior domain* Ω, the weighted Sobolev space $W^{1,p}(\Omega)$ is defined via replacing $\mathcal{D}'(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ in the definition of $W^{1,p}(\mathbb{R}^n)$, respectively, by $\mathcal{D}'(\Omega)$ and $L^p(\Omega)$, and the weighted Sobolev space $\mathring{W}^{1,p}(\Omega)$ is defined as the completion of $\mathcal{D}(\Omega)$ under the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. Moreover, for any $q \in (1, \infty)$, denote by $W^{-1,q}(\mathbb{R}^n)$, $W^{-1,q}(\Omega)$, and $\mathring{W}^{-1,q}(\Omega)$, respectively, the dual spaces of $W^{1,q'}(\mathbb{R}^n)$, $W^{1,q'}(\Omega)$, and $\mathring{W}^{1,q'}(\Omega)$, where $q' := q/(q-1)$.

We also recall some useful properties for the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $W^{1,p}(\Omega)$, and $\mathring{W}^{1,p}(\Omega)$ established in [1, 2] as following.

Remark 1.1. Let $n \ge 2$, $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain, and $p \in (1, \infty)$.

(i) $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$ and $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$. Here, $\mathcal{D}(\overline{\Omega})$ denotes the space of all infinitely differentiable functions with compact support in $\overline{\Omega}$. Furthermore, constants belong to $W^{1,p}(\mathbb{R}^n)$ or $W^{1,p}(\Omega)$ when $p \in [n, \infty)$, but constants do not belong to $W^{1,p}(\mathbb{R}^n)$ and $W^{1,p}(\Omega)$ when $p \in (1, n)$.

(ii) Let $U = \mathbb{R}^n$ or $U = \Omega$. For any $u \in W^{1,p}(U)$, define the semi-norm $[u]_{W^{1,p}(U)} := \|\,|\nabla u|\,\|_{L^p(U)}$. When $p \in (1, n)$, the semi-norm $[\cdot]_{W^{1,p}(U)}$ is a norm on $W^{1,p}(U)$ equivalent to the full norm $\|\cdot\|_{W^{1,p}(U)}$; when $p \in [n, \infty)$, the semi-norm $[\cdot]_{W^{1,p}(U)}$ defines on the quotient space $W^{1,p}(U)/C$ a norm which is equivalent to the quotient norm (see [2, Proposition 9.3] and [1, Theorem 1.1]). Moreover, the semi-norm $[\cdot]_{W^{1,p}(\Omega)}$ is a norm on $\mathring{W}^{1,p}(\Omega)$ that is equivalent to the full norm $\|\cdot\|_{W^{1,p}(\Omega)}$ for all $1 < p < \infty$ (see [1, Theorem 1.2]).

For a bounded Lipschitz domain $O \subset \mathbb{R}^n$ and $1 < p < \infty$, the Sobolev space $W^{1,p}(O)$ is defined as usual – that is, $f \in \mathcal{D}'(O)$ with

$$\|f\|_{W^{1,p}(O)} := \|f\|_{L^p(O)} + \|\,|\nabla f|\,\|_{L^p(O)} < \infty.$$

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Furthermore, $\mathring{W}^{1,p}(O)$ is defined to be the *closure* of $\mathcal{D}(O)$ in $W^{1,p}(O)$, and $W^{-1,p}(O)$ and $\mathring{W}^{-1,p}(O)$ are defined as the dual spaces of $W^{1,p'}(O)$ and $\mathring{W}^{1,p'}(O)$, respectively.

It is well known that the boundedness of the Riesz transform associated with some differential operators on various function spaces has important applications in harmonic analysis and partial differential equations and has aroused great interests in recent years (see, for instance, [3, 4, 7, 8, 10, 15, 17, 20, 25, 28]). In particular, let O be a bounded Lipschitz domain of $\mathbb{R}^n$. The sharp boundedness of the Riesz transform $\nabla\mathcal{L}_{D,O}^{-1/2}$ associated with the operator $\mathcal{L}_{D,O}$ having the Dirichlet boundary condition on the Lebesgue space $L^p(O)$ was established by Shen [25].

Compared with the boundedness of the Riesz transform associated with differential operators on bounded Lipschitz domains, there are relatively few literatures for the Riesz transform associated with differential operators on exterior Lipschitz domains. Since the heat kernel generated by $\mathcal{L}_D$ satisfies the Gaussian upper bound estimate, it follows from the results of Sikora [27] (see also [7]) that the Riesz transform $\nabla\mathcal{L}_D^{-1/2}$ is always bounded on $L^p(\Omega)$ for $p \in (1, 2]$. By studying weighted operators in the one dimension, Hassell and Sikora [10] discovered that the Riesz transform $\nabla\Delta_D^{-1/2}$ on the exterior of the unit ball is *not* bounded on L^p for $p \in (2, \infty)$ if $n = 2$, and $p \in [n, \infty)$ if $n \geq 3$; see also [22] for the case $n = 3$. Moreover, Killip, Visan and Zhang [20] proved that the Riesz transform $\nabla\Delta_D^{-1/2}$ on the exterior of a smooth convex obstacle in $\mathbb{R}^n$ ($n \geq 3$) is bounded for $p \in (1, n)$. Very recently, characterizations for the boundedness of the Riesz transform $\nabla\mathcal{L}_D^{-1/2}$ on $L^p(\Omega)$ with $p \in (2, n)$ was obtained in [16].

Let

$$p(\mathcal{L}) := \sup\{p > 2 : \ \nabla\mathcal{L}^{-1/2} \text{ is bounded on } L^p(\mathbb{R}^n)\}.$$

Furthermore, denote by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the set of *all locally integrable functions on* $\mathbb{R}^n$. Recall that the *space* $\mathrm{BMO}(\mathbb{R}^n)$ is defined as the set of all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \left| f(x) - \frac{1}{|B|}\int_B f(y)\,dy \right| dx < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ (see, for instance, [19, 29]). Moreover, the *space* $\mathrm{CMO}(\mathbb{R}^n)$ is defined as the completion of $\mathcal{D}(\mathbb{R}^n)$ in the space $\mathrm{BMO}(\mathbb{R}^n)$ (see, for instance, [6]). The *space* $\mathrm{VMO}(\mathbb{R}^n)$ is defined as the set of $f \in \mathrm{BMO}(\mathbb{R}^n)$ satisfying

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,r)|}\int_{B(x,r)} \left| f(y) - \frac{1}{|B(x,r)|}\int_{B(x,r)} f(z)\,dz \right| dy = 0.$$

Note that $\mathrm{CMO}(\mathbb{R}^n) \subsetneq \mathrm{VMO}(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n)$. Let us recall some results proved in [16, Theorems 1.3 and 1.4].

Theorem 1.2 [16]. *Let $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain, $n \geq 2$.*

(i) *For all $p \in (1, \infty)$, it holds for all $f \in \mathring{W}^{1,p}(\Omega)$ that*

$$\left\| \mathcal{L}_D^{1/2} f \right\|_{L^p(\Omega)} \leq C\|\nabla f\|_{L^p(\Omega)}. \tag{1.1}$$

(ii) *Suppose that $A \in \mathrm{VMO}(\mathbb{R}^n)$ and $n \geq 3$. There exist $\epsilon > 0$ and $C > 1$ such that, for all $f \in \mathring{W}^{1,p}(\Omega)$, it holds that*

$$C^{-1}\|\nabla f\|_{L^p(\Omega)} \leq \left\| \mathcal{L}_D^{1/2} f \right\|_{L^p(\Omega)} \leq C\|\nabla f\|_{L^p(\Omega)}, \tag{1.2}$$

where $1 < p < \min\{n, p(\mathcal{L}), 3 + \epsilon\}$. If Ω is C^1, then (1.2) holds for all $1 < p < \min\{n, p(\mathcal{L})\}$.

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Remark 1.3. The version of (1.1) for Neumann boundary operators $\mathcal{L}_N$ has been recently proved in [9] on complete manifolds with ends. Although the results in [9] were presented in smooth manifolds setting, their proofs extend to exterior Lipschitz domains almost identically and show that, for all $p \in (1,\infty)$,

$$\left\|\mathcal{L}_N^{1/2} f\right\|_{L^p(\Omega)} \le C\|\nabla f\|_{L^p(\Omega)}.$$

Note that the heat kernel satisfies two side Gaussian bounds; see [16, Proof of Theorem 1.2].

For the case $\mathcal{L} = \Delta$ being the Laplacian operator and Ω being C^1, $p(\mathcal{L}) = \infty$ and $\epsilon = \infty$. In this case, it follows from the above results that $\nabla\Delta_D^{-1/2}$ is bounded on $L^p(\Omega)$ for $1 < p < n$. By the unboundedness results on the Riesz transform $\nabla\Delta_D^{-1/2}$ established in [10], the range $(1, \min\{n, 3+\varepsilon\})$ of p for (1.2) is sharp; see also [16, 20].

The main purpose of this paper is to further investigate the case $p \ge n$. Note that from Theorem 1.2, the boundedness of the Riesz transform $\nabla\mathcal{L}_D^{-1/2}$ depends on n, $p(\mathcal{L})$, and the geometry of the boundary $\partial\Omega$. All the dependences are essential; see the characterizations obtained by [16, Theorem 1.1], the regularity dependence of the boundary by [18], and the counterexamples provided in [10, 20, 16]. However, for operator with nice coefficients and domain with nice boundary (C^1 or small Lipschitz constant) such that $p(\mathcal{L}), 3+\epsilon \ge n$, we can find a suitable substitution of $\mathring{W}^{1,p}(\Omega)$ space for the inequality (1.2) as following.

Let us assume that the matrix A in the operator $\mathcal{L}$ is in the space $\mathrm{VMO}(\mathbb{R}^n)$ and satisfies the perturbation

$$\fint_{B(x_0,r)} |A - I_{n\times n}|\,dx \le \frac{C}{r^\delta} \tag{GD}$$

for some $\delta > 0$, all $r > 1$, and all $x_0 \in \mathbb{R}^n$. Or we assume that $A \in \mathrm{CMO}(\mathbb{R}^n)$. In both cases, from [17] and [14, Theorem 1], respectively, it is known that

$$p(\mathcal{L}) = \infty.$$

We have the following replacement for the Riesz inequality for $p \ge n$ and $p > 2$.

Theorem 1.4. *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be an exterior C^1 domain. Assume that $A \in \mathrm{VMO}(\mathbb{R}^n)$ satisfies (GD) or $A \in \mathrm{CMO}(\mathbb{R}^n)$. Let $p > 2$ and $p \in [n,\infty)$.*

(i) *The kernel space $\mathcal{K}_p(\mathcal{L}_D^{1/2})$ of $\mathcal{L}_D^{1/2}$ in $\mathring{W}^{1,p}(\Omega)$ coincides with $\tilde{\mathcal{A}}_0^p(\Omega) := \{\phi \in \mathring{W}^{1,p}(\Omega) : \mathcal{L}_D f = 0\}$. Moreover, when $n \ge 3$, $\tilde{\mathcal{A}}_0^p(\Omega) = \mathcal{A}_0^p(\Omega) := \{c(u_0 - 1) : c \in \mathbb{R}\}$, where u_0 is the unique solution in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$ of the problem*

$$\begin{cases} -\mathrm{div}(A\nabla u_0) = 0 & \text{in } \Omega,\\ u_0 = 1 & \text{on } \partial\Omega; \end{cases}$$

when $n = 2$, $\tilde{\mathcal{A}}_0^p(\Omega) = \mathcal{A}_0^p(\Omega) := \{c(u_0 - u_1) : c \in \mathbb{R}\}$, where u_0 is the unique solution in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(A\nabla u_0) = 0 & \text{in } \Omega,\\ u_0 = u_1 & \text{on } \partial\Omega, \end{cases}$$

and $u_1 \in W^{1,p}(\mathbb{R}^2)$ is a solution of the problem $\mathcal{L}u = \frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega}$ in $\mathbb{R}^2$. Here and thereafter, $\sigma(\partial\Omega)$ denotes the surface measure of $\partial\Omega$, and $\delta_{\partial\Omega}$ is the distribution on $\mathcal{D}(\mathbb{R}^2)$ as

$$\langle \delta_{\partial\Omega}, \varphi\rangle := \int_{\partial\Omega} \varphi\, d\sigma,\ \forall\, \varphi \in \mathcal{D}(\mathbb{R}^2).$$

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(ii) *It holds for all* $f \in \mathring{W}^{1,p}(\Omega)$ *that*

$$\inf_{\phi \in \mathcal{K}_p(\mathcal{L}_D^{1/2})} \|\nabla f - \nabla \phi\|_{L^p(\Omega)} \le C \left\| \mathcal{L}_D^{1/2} f \right\|_{L^p(\Omega)}, \tag{1.3}$$

and consequently, it holds that

$$\inf_{\phi \in \mathcal{K}_p(\mathcal{L}_D^{1/2})} \|\nabla (f - \phi)\|_{L^p(\Omega)} \sim \left\| \mathcal{L}_D^{1/2} f \right\|_{L^p(\Omega)}.$$

The *symbol* $f \sim g$ means $f \lesssim g$ and $g \lesssim f$, which stands for $f \le Cg$ and $g \le Ch$. The main new ingredient that appeared in Theorem 1.4 is identifying the kernel $\mathcal{K}_p(\mathcal{L}_D^{1/2})$ of $\mathcal{L}_D^{1/2}$ in $\mathring{W}^{1,p}(\Omega)$ as the space $\mathcal{A}_0^p(\Omega)$, which is motivated by the work of Amrouche, Girault and Giroire [1]. We can actually establish a more general version of Theorem 1.4, provided that $p(\mathcal{L}) \ge n$ and the boundary $\partial\Omega$ is C^1 or with small Lipschitz constant; see Theorem 2.4 below.

Let us remark that we can have an explicit description in the exterior setting due to the boundedness of the Riesz transform in $\mathbb{R}^n$ for $1 < p < \infty$ and the special geometry of exterior domains. From previous results of Riesz transforms from [3, 8, 17], we know in case of $p \in (2, \infty)$, both local and global geometry can destroy the boundedness of the Riesz transform. In particular, a local perturbation of A may result in huge difference of behavior of the Riesz transform for $p > 2$; see [17], for instance. So generally speaking, it is hard (at least to us) to have an explicit description of the kernel space. For the case of exterior domains, under the assumption of $p(\mathcal{L}) = \infty$, we see that the kernel space that breaks down the boundedness of the Riesz transform for $p \ge n$ and $p > 2$ is actually only *one-dimensional subspace* of $\mathring{W}^{1,p}(\Omega)$.

Finally, let us apply Theorem 1.4 to the mapping property of the gradient of heat semigroup, which plays important roles in the study of of Schrödinger equations; see [11, 12, 13, 21, 22], for instance. For the operator $\sqrt{t}\nabla e^{-t\mathcal{L}_D}$, it was known that there are no uniform L^p-bounds in t for $p > n$; see [20, Proposition 8.1]. As an application of (1.3) of Theorem 1.4, we have the following substitution.

Theorem 1.5. *Let* $n \ge 2$ *and* $\Omega \subset \mathbb{R}^n$ *be an exterior* C^1 *domain. Assume that* $A \in \mathrm{VMO}(\mathbb{R}^n)$ *satisfies* (GD) *or* $A \in \mathrm{CMO}(\mathbb{R}^n)$*. Let* $p > 2$ *and* $p \in [n, \infty)$*. Then it holds that*

$$\inf_{\phi \in \mathcal{K}_p(\mathcal{L}_D^{1/2})} \left\| \sqrt{t}\nabla e^{-t\mathcal{L}_D} f - \nabla\phi \right\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)},\ \forall\, t > 0.$$

The proof is straightforward by using (1.3) and the analyticity of the heat semigroup, as

$$\inf_{\phi \in \mathcal{K}_p(\mathcal{L}_D^{1/2})} \left\| \sqrt{t}\nabla e^{-t\mathcal{L}_D} f - \nabla\phi \right\|_{L^p(\Omega)} \le C \left\| \sqrt{t}\mathcal{L}_D^{1/2} e^{-t\mathcal{L}_D} f \right\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)},\ \forall\, t > 0.$$

Moreover, noting that, for all $\phi \in \mathcal{K}_p(\mathcal{L}_D^{1/2})$,

$$\partial_t e^{-t\mathcal{L}_D}\phi = -\mathcal{L}_D e^{-t\mathcal{L}_D}\phi = -e^{-t\mathcal{L}_D}\mathcal{L}_D\phi = 0,$$

we find that $e^{-t\mathcal{L}_D}\phi = \phi$ for all $t > 0$.

In the particular case $\mathcal{L} := \Delta$ and $\Omega := \mathbb{R}^n \setminus \overline{B(0,1)}$, it is clear that the kernel space is exactly as

$$\mathcal{K}_p(\Delta_D^{1/2}) = \tilde{\mathcal{A}}_0^p(\Omega) = \begin{cases} \{c(1 - |x|^{2-n}) : \ |x| > 1,\ c \in \mathbb{R}\}, & n \ge 3, \\ \{c \log |x| : \ |x| > 1,\ c \in \mathbb{R}\}, & n = 2, \end{cases} \tag{1.4}$$

where $p \ge n$ and $p > 2$. We therefore have the following corollary.

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Corollary 1.6. *Let $n \geq 2$ and $\Omega := \mathbb{R}^n \setminus \overline{B(0,1)}$. Let $p > 2$ and $p \in [n, \infty)$. Then it holds that*

$$\inf_{c\in\mathbb{R}} \left\| \sqrt{t}\nabla e^{-t\Delta_D} f - \frac{cx}{|x|^n} \right\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \ \forall t > 0.$$

It is clear from [20, Proposition 8.1] that in the LHS of the last inequality, the infimum for large time t is not attained at $c = 0$. Moreover, since for $f \in L^p(\Omega)$, $\sqrt{t}\nabla e^{-t\mathcal{L}_D} f$ does belong to $L^p(\Omega)$ (without uniform bound in t), the infimum shall be attained at the finite c which depends on f and t.

We shall first prove an intermediate version of Theorem 1.4 in Section 2. We shall then show the equivalence of the spaces $\mathcal{A}_0^p(\Omega)$, $\tilde{\mathcal{A}}_0^p(\Omega)$ and $K_p(\mathcal{L}_D^{1/2})$ and complete the proof of Theorem 1.4 in Section 3.

Throughout the whole paper, we always denote by C or c a *positive constant* which is independent of the main parameters, but it may vary from line to line. Furthermore, for any $q \in [1, \infty]$, we denote by q' its *conjugate exponent* – namely, $1/q + 1/q' = 1$. Finally, for any measurable set $E \subset \mathbb{R}^n$ and (vector-valued or matrix-valued) function $f \in L^1(E)$, we denote the integral $\int_E |f(x)|\,dx$ simply by $\int_E |f|\,dx$ and, when $|E| < \infty$, we use the notation

$$(f)_E := \fint_E f(x)\,dx := \frac{1}{|E|}\int_E f(x)\,dx.$$

2. On boundedness of the Riesz transform

In this section, we prove the following more general version Theorem 2.4 of Theorem 1.4(ii) with $\mathcal{K}_p(\mathcal{L}_D^{1/2})$ replaced by $\tilde{\mathcal{A}}_0^p(\Omega)$, which is defined as

$$\tilde{\mathcal{A}}_0^p(\Omega) = \left\{ w \in \mathring{W}^{1,p}(\Omega) :\ \mathcal{L}_D w = 0 \right\}.$$

Let us begin with some necessary notations.

Definition 2.1. Let $\mathcal{L} := -\mathrm{div}(A\nabla\cdot)$ be a second-order divergence form elliptic operator on $\mathbb{R}^n$. Denote by $(q(\mathcal{L})', q(\mathcal{L}))$ the *interior of the maximal interval of exponents* $q \in [1, \infty]$ such that the operator $\nabla\mathcal{L}^{-1}\mathrm{div}$ is bounded on $L^q(\mathbb{R}^n)$.

Furthermore, let O be a bounded Lipschitz domain of $\mathbb{R}^n$ and let $\mathcal{L}_{D,O} := -\mathrm{div}(A\nabla\cdot)$ be a second-order divergence form elliptic operator on O subject to the Dirichlet boundary condition. Similarly, denote by $(q(\mathcal{L}_{D,O})', q(\mathcal{L}_{D,O}))$ the *interior of the maximal interval of exponents* $q \in [1, \infty]$ such that $\nabla\mathcal{L}_{D,O}^{-1}\mathrm{div}$ is bounded on $L^q(O)$.

Remark 2.2. It is well known that there exists a constant $\varepsilon_0 \in (0, \infty)$ depending on the matrix A and n such that $(2 - \varepsilon_0, 2 + \varepsilon_0) \subset (q(\mathcal{L})', q(\mathcal{L}))$ (see, for instance, [14]). Similarly, there exists a constant $\varepsilon_1 \in (0, \infty)$ depending on A, n, and the Lipschitz constant of O such that $(2 - \varepsilon_1, 2 + \varepsilon_1) \subset (q(\mathcal{L}_{D,O})', q(\mathcal{L}_{D,O}))$.

Remark 2.3. Note that $q(\mathcal{L}) = p(\mathcal{L})$. In fact, since for $1 < p < \infty$ it holds that

$$\left\|\mathcal{L}^{1/2} f\right\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}$$

(see [4]), one further has

$$\left\|\mathcal{L}^{-1/2}\mathrm{div}\right\|_{p\to p} = \left\|\mathcal{L}^{1/2}\mathcal{L}^{-1}\mathrm{div}\right\|_{p\to p} \leq C\left\|\nabla\mathcal{L}^{-1}\mathrm{div}\right\|_{p\to p},$$

which by duality implies that the L^p-boundedness of $\nabla\mathcal{L}^{-1}\mathrm{div}$ implies $L^{p'}$-boundedness of $\nabla\mathcal{L}^{-1/2}$. However, note that for $p \in (1, p(\mathcal{L}))$, $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$. Therefore, for $p \in (p(\mathcal{L})', p(\mathcal{L}))$, we have that

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$$\left\|\nabla\mathcal{L}^{-1}\mathrm{div}\right\|_{p\to p}=\left\|\nabla\mathcal{L}^{-1/2}\left(\mathcal{L}^{-1/2}\mathrm{div}\right)\right\|_{p\to p}\le\left\|\nabla\mathcal{L}^{-1/2}\right\|_{p\to p}\left\|\nabla\mathcal{L}^{-1/2}\right\|_{p'\to p'}<\infty.$$

Thus, we have $q(\mathcal{L})=p(\mathcal{L})$.

In what follows, for any $x\in\mathbb{R}^n$ and $r\in(0,\infty)$, we always let $B(x,r):=\{y\in\mathbb{R}^n:\ |y-x|<r\}$. Note that on $\mathbb{R}^n$, the maximal interval for the L^p-boundedness of the Riesz transform is open (see, for instance, [8]), so we may assume that $p(\mathcal{L})=q(\mathcal{L})>n$.

Theorem 2.4. *Let $n\ge 2$ and $\Omega\subset\mathbb{R}^n$ be an exterior Lipschitz domain. Take a large $R\in(0,\infty)$ such that $\Omega^c\subset B(0,R)$. Let $\Omega_R:=\Omega\cap B(0,R)$. Assume that $\min\{q(\mathcal{L}),q(\mathcal{L}_{D,\Omega_R})\}>n$ and $2<p\in[n,\min\{q(\mathcal{L}),q(\mathcal{L}_{D,\Omega_R})\})$. Then there exists a positive constant C such that, for any $f\in\mathring{W}^{1,p}(\Omega)$,*

$$\inf_{\phi\in\tilde{\mathcal{A}}_0^p(\Omega)}\|\nabla f-\nabla\phi\|_{L^p(\Omega)}\le C\left\|\mathcal{L}_D^{1/2}f\right\|_{L^p(\Omega)},$$

where $\tilde{\mathcal{A}}_0^p(\Omega)=\{\phi\in\mathring{W}^{1,p}(\Omega):\ \mathcal{L}_Df=0\}$.

To prove Theorem 2.4, let us first begin with the following several lemmas.

Let X be a Banach space and Y a closed subspace of X. Denote by X^* the *dual space* of X. Let

$$X^*\bot Y:=\{f\in X^*:\ \text{for all } x\in Y,\ \langle f,x\rangle\}=0,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X^* and X. That is, $X^*\bot Y$ denotes the subspace of X^* orthogonal to Y.

Meanwhile, for any given $m\in\mathbb{N}\cup\{0\}$, we denote by $\mathcal{P}_m$ the space of polynomials on $\mathbb{R}^n$ of degree less than or equal to m; if m is a strictly negative integer, we set by convention $\mathcal{P}_m=\{0\}$. Moreover, for any $s\in\mathbb{R}$, denote by $\lfloor s\rfloor$ the maximal integer not more than s.

Then we have the following conclusion on the isomorphism property of the divergence operator div which was obtained in [2, Propositions 4.1 and 9.2].

Lemma 2.5. *Let $n\ge 2$, $p\in(1,\infty)$ and $p'\in(1,\infty)$ be given by $1/p+1/p'=1$. Then the divergence operator* div *is an isomorphism from $L^p(\mathbb{R}^n)/H_p$ to $W^{-1,p}(\mathbb{R}^n)\bot\mathcal{P}_{\lfloor 1-n/p'\rfloor}$, where $H_p:=\{v\in L^p(\mathbb{R}^n):\ \mathrm{div}(v)=0$ in the sense of distributions$\}$.*

Lemma 2.6. *Let $p\in(2,\infty)$ and $f\in W^{-1,p}(\mathbb{R}^n)$ with compact support. Then $f\in W^{-1,2}(\mathbb{R}^n)$, and there exists a positive constant C, depending only on p and the support of f, such that*

$$\|f\|_{W^{-1,2}(\mathbb{R}^n)}\le C\|f\|_{W^{-1,p}(\mathbb{R}^n)}.$$

Lemma 2.6 is just [1, Lemma 2.1].

Lemma 2.7. *Let $n\ge 2$ and $p\in(2,q(\mathcal{L}))$. Assume that $f\in W^{-1,p}(\mathbb{R}^n)$ has compact support. When $n\ge 3$, the problem*

$$\mathcal{L}u=f\ \text{in}\ \mathbb{R}^n \tag{2.1}$$

has a unique solution u in $W^{1,2}(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n)$.

When $n=2$, if f further satisfies the compatibility condition $\langle f,1\rangle=0$, then the problem (2.1) has a unique solution u in $W^{1,2}(\mathbb{R}^2)\cap W^{1,p}(\mathbb{R}^2)$ up to constants.

Proof. When $n\ge 3$, by the assumption that $f\in W^{-1,p}(\mathbb{R}^n)$ has compact support and Lemma 2.6, we conclude that $f\in W^{-1,2}(\mathbb{R}^n)$. From Lemma 2.5, there exists $F\in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ such that $f=\mathrm{div}\,F$. From this, the definition of the interval $(q(\mathcal{L})',q(\mathcal{L}))$, and the assumption $p\in(2,q(\mathcal{L}))$, it follows that the equation (2.1) has a unique solution $u\in W^{1,2}(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n)$.

When $n=2$, by the assumption that $f\in W^{-1,p}(\mathbb{R}^n)$ has compact support and Lemma 2.6, we conclude that $f\in W^{-1,2}(\mathbb{R}^n)$. From this together with the compatibility condition $\langle f,1\rangle=0$, we deduce from Lemma 2.5 that there exists $F \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $f = \operatorname{div} F$. Using this and the assumption $p \in (2, q(\mathcal{L}))$ again, we conclude that the problem (2.1) has a unique solution $u \in W^{1,2}(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$ up to constants. This finishes the proof of Lemma 2.7. □

Lemma 2.8. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain. Take a large $R \in (0, \infty)$ such that $\Omega^c \subset B(0, R)$ and let $\Omega_R := \Omega \cap B(0, R)$. Let $p \in (2, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\})$. Assume that $f \in \mathring{W}^{-1,p}(\Omega)$ has compact support and its support is contained in $B(0, R)$. Then the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

has a unique solution u in $\mathring{W}^{1,2}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$.

Let $s \in (0, 1)$ and $p \in (1, \infty)$. For the exterior Lipschitz domain (or the bounded Lipschitz domain) Ω of $\mathbb{R}^n$, denote by $W^{s,p}(\partial\Omega)$ the *fractional Sobolev space on $\partial\Omega$* (see, for instance, [23, Section 2.4.3] for its definition). To show Lemma 2.8, we need the following conclusion.

Lemma 2.9. *Let $n \geq 2$ and $O \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $p \in (2, q(\mathcal{L}_{D,O}))$. Assume that $f \in \mathring{W}^{-1,p}(O)$ and $g \in W^{1/p',p}(\partial O)$. Then the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(A\nabla v) = f & \text{in } O, \\ v = g & \text{on } \partial O \end{cases} \tag{2.3}$$

has a unique solution v in $W^{1,p}(O)$.

Proof. We first prove that there exists a solution $v \in W^{1,p}(O)$ for the problem (2.3). Indeed, by $g \in W^{1/p',p}(\partial O)$ and the converse trace theorem for Sobolev spaces (see, for instance, [23, Section 2.5.7, Theorem 5.7]), we find that there exists a function $w_1 \in W^{1,p}(O)$ such that $w_1 = g$ on ∂O. Moreover, it is easy to find that $-\operatorname{div}(A\nabla w_1) \in \mathring{W}^{-1,p}(O)$. Furthermore, from the assumption that $p \in (2, q(\mathcal{L}_{D,O}))$, it follows that there exists a unique $w_2 \in \mathring{W}^{1,p}(O)$ satisfying

$$\begin{cases} -\operatorname{div}(A\nabla w_2) = f + \operatorname{div}(A\nabla w_1) & \text{in } O, \\ w_2 = 0 & \text{on } \partial O. \end{cases}$$

Thus, $v := w_1 + w_2 \in W^{1,p}(O)$ is a solution of the problem (2.3).

Now, we show that the solution of (2.3) is unique. Assume that $v_1, v_2 \in W^{1,p}(O)$ are solutions of (2.3). Then $\operatorname{div}(A\nabla(v_1 - v_2)) = 0$ in O and $v_1 - v_2 = 0$ on ∂O. Thus, $v_1 = v_2$ almost everywhere in O. This finishes the proof of Lemma 2.9. □

Now, we prove Lemma 2.8 by using Lemmas 2.7 and 2.9.

Proof of Lemma 2.8. Suppose that $\operatorname{supp} f \subset B(0, R)$ and take a bump function ψ_R such that $\psi_R = 1$ on $B(0, R)$, $\operatorname{supp} \psi_R \subset B(0, R+1)$ and $|\nabla \psi_R| \leq 1$.

For any $g \in \mathcal{D}(\Omega)$, by the fact $\|g\|_{\mathring{W}^{1,p'}(\Omega)}$ is equivalent to $\|\nabla g\|_{L^{p'}(\Omega)}$ (see Remark 1.1), we have

$$\begin{aligned} |\langle f, g\rangle| &= |\langle f, g\psi_R\rangle| \leq \|f\|_{\mathring{W}^{-1,p}(\Omega)} \|\nabla(g\psi_R)\|_{L^{p'}(\Omega)} \\ &\leq \|f\|_{\mathring{W}^{-1,p}(\Omega)} \left[\|\nabla g\|_{L^{p'}(B(0,R+1)\cap\Omega)} + C(R)\|g\|_{L^{p'}(B(0,R+1)\cap\Omega)}\right] \\ &\leq C(R)\|f\|_{\mathring{W}^{-1,p}(\Omega)} \left[\|\nabla g\|_{L^2(\Omega)} + \left(\int_\Omega \frac{|g(x)|^2}{1+|x|^2}\,dx\right)^{1/2}\right] \\ &\leq C(R)\|f\|_{\mathring{W}^{-1,p}(\Omega)} \|g\|_{\mathring{W}^{1,2}(\Omega)}, \end{aligned} \tag{2.4}$$

for $n \geq 3$, which implies that $f \in \mathring{W}^{-1,2}(\Omega)$. For $n = 2$, simply replacing $\int_\Omega \frac{|g(x)|^2}{1+|x|^2}\,dx$ by $\int_\Omega \frac{|g(x)|^2}{(1+|x|^2)\ln^2(2+|x|^2)}\,dx$ gives the same conclusion.

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Thus, $f \in \mathring{W}^{-1,2}(\Omega)$, which, together with the Lax–Milgram theorem, further implies that the Dirichlet problem (2.2) has a unique solution $u \in \mathring{W}^{1,2}(\Omega)$.

Next, we show $u \in \mathring{W}^{1,p}(\Omega)$. We first assume that

$$p \in \left(2, \min\left\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R}), \frac{2n}{n-2}\right\}\right)$$

when $n \geq 3$ or $p \in (2, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\})$ when $n = 2$.

Let $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^n)$ satisfy $0 \leq \varphi_1, \varphi_2 \leq 1$, $\operatorname{supp}(\varphi_1) \subset B(0, R+1)$, $\varphi_1 \equiv 1$ on $B(0,R)$, and $\varphi_1 + \varphi_2 \equiv 1$ in $\mathbb{R}^n$. Extend u by zero in Ω^c and let $u = u_1 + u_2$, where $u_1 := u\varphi_1$ and $u_2 := u\varphi_2$. Then

$$\operatorname{div}(A\nabla u_2) = \operatorname{div}(A\nabla(u\varphi_2)) \text{ in } \mathbb{R}^n.$$

From $u \in \mathring{W}^{1,2}(\Omega)$ and the assumptions that $\varphi_2 \in C^\infty(\mathbb{R}^n)$ and $\varphi_2 \equiv 1$ on $\mathbb{R}^n \backslash B(0, R+1)$, we infer that $u\varphi_2 \in W^{1,2}(\Omega \cap B(0, R+1))$ and hence $\nabla(u\varphi_2) \in L^2(\mathbb{R}^n)$, which, together with the assumption $A \in L^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n})$, further implies that $A\nabla(u\varphi_2) \in L^2(\mathbb{R}^n)$.

Furthermore, it is straight to see that

$$\operatorname{div}(A\nabla(u\varphi_2)) = f\varphi_2 - A\nabla u \cdot \nabla\varphi_2 - \operatorname{div}(uA\nabla\varphi_2) := g$$

in the weak sense. By the assumption $f \in \mathring{W}^{-1,p}(\Omega)$, we conclude that $f\varphi_2 \in \mathring{W}^{-1,p}(\Omega)$. Meanwhile, from $u \in \mathring{W}^{1,2}(\Omega)$ and the assumptions that $\varphi_2 \in C^\infty(\mathbb{R}^n)$, $0 \leq \varphi_2 \leq 1$, and $\varphi_2 \equiv 1$ on $\mathbb{R}^n \backslash B(0, R+1)$, we deduce that $A\nabla u \cdot \nabla\varphi_2 \in L^2(\Omega \cap B(0, R+1))$, which, combined with the Sobolev inequality, further implies that $A\nabla u \cdot \nabla\varphi_2 \in W^{-1,p}(B(0, R+1))$. Moreover, by $u \in W^{1,2}(\Omega)$, we find that $u \in L^p_{\mathrm{loc}}(\Omega)$, which, together with the assumptions that $\varphi_2 \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\nabla\varphi_2) \subset B(0, R+1)$, further implies that $Au\nabla\varphi_2 \in L^p(\Omega \cap B(0, R+1))$ and hence $\operatorname{div}(Au\nabla\varphi_2) \in \mathring{W}^{-1,p}(\Omega \cap B(0, R+1))$. Thus, we have $g \in \mathring{W}^{-1,p}(\Omega \cap B(0, R+1))$. Extend g by zero in Ω^c. Then $g \in W^{-1,p}(\mathbb{R}^n)$. Therefore,

$$-\operatorname{div}(A\nabla u_2) = g \text{ in } \mathbb{R}^n,$$

which, combined with Lemma 2.5 and $p \in (2, q(\mathcal{L}))$, further implies that $u_2 \in W^{1,p}(\mathbb{R}^n)$.

Furthermore, from $u = u_2$ on $\partial B(0, R+1)$ and the trace theorem for Sobolev spaces (see, for instance, [23, Section 2.5.4, Theorem 5.5]), it follows that $u \in W^{1/p',p}(\partial B(0, R+1))$. Meanwhile, we have

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega_{R+1}, \\ u = 0 & \text{on } \partial\Omega, \\ u = u_2 & \text{on } \partial B(0, R+1), \end{cases} \tag{2.5}$$

where $\Omega_{R+1} := \Omega \cap B(0, R+1)$. By the assumption $p < q(\mathcal{L}_{D,\Omega_{R+1}})$ and Lemma 2.9, we conclude that the problem (2.5) has a unique solution in $W^{1,p}(\Omega_{R+1})$, which further implies that $u \in W^{1,p}(\Omega_{R+1})$. From this, $u \in \mathring{W}^{1,2}(\Omega)$, $u_2 \in W^{1,p}(\mathbb{R}^n)$, and the fact that $u = u_2$ on $\mathbb{R}^n \backslash B(0, R+1)$, we deduce that $u \in W^{1,p}(\Omega)$ with any given

$$p \in \left(2, \min\left\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_{R+1}}), \frac{2n}{n-2}\right\}\right)$$

when $n \geq 3$ or any given $p \in (2, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_{R+1}})\})$ when $n = 2$. Then, using a bootstrap argument (see, for instance, [1, p. 63]), we find that $u \in \mathring{W}^{1,p}(\Omega)$ with any given $p \in (2, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_{R+1}})\})$. This finishes the proof of Lemma 2.8. □

Lemma 2.10. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain. Take a large $R \in (0, \infty)$ such that $\Omega^c \subset B(0, R)$ and let $\Omega_R := \Omega \cap B(0, R)$. Assume that $\min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\} > n$. Let $p > 2$ and $p \in [n, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\})$. Assume further that $f \in \mathring{W}^{-1,p}(\Omega)$ and*

$$\tilde{\mathcal{A}}_0^p(\Omega) = \left\{w \in \mathring{W}^{1,p}(\Omega) : \mathcal{L}_D w = 0\right\}.$$

Then the problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

has a unique solution u in $\mathring{W}^{1,p}(\Omega)/\tilde{\mathcal{A}}_0^p(\Omega)$, and there exists a positive constant C independent of f such that

$$\inf_{\phi \in \tilde{\mathcal{A}}_0^p(\Omega)} \|\nabla u - \nabla\phi\|_{L^p(\Omega)} \le C\|f\|_{\mathring{W}^{-1,p}(\Omega)}.$$

Remark 2.11. Note that the above lemma is nontrivial only if $\min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\} \ge n$ and $\min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\} > 2$. This is not surprise, since by [16, Theorem 1.1] and a similar proof of [16, Theorem 1.4] via using the role of $q(\mathcal{L}_{D,\Omega_R})$ instead of using [25, Theorem B & Theorem C] there, the Riesz operator $\nabla\mathcal{L}^{-1/2}$ is bounded for $p \in (1,n) \cup (1,2]$. In this case, the kernel $\mathcal{A}_0^p(\Omega)$ must be trivial (i.e., equal zero).

Proof of Lemma 2.10. By the Closed Range Theorem of Banach (see, for instance, [5, Theorem 5.11-5]), we find that there exists a vector-valued function $F \in L^p(\Omega)$ such that $f = \mathrm{div}F$ in Ω. Extend F by zero in Ω^c and still denote this extension by F. Let $\widetilde{f} := \mathrm{div}F$. Then $\widetilde{f} \in W^{-1,p}(\mathbb{R}^n)$. From Lemma 2.5, the assumption that $p \in (2, q(\mathcal{L}))$, and the definition of the interval $(q(\mathcal{L})', q(\mathcal{L}))$, it follows that there exists a unique $w \in W^{1,p}(\mathbb{R}^n)$ up to constants such that

$$\mathcal{L}w = \widetilde{f} \text{ in } \mathbb{R}^n.$$

Moreover, consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}(A\nabla z) = 0 & \text{in } \Omega, \\ z = -w & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Then the problem (2.7) has a unique solution $z \in W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$. Indeed, take a large $R \in (0,\infty)$ such that $\Omega^c \subset B(0,R)$ and let $\Omega_R := \Omega \cap B(0,R)$. By $w \in W^{1,p}(\mathbb{R}^n)$, we conclude that $w \in W^{1/p',p}(\partial\Omega)$. Let u_z satisfy

$$\begin{cases} -\mathrm{div}(A\nabla u_z) = 0 & \text{in } \Omega_R, \\ u_z = -w & \text{on } \partial\Omega, \\ u_z = 0 & \text{on } \partial B(0,R). \end{cases} \tag{2.8}$$

Then, from Lemma 2.9, we infer that the problem (2.8) has a unique solution $u_z \in W^{1,p}(\Omega_R)$. Extend u_z by zero on $\mathbb{R}^n \backslash B(0,R)$. Then $u_z \in W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$. Let v satisfy

$$\begin{cases} -\mathrm{div}(A\nabla v) = \mathrm{div}(A\nabla u_z) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

By $u_z \in W^{1,p}(\Omega)$ and $u_z \equiv 0$ on $\mathbb{R}^n \backslash B(0,R)$, we conclude that $\mathrm{div}(A\nabla u_z) \in \mathring{W}^{-1,p}(\Omega)$ has compact support. From this and Lemma 2.8, it follows that the problem (2.9) has a unique solution $v \in \mathring{W}^{1,2}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$. Thus, the problem (2.7) has a unique solution $z = u_z + v \in W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$. Then $u = w + z \in \mathring{W}^{1,p}(\Omega)$ is a solution of the problem (2.6).

Meanwhile, by the definition of the space $\tilde{\mathcal{A}}_0^p(\Omega)$, we find that the problem (2.6) has a unique solution in $\mathring{W}^{1,p}(\Omega)/\tilde{\mathcal{A}}_0^p(\Omega)$. Furthermore, a duality argument shows that

$$\|f\|_{\mathring{W}^{-1,p}(\Omega)} = \|\mathrm{div}(A\nabla u)\|_{\mathring{W}^{-1,p}(\Omega)} \lesssim \|u\|_{\mathring{W}^{1,p}(\Omega)/\tilde{\mathcal{A}}_0^p(\Omega)},$$

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which, together with the Open Mapping Theorem of Banach (see, for instance, [5, Theorem 5.6-2]), further implies that

$$\|u\|_{\mathring{W}^{1,p}(\Omega)/\tilde{\mathcal{A}}_0^p(\Omega)} \lesssim \|f\|_{\mathring{W}^{-1,p}(\Omega)},$$

namely,

$$\inf_{\phi\in\tilde{\mathcal{A}}_0^p(\Omega)} \|\nabla u - \nabla\phi\|_{L^p(\Omega)} \lesssim \|f\|_{\mathring{W}^{-1,p}(\Omega)}.$$

This finishes the proof of Lemma 2.10. □

Now, we prove Theorem 2.4 by using Lemma 2.10 and Theorem 1.2(i).

Proof of Theorem 2.4. Let $2 < p \in [n, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\})$. By Theorem 1.2(i) together with a duality argument, we see that, for any given $q \in (1,\infty)$ and any $g \in L^q(\Omega)$,

$$\left\|\mathcal{L}_D^{1/2} g\right\|_{\mathring{W}^{-1,q}(\Omega)} \lesssim \|g\|_{L^q(\Omega)}.$$

From this and Lemma 2.10, we infer that

$$\inf_{\phi\in\tilde{\mathcal{A}}_0^p(\Omega)} \|\nabla f - \nabla\phi\|_{L^p(\Omega)} \lesssim \|\mathcal{L}_D f\|_{\mathring{W}^{-1,p}(\Omega)} = \left\|\mathcal{L}_D^{1/2}\mathcal{L}_D^{1/2} f\right\|_{\mathring{W}^{-1,p}(\Omega)} \lesssim \left\|\mathcal{L}_D^{1/2} f\right\|_{L^p(\Omega)}.$$

This finishes the proof of Theorem 2.4. □

3. On the kernel space and completion of the proof

In this section, we first identify $\mathcal{A}_0^p(\Omega)$ with $\tilde{\mathcal{A}}_0^p(\Omega)$, and then with $\mathcal{K}_p(\mathcal{L}_D^{1/2})$, and finally complete the proof of Theorem 1.4.

Lemma 3.1. *Let $n = 2$, $\Omega \subset \mathbb{R}^2$ be an exterior Lipschitz domain, and $p \in (2, q(\mathcal{L}))$. Then the problem*

$$\mathcal{L}u = \frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega} \text{ in } \mathbb{R}^2 \tag{3.1}$$

has a unique solution $u \in W^{1,p}(\mathbb{R}^2)$ up to constants.

Proof. Let $q \in (2,\infty)$ and $t \in (1,\infty)$ be given by $\frac{1}{t} = \frac{2}{q'} - 1$. Then, by the Sobolev trace embedding theorem (see, for instance, [23, Section 2.4.2, Theorem 4.2]), we find that, for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned}\left|\left\langle \frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega}, \varphi\right\rangle\right| &= \left|\frac{1}{\sigma(\partial\Omega)}\int_{\partial\Omega}\varphi(x)\,d\sigma(x)\right| \le \frac{1}{[\sigma(\partial\Omega)]^{1/t}}\|\varphi\|_{L^t(\partial\Omega)}\\ &\lesssim \frac{1}{[\sigma(\partial\Omega)]^{1/t}}\|\varphi\|_{W^{1,q'}(\Omega^c)} \lesssim \frac{1}{[\sigma(\partial\Omega)]^{1/t}}\|\varphi\|_{W^{1,q'}(\mathbb{R}^2)},\end{aligned}$$

which, together with the fact that $\mathcal{D}(\mathbb{R}^2)$ is dense in $W^{1,q'}(\mathbb{R}^2)$, implies that $\frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega} \in W^{-1,q}(\mathbb{R}^2)$ with any given $q \in (2,\infty)$.

Let $p \in (2, q(\mathcal{L}))$. From Lemma 2.5 with $n = 2$, we deduce that there exists $f \in L^p(\mathbb{R}^2)$ such that $\operatorname{div} f = \frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega}$. By this and the assumption $p \in (2, q(\mathcal{L}))$, we further conclude that there exists $u \in W^{1,p}(\mathbb{R}^2)$ such that $\mathcal{L}u = \frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega}$.

Moreover, if there exist $u_1, u_2 \in W^{1,p}(\mathbb{R}^2)$ satisfying $\mathcal{L}u_1 = \frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega} = \mathcal{L}u_2$, then $u_1 - u_2 \in W^{1,p}(\mathbb{R}^2)$ and $\mathcal{L}(u_1 - u_2) = 0$, which, together with $p \in (q(\mathcal{L})', q(\mathcal{L}))$, further implies that $u_1 - u_2 = c$.

Thus, the problem (3.1) has a unique solution $u \in W^{1,p}(\mathbb{R}^2)$ up to constants. This finishes the proof of Lemma 3.1. □

The following was essentially obtained in [25].

Lemma 3.2. *Let $n \geq 2$ and $O \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $A \in \mathrm{VMO}(\mathbb{R}^n)$, then there exists a positive constant ε_0, depending only on n and the Lipschitz constant of O, such that, for any given $p \in (\frac{4+\varepsilon_0}{3+\varepsilon_0}, 4+\varepsilon_0)$ when $n = 2$ or $p \in (\frac{3+\varepsilon_0}{2+\varepsilon_0}, 3+\varepsilon_0)$ when $n \geq 3$, $\nabla \mathcal{L}_{D,O}^{-1}\mathrm{div}$ is bounded on $L^p(O)$. In particular, if $\partial O \in C^1$, it holds that $\varepsilon_0 = \infty$; that is, $\nabla \mathcal{L}_{D,O}^{-1}\mathrm{div}$ is bounded on $L^p(O)$ for any $p \in (1,\infty)$.*

We now identify $\mathcal{A}_0^p(\Omega)$ with $\tilde{\mathcal{A}}_0^p(\Omega)$.

Proposition 3.3. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain, and $p \in (1,\infty)$. Take a large $R \in (0,\infty)$ such that $\Omega^c \subset B(0,R)$ and let $\Omega_R := \Omega \cap B(0,R)$. Assume that $\min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\} > n$ and $2 < p \in [n, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\})$. When $n \geq 3$,*

$$\tilde{\mathcal{A}}_0^p(\Omega) = \mathcal{A}_0^p(\Omega) = \{c(u_0 - 1): \ c \in \mathbb{R}\},$$

where u_0 is the unique solution in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(A\nabla u_0) = 0 & \text{in } \Omega, \\ u_0 = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

When $n = 2$,

$$\tilde{\mathcal{A}}_0^p(\Omega) = \mathcal{A}_0^p(\Omega) = \{c(u_0 - u_1): \ c \in \mathbb{R}\},$$

where u_1 is a solution of the problem (3.1) and u_0 is the unique solution in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(A\nabla u_0) = 0 & \text{in } \Omega, \\ u_0 = u_1 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Proof. For $2 < p \in [n, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\})$ and $\phi \in \tilde{\mathcal{A}}_0^p(\Omega)$, extend ϕ by zero in Ω^c. Then the extension of ϕ, still denoted by ϕ, belongs to $W^{1,p}(\mathbb{R}^n)$ and satisfies that

$$\mathrm{div}(A\nabla\phi) = 0 \text{ in } \Omega, \ \mathrm{div}(A\nabla\phi) = 0 \text{ in } \Omega^c, \text{ and } \phi = 0 \text{ on } \partial\Omega.$$

Since $\phi \in W^{1,p}(\mathbb{R}^n)$, it follows that $\frac{\partial\phi}{\partial\nu} \in W^{-1/p,p}(\partial\Omega)$, where $\frac{\partial\phi}{\partial\nu} := (A\nabla\phi)\cdot\boldsymbol{\nu}$ denotes the *conormal derivative* of ϕ on $\partial\Omega$, and $W^{-1/p,p}(\partial\Omega)$ denotes the dual space of $W^{1/p,p'}(\partial\Omega)$. Moreover, it is easy to show that $\mathrm{div}(A\nabla\phi)$, as a distribution in $\mathbb{R}^n$, satisfies that, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle \mathrm{div}(A\nabla\phi), \varphi\rangle = -\left\langle \frac{\partial\phi}{\partial\boldsymbol{\nu}}, \varphi\right\rangle_{\partial\Omega}, \tag{3.4}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality pairing between $W^{-1/p,p}(\partial\Omega)$ and $W^{1/p,p'}(\partial\Omega)$. Furthermore, let h denote the distribution defined by $\mathrm{div}(A\nabla\phi)$; that is, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle h, \varphi\rangle = \langle \mathrm{div}(A\nabla\phi), \varphi\rangle = -\left\langle \frac{\partial\phi}{\partial\boldsymbol{\nu}}, \varphi\right\rangle_{\partial\Omega},$$

which, combined with the Sobolev trace embedding theorem (see, for instance, [23, Section 2.5.4, Theorem 5.5]), further implies that, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

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$$|\langle h,\varphi\rangle| \le \left\|\frac{\partial\phi}{\partial\nu}\right\|_{W^{-1/p,p}(\partial\Omega)}\|\varphi\|_{W^{1/p,p'}(\partial\Omega)} \lesssim \left\|\frac{\partial\phi}{\partial\nu}\right\|_{W^{-1/p,p}(\partial\Omega)}\|\varphi\|_{W^{1,p'}(\Omega^c)}$$
$$\lesssim \left\|\frac{\partial\phi}{\partial\nu}\right\|_{W^{-1/p,p}(\partial\Omega)}\|\varphi\|_{W^{1,p'}(\mathbb{R}^n)}.$$

By this and the fact that $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{1,p'}(\mathbb{R}^n)$, we conclude that $h \in W^{-1,p}(\mathbb{R}^n)$ and h has a compact support.

When $n \ge 3$, from Lemma 2.7, it follows that the problem, that $\mathrm{div}(A\nabla w) = h$ in $\mathbb{R}^n$, has a unique solution in $W^{1,2}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$. Therefore, $w - \phi \in W^{1,p}(\mathbb{R}^n)$ and $\mathrm{div}(A\nabla(w-\phi)) = 0$ in $\mathbb{R}^n$. By this and the assumption $2 < p \in [n, \min\{q(\mathcal{L}), q(\mathcal{L}_{D,\Omega_R})\})$, we find that $w - \phi = c$ with $c \in \mathbb{R}$, which, together with the fact that $\mathrm{div}(A\nabla\phi) = 0$ in Ω, implies that the restriction of w to Ω is the unique solution in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$ of the problem that $\mathrm{div}(A\nabla w) = 0$ in Ω and $w = c$ on $\partial\Omega$. Thus, $w = cu_0$ with u_0 being the same as in (3.2) and $\phi = c(u_0 - 1)$. This shows $\tilde{\mathcal{A}}_0^p(\Omega) \subset \mathcal{A}_0^p(\Omega)$.

When $n = 2$, without the loss of generality, we can assume that $\langle h, 1\rangle \neq 0$. Otherwise, $\langle h, 1\rangle = 0$, and we see that $h \in W^{-1,2}(\mathbb{R}^2) \cap W^{-1,p}(\mathbb{R}^2)$. Then the same proof as in the case of $n \ge 3$ yields that $\phi = c(u_0 - 1)$ with u_0 obtained by (3.2). However, since $1 \in W^{1,2}(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$, the uniqueness then implies $u_0 = 1$ and $\phi = 0$.

Suppose now $\langle h, 1\rangle \neq 0$. Let $u_1 \in W^{1,p}(\mathbb{R}^2)$ be a solution of the problem (3.1). Then u_1 satisfies

$$\mathrm{div}(A\nabla u_1) = 0 \text{ in } \Omega, \quad \mathrm{div}(A\nabla u_1) = 0 \text{ in } \Omega^c, \quad \text{and } \langle -\mathrm{div}(A\nabla u_1), 1\rangle = 1.$$

Let $w \in W^{1,2}(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$ satisfy

$$\mathrm{div}(A\nabla w) = h + \langle h, 1\rangle \mathrm{div}(A\nabla u_1) \text{ in } \mathbb{R}^2. \tag{3.5}$$

Indeed, by $h \in W^{-1,p}(\mathbb{R}^2)$ and

$$-\mathrm{div}(A\nabla u_1) = \frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega} \in W^{-1,p}(\mathbb{R}^2),$$

we conclude that the right-hand side of (3.5) belongs to $W^{-1,p}(\mathbb{R}^2)$. This, together with the fact that both h and $\frac{1}{\sigma(\partial\Omega)}\delta_{\partial\Omega}$ have compact supports and Lemma 2.6, further implies that

$$h + \langle h, 1\rangle \mathrm{div}(A\nabla u_1) \in W^{-1,2}(\mathbb{R}^2).$$

Moreover, it is easy to find that

$$\langle [h + \langle h, 1\rangle \mathrm{div}(A\nabla u_1)], 1\rangle = 0.$$

Therefore, from this and Lemma 2.7, we deduce that the problem (3.5) has a unique solution $w \in W^{1,2}(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$ up to constants. Then, by the fact that w satisfies (3.5), we conclude that $w - \langle h, 1\rangle u_1 - \phi \in W^{1,p}(\mathbb{R}^2)$ and

$$\mathrm{div}(A\nabla[w - \langle h, 1\rangle u_1 - \phi]) = 0 \text{ in } \mathbb{R}^2,$$

where ϕ is as in (3.4), which, combined with $p \in (2, q(\mathcal{L}))$, implies that

$$\phi = w - \langle h, 1\rangle u_1 - c \text{ in } \mathbb{R}^2, \tag{3.6}$$

where c is a constant.

From the boundary condition that $\phi = 0$ on $\partial\Omega$ and (3.6), we deduce that $w = c + \langle h, 1\rangle u_1$ on $\partial\Omega$. Then the restriction of w to Ω is the unique solution in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$ of the problem that

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$\operatorname{div}(A\nabla w)=0$ in Ω and $w=c+\langle h,1\rangle u_1$ on $\partial\Omega$. Moreover, let $w_1\in W^{1,2}(\Omega)\cap W^{1,p}(\Omega)$ be the unique solution of the problem

$$\begin{cases}\operatorname{div}(A\nabla w_1)=0 & \text{in } \Omega,\\ w_1=\langle h,1\rangle u_1 & \text{on } \partial\Omega,\end{cases}$$

that is, $w_1=\langle h,1\rangle u_0$. Then, we find that $w=w_1+c=\langle h,1\rangle u_0+c$. This, combined with (3.6), further concludes that

$$\phi=\langle h,1\rangle(u_0-u_1).$$

This shows $\tilde{\mathcal{A}}_0^p(\Omega)\subset\mathcal{A}_0^p(\Omega)$ for $n=2$.

The converse inclusion $\mathcal{A}_0^p(\Omega)\subset\tilde{\mathcal{A}}_0^p(\Omega)$ is obvious, since constants belongs to $W^{1,p}(\mathbb{R}^n)$ for $p\geq n$ and $u_1\in W^{1,p}(\Omega)$ for $n=2$. □

Lemma 3.4. *Let $n\geq 2$ and $p\in(1,\infty)$. Suppose that $A\in \mathrm{VMO}(\mathbb{R}^n)$ satisfies (GD), or $A\in\mathrm{CMO}(\mathbb{R}^n)$. Then the operator $\nabla\mathcal{L}^{-1}\operatorname{div}$ is bounded on $L^p(\mathbb{R}^n)$.*

Proof. The case $A\in\mathrm{CMO}(\mathbb{R}^n)$ follows from [14, Theorem 1]. For the case $A\in\mathrm{VMO}(\mathbb{R}^n)$ satisfying (GD), it follows from [17] (see also [16, Theorem 5.1 and Proposition 5.2]) that $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Thus, we have

$$\left\|\nabla\mathcal{L}^{-1}\operatorname{div}\right\|_{p\to p}<\infty,$$

which gives the desired conclusion. □

We also point out that when both the matrix A and Ω have nice smoothness, the function u as in (3.1) can be represented by using the fundamental solution associated with $\mathcal{L}$ (see, for instance, [1, Theorem 2.7]).

Proposition 3.5. *Let $n\geq 2$, $\Omega\subset\mathbb{R}^n$ be an exterior Lipschitz domain, and $p\in(1,\infty)$.*

(i) *If Ω is C^1, and $A\in\mathrm{CMO}(\mathbb{R}^n)$, or $A\in\mathrm{VMO}(\mathbb{R}^n)$ satisfies (GD), then, when $n\geq 3$, for any $p\in[n,\infty)$, $\tilde{\mathcal{A}}_0^p(\Omega)=\{c(u_0-1):\ c\in\mathbb{R}\}$ with u_0 being the same as in (3.2); when $n=2$, for any $p\in(2,\infty)$, $\tilde{\mathcal{A}}_0^p(\Omega)=\{c(u_0-u_1):\ c\in\mathbb{R}\}$ with u_0 being the same as in (3.2) and u_1 being a solution of the problem (3.1).*

(ii) *Assume $\mathcal{L}_D:=\Delta_D$ and Ω is C^1. If $n\geq 3$ and $p\in[n,\infty)$, then $\tilde{\mathcal{A}}_0^p(\Omega)=\{c\phi_*:\ c\in\mathbb{R}\}$, where ϕ_* is the unique solution of the Dirichlet problem*

$$\begin{cases}\Delta\phi_*=0 & \text{in } \Omega,\\ \phi_*=0 & \text{on } \partial\Omega,\\ \phi_*(x)\to 1 & \text{as } |x|\to\infty.\end{cases}$$

If $n=2$ and $p\in(2,\infty)$, then $\tilde{\mathcal{A}}_0^p(\Omega)=\{c\phi_*:\ c\in\mathbb{R}\}$, where ϕ_* is a harmonic function in Ω satisfying that $\phi_*=0$ on $\partial\Omega$ and

$$\begin{cases}\phi_*(x)=-c_0\ln|x|+O(|x|^{-1}),\\ \nabla\phi_*(x)=-c_0\nabla\ln|x|+O(|x|^{-2}),\\ \nabla^2\phi_*(x)=O(|x|^{-2}),\end{cases}$$

as $|x|\to\infty$. Here, c_0 is a constant, and the notation $O(|x|^{-2})$ means that $\lim_{|x|\to\infty}\frac{|x|^{-2}}{O(|x|^{-2})}$ exists and is finite.

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Proof. If $A \in \mathrm{CMO}(\mathbb{R}^n)$, or $A \in \mathrm{VMO}(\mathbb{R}^n)$ satisfies (GD), from Lemma 3.4, we infer that $q(\mathcal{L}) = \infty$. Moreover, by Lemma 3.2, it holds that $q(\mathcal{L}_{D,\Omega_R}) = \infty$. Therefore, by Proposition 3.3, we find that (i) holds.

The conclusion of (ii) was obtained in [1, Theorem 2.7 and Remark 2.8] (see also [26, Remarks 5.3, 5.4, and 5.5]), and we omit the details here. This finishes the proof of Proposition 3.5. □

We prove Theorem 1.4 by using Theorem 2.4 and Proposition 3.5.

Proof of Theorem 1.4. (i) Assume that $A \in \mathrm{VMO}(\mathbb{R}^n)$ satisfies (GD), or $A \in \mathrm{CMO}(\mathbb{R}^n)$. Let $2 < p \in [n, \infty)$. Let us show that $\mathcal{K}_p(\mathcal{L}_D^{1/2})$ coincides with $\tilde{\mathcal{A}}_0^p(\Omega)$.

Take a large constant $R \in (0, \infty)$ such that $\Omega^c \subset B(0, R-1)$ and let $\Omega_R := \Omega \cap B(0, R)$. Then Ω_R is a bounded C^1 domain of $\mathbb{R}^n$. By Lemma 3.4, we find that $q(\mathcal{L}) = \infty$. Moreover, from Lemma 3.2, we infer that $q(\mathcal{L}_{D,\Omega_R}) = \infty$. Therefore, by Theorem 2.4 and Proposition 3.5, it holds for any $f \in \mathring{W}^{1,p}(\Omega)$ that

$$\inf_{\phi \in \tilde{\mathcal{A}}_0^p(\Omega)} \|\nabla f - \nabla \phi\|_{L^p(\Omega)} \leq C \left\| \mathcal{L}_D^{1/2} f \right\|_{L^p(\Omega)}.$$

This implies that $\mathcal{K}_p(\mathcal{L}_D^{1/2}) \subset \tilde{\mathcal{A}}_0^p(\Omega)$.

Let us show the converse inclusion. Let $u \in \mathcal{A}_0^p(\Omega)$. By Theorem 1.2(i), we see that $\mathcal{L}_D^{1/2} u \in L^p(\Omega)$. Denote by $\{p_t^D\}_{t>0}$ the heat kernels of the heat semigroup $\{e^{-t\mathcal{L}_D}\}_{t>0}$. By [7, Lemma 2.3], we further find that there exists $\gamma > 0$ such that, for all $t > 0$,

$$\int_\Omega \left|\nabla_x p_t^D(x, y)\right|^2 \exp\left\{\gamma |x - y|^2/t\right\} dx \leq C t^{1-\frac{n}{2}},$$

which implies for $1 < q < 2$ that

$$\int_\Omega \left|\nabla_x p_t^D(x, y)\right|^q \exp\left\{\gamma |x - y|^2/(2t)\right\} dx \leq C t^{\frac{q}{2}-\frac{n}{2}}.$$

Thus, $p_t^D(x, \cdot) \in \mathring{W}^{1,q}(\Omega)$ for $1 < q < 2$ and all $t > 0$. Therefore, for each $t > 0$, $\mathcal{L}_D e^{-t\mathcal{L}_D} u$ satisfies that, for all $x \in \Omega$,

$$\begin{aligned}\mathcal{L}_D e^{-t\mathcal{L}_D} u(x) &= \int_\Omega (\mathcal{L}_D)_x p_t^D(x, y) u(y)\, dy = \int_\Omega (\mathcal{L}_D)_y p_t^D(x, y) u(y)\, dy \\ &= -\int_\Omega A(y) \nabla_y p_t^D(x, y) \cdot \nabla u(y)\, dy = \int_\Omega p_t^D(x, y) \mathcal{L}_D u(y)\, dy = 0,\end{aligned}$$

where the second equality by symmetry of the heat kernel, the third equality by $u \in \mathring{W}^{1,p}(\Omega)$ and $p_t^D(x, \cdot) \in \mathring{W}^{1,p'}(\Omega)$, $1/p + 1/p' = 1$, $p > n$ when $n = 2$ and $p \geq n$ when $n \geq 3$. We thus see that

$$\mathcal{L}_D^{1/2} u = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathcal{L}_D e^{-s\mathcal{L}_D} u \frac{ds}{\sqrt{s}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s\mathcal{L}_D} \mathcal{L}_D u \frac{ds}{\sqrt{s}} = 0,$$

which implies that $\tilde{\mathcal{A}}_0^p(\Omega) \subset \mathcal{K}_p(\mathcal{L}_D^{1/2})$.

(ii) By (i) and Theorem 2.4, we conclude that, for all $f \in \mathring{W}^{1,p}(\Omega)$,

$$\inf_{\phi \in \mathcal{K}_p(\mathcal{L}_D^{1/2})} \|\nabla f - \nabla \phi\|_{L^p(\Omega)} \leq C \left\| \mathcal{L}_D^{1/2} f \right\|_{L^p(\Omega)}.$$

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This further implies that, for all $f \in \mathring{W}^{1,p}(\Omega)$,

$$\inf_{\phi \in \mathcal{K}_p(\mathcal{L}_D^{1/2})} \|\nabla f - \nabla \phi\|_{L^p(\Omega)} \le C \left\| \mathcal{L}_D^{1/2} f \right\|_{L^p(\Omega)}.$$

Moreover, by Theorem 1.2(i), it holds for all $f \in \mathring{W}^{1,p}(\Omega)$ and $\phi \in \mathcal{K}_p(\mathcal{L}_D^{1/2})$ that

$$\left\| \mathcal{L}_D^{1/2} f \right\|_{L^p(\Omega)} = \left\| \mathcal{L}_D^{1/2} (f - \phi) \right\|_{L^p(\Omega)} \le C \|\nabla (f - \phi)\|_{L^p(\Omega)}.$$

The last two inequalities give the desired conclusion and complete the proof. □

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Competing interest. The authors have no competing interests to declare.

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