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ARTICLE

On the choosability of H-minor-free graphs

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Abstract

Given a graph H, let us denote by $f_\chi(H)$ and $f_\ell(H)$, respectively, the maximum chromatic number and the maximum list chromatic number of H-minor-free graphs. Hadwiger's famous colouring conjecture from 1943 states that $f_\chi(K_t) = t-1$ for every $t \geq 2$. A closely related problem that has received significant attention in the past concerns $f_\ell(K_t)$, for which it is known that $2t - o(t) \leq f_\ell(K_t) \leq O(t(\log\log t)^6)$. Thus, $f_\ell(K_t)$ is bounded away from the conjectured value $t-1$ for $f_\chi(K_t)$ by at least a constant factor. The so-called H-Hadwiger's conjecture, proposed by Seymour, asks to prove that $f_\chi(H) = \mathsf{v}(H) - 1$ for a given graph H (which would be implied by Hadwiger's conjecture).

In this paper, we prove several new lower bounds on $f_\ell(H)$, thus exploring the limits of a list colouring extension of H-Hadwiger's conjecture. Our main results are:

- For every $\varepsilon > 0$ and all sufficiently large graphs H we have $f_\ell(H) \geq (1-\varepsilon)(\mathsf{v}(H) + \kappa(H))$, where $\kappa(H)$ denotes the vertex-connectivity of H.
- For every $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that asymptotically almost every n-vertex graph H with $\lceil Cn \log n \rceil$ edges satisfies $f_\ell(H) \geq (2-\varepsilon)n$.

The first result generalizes recent results on complete and complete bipartite graphs and shows that the list chromatic number of H-minor-free graphs is separated from the desired value of $(\mathsf{v}(H)-1)$ by a constant factor for all large graphs H of linear connectivity. The second result tells us that for almost all graphs H with superlogarithmic average degree $f_\ell(H)$ is separated from $(\mathsf{v}(H)-1)$ by a constant factor arbitrarily close to 2. Conceptually these results indicate that the graphs H for which $f_\ell(H)$ is close to the conjectured value $(\mathsf{v}(H)-1)$ for $f_\chi(H)$ are typically rather sparse.

Keywords: chromatic number; graph minors; list coloring; Hadwiger's conjecture

2020 MSC Codes: Primary: 05C15, 05C80, 05C83

1. Introduction

All graphs considered in this paper are finite, have no loops and no parallel edges. Given graphs G and H, we say that G contains H as a *minor*, in symbols, $G \succeq H$, if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

Hadwiger's colouring conjecture, first stated in 1943 by Hugo Hadwiger [8], is among the most famous and important open problems in graph theory. It claims a deep relationship between the chromatic number of graphs and their containment of graph minors, as follows.

Conjecture 1 (Hadwiger [8]). *Let $t \in \mathbb{N}$. If a graph G is K_t-minor-free, then $\chi(G) \leq t-1$.*

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Hadwiger's conjecture has been proved for all values $t \leq 6$, see [26] for the most recent result in this sequence, resolving the K_6-minor-free case. For $t = 5$, the conjecture states that K_5-minor-free graphs are 4-colorable. Since planar graphs are K_5-minor-free, this special case already generalises the famous four colour theorem that was proved in 1976 by Appel, Haken and Koch [1, 2].

Given that during 80 years of study little progress has been made towards resolving Hadwiger's conjecture for $t \geq 7$, it seems natural to approach the conjecture via meaningful relaxations. For instance, much of recent work has focused on its asymptotic version. The so-called *linear Hadwiger conjecture* states that for some absolute constant $C \geq 1$, every K_t-minor-free graph is $\lfloor Ct \rfloor$-colorable. Starting with a breakthrough result by Norin, Postle and Song [20] in 2019, there has been a set of papers providing some exciting progress towards this conjecture [21, 23–25]. This culminated in the currently best known upper bound of $O(t \log \log t)$ for the chromatic number of K_t-minor-free graphs by Delcourt and Postle [6] in 2021.

Another natural relaxation, proposed by Seymour [27, 28], suggests replacing the condition that the considered graphs exclude K_t as a minor by the stronger condition that they exclude a particular, possibly non-complete graph H on t vertices as a minor.

Conjecture 2 (H-Hadwiger's conjecture [27, 28]). *H-minor-free graphs are* $(\mathsf{v}(H) - 1)$*-colorable.*

Note that Hadwiger's conjecture would imply the truth of this statement for every H. Also note that this upper bound on the chromatic number would be best possible for every H, as the complete graph $K_{\mathsf{v}(H)-1}$ has chromatic number $\mathsf{v}(H) - 1$ but is too small to host an H-minor.

H-Hadwiger's conjecture can easily be verified using a degeneracy-colouring approach if H is a forest, and it is also known to be true for spanning subgraphs of the Petersen graph [9]. A particular case of H-Hadwiger's conjecture which has received special attention in the past is when $H = K_{s,t}$ is a complete bipartite graph. Woodall [37] conjectured in 2001 that every $K_{s,t}$-minor-free graph is $(s + t - 1)$-colorable. Also this problem remains open, but if true it would resolve H-Hadwiger's conjecture for all bipartite H. Several special cases of this conjecture have been solved by now. Most notably, Kostochka [14, 15] proved that for some function $t_0(s) = O(s^3 \log^3 s)$, H-Hadwiger's conjecture holds whenever $H = K_{s,t}$ and $t \geq t_0(s)$. The conjecture is also true for $H = K_{3,3}$, which can be seen using the structure theorem for $K_{3,3}$-minor-free graphs by Wagner [35] and the fact that planar graphs are 5-colorable. In addition, the statement has been proved for $H = K_{2,t}$ when $t \geq 1$ [5, 18, 37, 38], for $H = K_{3,t}$ when $t \geq 6300$ [16] and for $H = K_{3,4}$ [10]. In a different direction, Norin and Turcotte [22] recently proved H-Hadwiger's conjecture for all sufficiently large bipartite graphs of bounded maximum degree that belong to a class of graphs with strongly sublinear separators.

1.1. List colouring H-minor-free graphs

In this paper, we shall be concerned with the *list chromatic number* of graphs that exclude a fixed graph H as a minor. List colouring is a well-known and popular subject in the area of graph colouring, whose introduction dates back to the seminal paper of Erdős, Rubin and Taylor [7]. A *list assignment* for a graph G is a mapping $L : V(G) \to 2^{\mathbb{N}}$ assigning to every vertex $v \in V(G)$ a finite set $L(v)$ of colours, also called the *list* of v. An *L-colouring* of G is a proper colouring $c : V(G) \to \mathbb{N}$ for which every vertex must choose a colour from its list, that is, $c(v) \in L(v)$ for every $v \in V(G)$. Finally, we say that G is *k-choosable* for some integer $k \geq 1$ if there exists a proper L-colouring for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$. The *list chromatic number* of G, denoted $\chi_\ell(G)$, is the smallest integer k such that G is k-choosable. Note that trivially $\chi(G) \leq \chi_\ell(G)$ for every graph G, but conversely no relationship holds, as $\chi_\ell(G)$ is unbounded even on bipartite graphs G, see [7].

The first open problem regarding list colouring of minor-closed graph classes was raised already in 1979 in the seminal paper by Erdős, Rubin and Taylor [7], who asked to determine

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the maximum list chromatic number of planar graphs. This question was answered in the 1990s in work of Thomassen [33] and Voigt [34]. Thomassen proved that every planar graph is 5-choosable, and Voigt gave the first examples of planar graphs G with list chromatic number $\chi_\ell(G) = 5$.

The latter result also answered a question by Borowiecki [4] in the negative, who had asked whether one could potentially strengthen Hadwiger's conjecture to the list colouring setting by asserting that every K_t-minor-free graph G satisfies $\chi_\ell(G) \leq t - 1$.

Given the previous discussion, it is natural to study the maximum list chromatic number of K_t-minor-free graphs, see also [39] for an open problem garden entry about this problem. To make the following presentation more convenient, for every graph H we denote by $f_\chi(H)$ and $f_\ell(H)$, respectively, the maximum (list) chromatic number of H-minor-free graphs. Note that with this notation, the H-Hadwiger's conjecture amounts to saying that $f_\chi(H) = \mathsf{v}(H) - 1$.

Let us briefly summarise previous work regarding bounds on $f_\ell(K_t)$. The construction of Voigt mentioned above shows that $f_\ell(K_5) \geq 5$. Thomassen's result regarding the 5-choosability of planar graphs was later extended by Škrekovski [29] to K_5-minor-free graphs, thus proving that $f_\ell(K_5) = 5$. Until today none of the values $f_\ell(K_t)$ with $t \geq 6$ have been determined precisely, a list of the currently best known lower and upper bounds for $f_\ell(K_t)$ for small values of t can be found in [3]. In 2007, Kawarabayashi and Mohar [12] made two conjectures regarding the asymptotic behaviour of $f_\ell(K_t)$, namely that (A) $f_\ell(K_t) = O(t)$, this is known as the *list linear Hadwiger conjecture*, and that (B) $f_\ell(K_t) \leq \frac{3}{2}t$ for every t. In 2010, Wood [36], inspired by the fact that $f_\ell(K_5) = 5$, proposed an even stronger conjecture stating that $f_\ell(K_t) = t$ for every $t \geq 5$. This strong conjecture was refuted in 2011 by Barát, Joret and Wood, who gave a construction showing that $f_\ell(K_t) \geq \frac{4}{3}t - O(1)$. However, the weaker conjecture (B) by Kawarabayashi and Mohar still remained open. Recently, a new lower bound of $f_\ell(K_t) \geq 2t - o(t)$ was established by the second author [30], thus refuting conjecture (B). As for upper bounds, the best currently known bound is $f_\ell(K_t) \leq Ct(\log \log t)^6$, which was established in 2020 by Postle [25]. Some previous work also addressed bounds on $f_\ell(H)$ when H is non-complete. In particular, Woodall [27] conjectured in 2001 that $f_\ell(K_{s,t}) = s + t - 1$ for all integers $s, t \geq 1$, and proved this in the case when $s = t = 3$. From the previously mentioned works [5, 18, 37, 38] it was also known that $f_\ell(K_{2,t}) = t + 1$ for $t \geq 1$. Additionally, a result by Jørgensen [10] implied the truth of the conjecture for $K_{3,4}$, and Kawarabayashi [11] proved that $f_\ell(K_{4,t}) \leq 4t$ for every t. Despite this positive evidence, Woodall's conjecture was recently disproved by the second author [31] showing that $f_\ell(K_{s,t}) \geq (1 - o(1))(2s + t)$ for all large values of $s \leq t$. A positive result comes from the aforementioned result of Norin and Turcotte [22], which also works for list colourings and shows that $f_\ell(H) = \mathsf{v}(H) - 1$ for all large bipartite graphs H of bounded maximum degree in a graph class with strongly sublinear separators.

1.2. Our contribution

The above discussion shows that when excluding a sufficiently large complete or a sufficiently large balanced complete bipartite graph H, the value of $f_\ell(H)$ exceeds the trivial lower bound $f_\ell(H) \geq \mathsf{v}(H) - 1$ by at least a constant factor. This means that, in a strong sense, one cannot hope for extending Hadwiger's conjecture to list colouring with the same quantitative bounds. However, note that if H is a complete or a balanced complete bipartite graph, then H is quite dense in the sense that it has a quadratic number of edges. On the other extreme of the spectrum, the previously mentioned result by Norin and Turcotte [22] shows that $f_\ell(H) = \mathsf{v}(H) - 1$ *does* hold for large classes of graphs H with a constant maximum degree (and thus, with a linear number of edges). This naturally opens up a new question, as follows: How sparse must the desired minor H be, such that one *can* hope for a list colouring extension of H-Hadwiger's conjecture? Concretely, which structural and density properties of graphs H guarantee that $f_\ell(H) = \mathsf{v}(H) - 1$? While one might

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be tempted to hope for a nice description of the class of all graphs H satisfying $f_\ell(H) = \mathsf{v}(H) - 1$, Theorem 3 below speaks a word of caution: Any given graph F can be augmented, by the addition of sufficiently many isolated vertices, to a graph H in this class.

Theorem 3. *For every graph F there exists $k_0 = k_0(F)$ such that for every $k \geq k_0$ the graph H obtained from F by the addition of k isolated vertices satisfies $f_\ell(H) = \mathsf{v}(H) - 1$. In fact, every H-minor-free graph is $(\mathsf{v}(H) - 2)$-degenerate.*

This shows that arbitrary graphs F can show up as induced subgraphs of graphs H with $f_\ell(H) = \mathsf{v}(H) - 1$. To avoid such artificial constructions and to make a nice structural description of the graph class at hand more likely, it seems natural to ask for the largest class that is *closed under taking subgraphs* such that all members H of this class satisfy $f_\ell(H) = \mathsf{v}(H) - 1$.[1]

Problem 4. *Characterise the class $\mathcal{H}$ of graphs H such that $f_\ell(H') = \mathsf{v}(H') - 1$ for all $H' \subseteq H$.*

The main contributions of this paper are Theorems 5 and 7 below, which establish new lower bounds on $f_\ell(H)$ and strongly limit the horizon for positive instances of Problem 4. The first result proves a lower bound on $f_\ell(H)$ in terms of $\mathsf{v}(H)$ and the vertex-connectivity $\kappa(H)$, implying that $f_\ell(H)$ exceeds $\mathsf{v}(H)$ by a constant factor for all large graphs of linear connectivity.[2]

Theorem 5. *For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that every graph H on at least n_0 vertices satisfies $f_\ell(H) \geq (1 - \varepsilon)(\mathsf{v}(H) + \kappa(H))$.*

In particular, this result immediately generalises both of the lower bounds of $f_\ell(K_t) \geq 2t - o(t)$ and $f_\ell(K_{s,t}) \geq (1 - o(1))(2s + t)$ previously established by the second author in [30, 31] by noting that $\kappa(K_t) = t - 1$ and $\kappa(K_{s,t}) = s$ for $s \leq t$. It also has the following simple consequence, showing that the graphs in $\mathcal{H}$ have a subquadratic number of edges.

Corollary 6. *For every $n \in \mathbb{N}$, let $h(n)$ denote the maximum possible number of edges of an n-vertex graph in $\mathcal{H}$. Then $\lim_{n\to\infty} \frac{h(n)}{n^2} = 0$.*

Proof. Towards a contradiction, suppose the statement is not true. Then there is some constant $\delta > 0$ such that there exist arbitrarily large graphs $H \in \mathcal{H}$ with average degree at least $\delta \mathsf{v}(H)$. By a classical result of Mader [17], every graph of average degree at least $4(k - 1)$ for some integer $k \geq 2$ contains a k-connected subgraph. As $\mathcal{H}$ is closed under subgraphs, this implies that there are arbitrarily large graphs $H \in \mathcal{H}$ with connectivity at least $\frac{\delta}{4}\mathsf{v}(H)$. Then, using $\varepsilon := \frac{\delta}{8}$ and Theorem 5, for sufficiently large $H \in \mathcal{H}$ with average degree at least $\delta \mathsf{v}(H)$, we have $f_\ell(H) \geq (1 - \varepsilon)(1 + \frac{\delta}{4})\mathsf{v}(H) = (1 + \frac{\delta}{8} - \frac{\delta^2}{32})\mathsf{v}(H) > \mathsf{v}(H)$. However, we have $f_\ell(H) = \mathsf{v}(H) - 1$ by the definition of $\mathcal{H}$, which yields the desired contradiction and concludes the proof. □

Our second result addresses to what extent sparsity of H can push $f_\ell(H)$ closer to the trivial lower bound $\mathsf{v}(H) - 1$, by showing that for any fixed $\varepsilon > 0$, asymptotically almost all n-vertex graphs H with average degree of order $C \log n$ for a sufficiently large constant C are far from being in $\mathcal{H}$, in the sense that $f_\ell(H)$ is separated from $\mathsf{v}(H) - 1$ by a factor of at least $2 - \varepsilon$.

Theorem 7. *For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that asymptotically almost every graph H on n vertices with $\lceil Cn \log n \rceil$ edges satisfies $f_\ell(H) \geq (2 - \varepsilon)n$.*

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[1] This is done in the spirit of the definition of perfect graphs, where a nice characterisation of graphs with $\chi(G) = \omega(G)$ seems elusive, but the largest class of graphs with this property that is closed under taking induced subgraphs admits a beautiful structural description by the strong perfect graph theorem.

[2] Here, by graphs of linear connectivity we mean n-vertex graphs H that are αn-connected for some small but absolute constant $\alpha > 0$.

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Together with Corollary 6, this hints at the graphs in $\mathcal{H}$ typically being quite sparse. It also shows that the lower bound $f_\ell(K_t) \geq 2t - o(t)$ for complete graphs from [30] applies in equal strength to almost all t-vertex graphs H with $\omega(t \log t)$ edges, despite them being (much) sparser than K_t.

Our proofs of Theorems 5 and 7 are based on several extensions and refinements of the probabilistic approach for lower bounding $f_\ell(K_t)$ and $f_\ell(K_{s,t})$ introduced by the second author in [30, 31]. However, several new ideas are required to overcome obstacles arising from the largely increased generality of the setup. For instance, to prove Theorem 7 one has to construct graphs avoiding rather sparse graphs H as a minor. While the constructions in [30, 31] were based on the fact that clique sums of graphs under mild assumptions preserve K_t- and $K_{s,t}$-minor-freeness, a corresponding statement is no longer true for sparse graphs H of much lower connectivity.

1.3. Organisation of the paper

In Section 2 we prove two probabilistic results on random bipartite graphs that exhibit properties of these graphs that are crucial for our constructions in the proofs of Theorems 5 and 7. We then present the proofs of our main results Theorem 5 and Theorem 7 in, respectively, Section 3 and Section 4. Finally, in Section 5 we separately prove Theorem 3. The latter proof is self-contained and independent of the results in the other three sections.

1.4. Notation and terminology

By $\kappa(G)$ we denote the *vertex-connectivity* of a graph G, i.e., the minimum k such that G is k-connected. Given integers $m, n \geq 1$ and an edge-probability $p \in [0, 1]$, we use $G(m, n;p)$ to denote the bipartite Erdős-Rényi random graph with bipartition classes A and B of sizes m and n, respectively, and in which a pair ab with $a \in A$ and $b \in B$ is chosen as an edge of $G(m, n;p)$ independently with probability p. For integers $m, n \geq 1$ we denote by $G(n;m)$ a random graph drawn uniformly from all graphs on vertex set $[n] = \{1, \ldots, n\}$ with exactly m edges.

While the original definition of the graph minor-containment relation $\succeq$ is via edge contractions and deletions, for proving the results in this paper it will be more convenient to think about *graph minor models*. Given a graph G and a graph H, an *H-minor model* is a collection $(Z_h)_{h \in V(H)}$ of pairwise disjoint and non-empty subsets of $V(G)$ with the property that $G[Z_h]$ is a connected graph for every $h \in V(H)$ and such that for every edge $h_1 h_2 \in E(H)$, there exists at least one edge in G with endpoints in Z_{h_1} and Z_{h_2}. The sets $Z_h, h \in V(H)$ are also called the *branch sets* of the minor model. It is well-known and easy to see that for every pair of graphs G and H we have $G \succeq H$ if and only if there exists an H-minor model in G.

2. Probabilistic lemmas

In this short preparatory section we prove two simple auxiliary results (Lemmas 9 and 11) that will be used in the proofs of both our main results in Section 3 and 4. The lemmas capture two simple but important properties exhibited by bipartite Erdős-Rényi random graphs. These properties will later be used to lower bound the list chromatic number of the graphs in our constructions for Theorems 5 and 7 and to argue that they exclude a given graph as a minor.

Two basic tools from probability theory that we will use in the following are the classical Chernoff concentration bounds, stated below. A standard application of the Chernoff bounds yields an upper bound on the maximum degree of bipartite graphs with linear expected degree, stated below without proof.

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Lemma 8 (Chernoff). *Let X be a binomially distributed random variable. Then the following bounds hold for every $\delta \in (0, 1]$:*

$$\mathbb{P}(X \geq (1+\delta)\mathbb{E}(X)) \leq \exp\left(-\frac{\delta^2}{3}\mathbb{E}(X)\right), \ \mathbb{P}(X \leq (1-\delta)\mathbb{E}(X)) \leq \exp\left(-\frac{\delta^2}{2}\mathbb{E}(X)\right).$$

Lemma 9. *Let $p \in (0, 1]$ be a constant. Then w.h.p. the random bipartite graph $G = G(n, n;p)$ has maximum degree at most $2pn$.*

In order to compactly state and refer to our next lemma below, it is convenient for us to introduce a technical definition for the following relationship between a graph H and a bipartite graph G with vertex bipartition A, B. Let $G^{\complement}$ denote the graph complement of G. We are interested in the existence of $\widetilde{H}$-minor models in $G^{\complement}$, where $\widetilde{H}$ is a subgraph of H. For fixed integers k, l, consider the situation where $X_1, \ldots, X_k \subseteq A$, $Y_1, \ldots, Y_k \subseteq B$ are pairwise disjoint subsets of A and B and $x_1, \ldots, x_k, y_1, \ldots, y_l \in V(H)$ are distinct vertices of H. Let $\widetilde{H}$ be the induced subgraph by the vertices $\{x_1, \ldots, x_k, y_1, \ldots, y_l\}$ and let $Z_{x_1} := X_1, \ldots, Z_{x_k} := X_k, Z_{y_1} := Y_1, \ldots, Z_{y_l} := Y_l$. Then the branch sets $(Z_v)_{v \in V(\widetilde{H})}$ form an $\widetilde{H}$-minor model in $G^{\complement}$ if and only if each branch set is connected and for each edge of the form $x_iy_j \in E(H)$ there is an edge between Z_{x_i} and Z_{y_j} in $G^{\complement}$. Therefore, $(Z_v)_{v \in V(\widetilde{H})}$ is not an $\widetilde{H}$-minor model in $G^{\complement}$ if there is an edge $x_iy_j \in E(H)$ such that G contains all edges between X_i and Y_j. This relationship, with some additional constraints on branch set size and subgraph size, is captured by the following property.

Definition 10 (Property P). *Let $0 < \delta < 1$, $s \in \mathbb{N}$ and let H be a graph on n vertices. We say that a bipartite graph G with bipartition $\{A, B\}$ satisfies property $\mathsf{P}(H, \delta, s)$ if for all integers $k, l \geq \delta n$ the following holds:*

If $x_1, \ldots, x_k, y_1, \ldots, y_l \in V(H)$ are distinct vertices satisfying $\mathsf{e}_H(\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_l\}) \geq s$ and $X_1, \ldots, X_k \subseteq A$, $Y_1, \ldots, Y_l \subseteq B$ are pairwise disjoint sets of size at most $\frac{1}{\delta}$ each, then there exists an index pair $(i, j) \in [k] \times [l]$ such that $x_iy_j \in E(H)$ and $xy \in E(G)$ for every $(x, y) \in X_i \times Y_j$.

Lemma 11. *Let $\delta, p \in (0, 1)$ be constants. Then there exists a constant $D = D(\delta, p) > 1$ and a sequence $q_n = 1 - o(1)$ such that with $s = s(n) := \lceil Dn \log n \rceil$ for every n-vertex graph H the random bipartite graph $G = G(n, n;p)$ satisfies $P(H, \delta, s)$ with probability at least q_n.*

Proof. Choose any constant $D > \max\{1, 3p^{-(1/\delta^2)}\}$. Let A, B be the vertex bipartition of G with $|A| = |B| = n$, let H be an n-vertex graph and let $k, l \geq \delta n$. There are at most n^n choices of distinct vertices $x_1, \ldots, x_k, y_1, \ldots, y_l \in V(H)$ and at most n^{2n} choices of disjoint vertex sets $X_1, \ldots, X_k \subseteq A$, $Y_1, \ldots, Y_l \subseteq B$. Consider a fixed such choice satisfying the premises in Definition 10 and the random event that for every pair $(i, j) \in [k] \times [l]$ such that $x_iy_j \in E(H)$, not all of the potential edges between X_i and Y_j are included in G. The probability that this holds is $\prod_{x_iy_j \in E(H)} (1 - p^{|X_i||Y_j|}) \leq (1 - p^{(1/\delta^2)})^{Dn \log n}$, where we used the premises that the sets X_i, Y_j are of size at most $\frac{1}{\delta}$ and that there are at least $s \geq Dn \log n$ edges of the form $x_iy_j \in E(H)$. Using a union bound over the choices described above, we have

$$\begin{aligned}\mathbb{P}(G \text{ does not satisfy property } \mathsf{P}(H, \delta, s)) &\leq n^{3n}(1 - p^{(1/\delta^2)})^{Dn \log n} \\ &\leq \exp(3n \log n - p^{(1/\delta^2)} Dn \log n).\end{aligned}$$

We have $3 - p^{(1/\delta^2)}D < 0$ and thus the above expression tends to 0 as $n \to \infty$. Setting $q_n := 1 - \exp((3 - p^{(1/\delta^2)}D)n \log n)$ then concludes the proof of the lemma. □

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3. Proof of Theorem 5

In this section, we present the proof of Theorem 5. We start off by making use of Lemmas 9 and 11 from the previous section to establish the existence of small H-minor-free graphs that are in a sense "almost complete", as follows.

Lemma 12. *For every $\varepsilon \in (0, \frac{1}{2})$ there exists an integer $N = N(\varepsilon)$ such that for every $n \geq N$ and every n-vertex graph H with $\kappa(H) \geq \varepsilon n$ there exists a graph F with the following properties:*

- *The vertex set of F can be partitioned into two disjoint sets A and B such that both A and B form cliques in F and $|A| = \lfloor (1 - 2\varepsilon)\kappa(H) \rfloor$, $|B| = \lfloor (1 - 2\varepsilon)n \rfloor$.*
- *Every vertex in B has at most εn non-neighbors in F.*
- *F is H-minor-free.*

Proof. Define $p := \frac{\varepsilon}{2}$ and $\delta := \varepsilon^2$. By Lemma 9 there is a sequence $p_n = 1 - o(1)$ such that $G(n, n;p)$ has maximum degree at most $2pn = \varepsilon n$ with probability at least p_n, and by Lemma 11 there exists an absolute constant $D > 0$ and a sequence $q_n = 1 - o(1)$ such that for every n-vertex graph H the probability that $G(n, n;p)$ satisfies property $\mathsf{P}(H, \delta, \lceil Dn \log n \rceil)$ is at least q_n. Let n_1 be such that $p_n, q_n > \frac{1}{2}$ for every $n \geq n_1$. Moreover, let $n_2 \in \mathbb{N}$ be chosen large enough such that the inequality $\delta^2 n^2 \geq Dn \log n$ holds for every $n \geq n_2$. Finally, we put $N := \max\{n_1, n_2\}$ and let $n \geq N$ be arbitrary. By our choice of N, there then exists at least one bipartite graph G with bipartition $\{A', B'\}$ such that $|A'| = |B'| = n$, G has maximum degree at most εn, and G satisfies property $\mathsf{P}(H, \delta, \lceil Dn \log n \rceil)$. Let $A \subseteq A', B \subseteq B'$ be chosen (arbitrarily) such that $|A| = \lfloor (1 - 2\varepsilon)\kappa(H) \rfloor$, $|B| = \lfloor (1 - 2\varepsilon)n \rfloor$. Note that this is possible as $\kappa(H) < \mathsf{v}(H) = n$. We now define F as the graph complement of the induced subgraph $G[A \cup B]$ of G. Since A and B are independent sets in G, they form cliques in F. Thus the first item of the lemma is satisfied. To verify the second item, it suffices to note that since G has maximum degree at most εn, the same is true for $G[A \cup B]$, and thus every vertex in F can have at most εn non-neighbors in F.

It thus remains to prove that F is indeed H-minor-free. Towards a contradiction, suppose that there exists an H-minor model $(Z_h)_{h \in V(H)}$ in F. Let X_A, X_B, X_{AB} be the partition of $V(H)$ defined as follows: $X_A := \{h \in V(H) | Z_h \subseteq A\}$ and $X_B := \{h \in V(H) | Z_h \subseteq B\}$ contain those branch sets which are subsets of A or of B, respectively, and $X_{AB} := \{h \in V(H) | Z_h \cap A \neq \emptyset \neq Z_h \cap B\}$ contains the branch sets which overlap with both A and B.

Our goal now is to find at least δn vertices in X_A and in X_B whose corresponding vertex subsets of A or of B have size at most $\frac{1}{\delta}$ and which have at least $Dn \log n$ edges between them. We will then be able to use property $\mathsf{P}(H, \delta, \lceil Dn \log n \rceil)$ to complete the proof.

Note that we have $|X_B| + |X_{AB}| \leq |B| \leq (1 - 2\varepsilon)n$ as the sets in $(Z_h)_{h \in V(H)}$ are pairwise disjoint. Given that $|X_A| + |X_B| + |X_{AB}| = \mathsf{v}(H) = n$, this implies that $|X_A| \geq 2\varepsilon n$. Since the sets $(Z_h)_{h \in X_A}$ are disjoint and since $|A| \leq (1 - 2\varepsilon)\kappa(H) < (1 - 2\varepsilon)n < n$, there cannot be more than δn sets of size greater than $\frac{1}{\delta}$ in the collection $(Z_h)_{h \in X_A}$. Hence, there exists $k \geq 2\varepsilon n - \delta n \geq \delta n$ and k distinct vertices $x_1, \ldots, x_k \in X_A$ such that $|Z_{x_i}| \leq \frac{1}{\delta}$ for $i = 1, \ldots, k$. Note that H has minimum degree at least $\kappa(H)$, for otherwise one could separate a vertex in H from the rest of the graph by deleting fewer than $\kappa(H)$ vertices. Using this, we have

$$|N_H(x_i) \cap X_B| \geq \deg_H(x_i) - |X_A \cup X_{AB}| \geq \delta(H) - |A|$$

$$\geq \kappa(H) - (1 - 2\varepsilon)\kappa(H) = 2\varepsilon\kappa(H) \geq 2\varepsilon^2 n = 2\delta n$$

for every $i = 1, \ldots, k$, where in the last step we used that $\kappa(H) \geq \varepsilon n$ by assumption. Consider for any fixed index $i \in [k]$ the set collection $(Z_h)_{h \in N_H(x_i) \cap X_B}$. Since the sets are pairwise disjoint and contained in the set B of size at most n, as above it follows that at most δn sets in this collection can be of size greater than $\frac{1}{\delta}$. Consequently, for each $i \in [k]$ there exists a subset $N_i \subseteq N_H(x_i) \cap X_B$ of size at least $2\delta n - \delta n = \delta n$ such that $|Z_h| \leq \frac{1}{\delta}$ for every $h \in N_i$ and $i \in [k]$. Let $y_1, \ldots, y_l \in X_B$

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be distinct vertices such that $\{y_1, \ldots, y_l\} = \bigcup_{i=1}^{k} N_i$. Then clearly, $l \geq |N_1| \geq \delta n$. Furthermore, we have

$$\mathrm{e}_H(\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_l\}) \geq \sum_{i=1}^{k} |N_i| \geq k \cdot \delta n \geq \delta^2 n^2 \geq Dn \log n,$$

where in the last step we used our assumption that $n \geq N \geq n_2$.

We can now use that G satisfies property $\mathsf{P}(H, \delta, \lceil Dn \log n \rceil)$, which directly implies that there exists a pair $(i, j) \in [k]^2$ such that $x_i y_j \in E(H)$ and G contains all edges of the form xy where $(x, y) \in Z_{x_i} \times Z_{y_j}$. However, by definition of F this means that there exists no edge in F which has endpoints in both Z_{x_i} and Z_{y_j}. This is a contradiction to our initial assumption that $(Z_h)_{h \in V(H)}$ form an H-minor model in F. Thus, F does not contain H as a minor, which establishes the third item of the lemma and concludes the proof. □

Our next lemma below guarantees that for sufficiently well-connected graphs H, the property of being H-minor-free is preserved when pasting together two graphs along a sufficiently small clique. This statement will then be used in the proof of Theorem 5 to glue several copies of the H-minor-free graph from Lemma 12 along a common clique, thus eventually creating a graph that is still H-minor-free but has an increased list chromatic number. The lemma is folklore in the graph minors community, see also Section 3.1 in [19].

Lemma 13. *Let G_1, G_2 be H-minor-free graphs and let $C := V(G_1) \cap V(G_2)$. If C forms a clique in both G_1 and G_2 and if $|C| < \kappa(H)$, then the graph union $G_1 \cup G_2$ is also H-minor-free.*

The last ingredient required to complete the proof of Theorem 5 is a simple but important idea on how to lower-bound the list chromatic number of a graph that is obtained from a fixed graph F by repeated pasting along the same clique. Since the statement will also be reused for the proof of Theorem 7 in the next section, we decided to isolate it here. We use the following terminology:

Definition 14 (Pasting). *Let F be a graph, let $S \subseteq V(F)$ and $K \in \mathbb{N}$. A K-fold pasting of F at S is any graph that can be expressed as the union of K isomorphic copies $F_1, \ldots, F_K$ of F with the property that $V(F_i) \cap V(F_j) = S$ for all $1 \leq i < j \leq K$.*

Lemma 15. *Let $m, n, d \in \mathbb{N}$ with $d \leq m$ and let F be a graph whose vertex set is partitioned into two cliques A, B such that every vertex in B has at least $|A| - d$ neighbours in A. Let $K = (|A| + |B| - 1)^{|A|}$ and let $F^{(K)}$ be a K-fold pasting of F at A. Then $\chi_\ell(F^{(K)}) \geq |A| + |B| - d$.*

Proof. Let $F_1, \ldots, F_K$ be an ordering of the copies of F in the pasting graph $F^{(K)}$, and let $B_1, \ldots, B_K$ be the corresponding copies of B. Let $f{:}[|A| + |B| - 1]^A \to [K]$ be an arbitrary bijection and let $c_1, \ldots, c_K{:}A \to [|A| + |B| - 1]$ be the ordering of colour assignments to A that satisfies $f(c_i) = i$ for all $i \in [K]$. Consider the list assignment $L{:}V(F^{(K)}) \to 2^{[|A|+|B|-1]}$ defined as follows:

- $L(a) := [|A| + |B| - 1]$ for all $a \in A$
- $L(b) := [|A| + |B| - 1] \setminus \{c_i(a) \mid a \in A \setminus N_{F_i}(b)\}$ for all $b \in B_i$ for all $i \in [K]$

Given that every vertex in B by assumption has at most d non-neighbors in F, we have $|L(v)| \geq |A| + |B| - 1 - d$ for all $v \in V(F^{(K)})$. Now assume towards a contradiction that $F^{(K)}$ admits a proper L-colouring c and let $i \in [K]$ be the unique index satisfying $c_i(a) = c(a)$ for all $a \in A$. Then let $c_{F_i}{:}A \cup B_i \to [|A| + |B| - 1]$ be the colouring c restricted to the graph F_i. Since $\mathsf{v}(F_i) = |A| + |B|$, there exist by the pigeonhole principle vertices $u, v \in V(F_i)$ with $c(u) = c(v)$. Since c is proper and A, B are cliques, we have $uv \notin E(F_i)$ and $u \in A$, $v \in B$ without loss of generality. However, $c(u) \notin L(v)$ by the construction of L, a contradiction. □

By assembling the previously established pieces, we can now easily deduce Theorem 5.

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Proof of Theorem 5. Let a constant $\varepsilon > 0$ be given choose $\tilde{\varepsilon} \in (0, \frac{\varepsilon}{4})$. Let $N = N(\tilde{\varepsilon})$ be as in Lemma 12. We now set $n_0 := \max\{N, \lceil \frac{4}{\varepsilon^2} \rceil\}$ and claim that Theorem 5 holds for this choice of n_0.

Let H be a graph on $n \geq n_0$ vertices. We have to prove that $f_\ell(H) \geq (1-\varepsilon)(n + \kappa(H))$. If $\kappa(H) < \varepsilon n$, then this follows directly from the trivial lower bound via

$$f_\ell(H) \geq \mathsf{v}(H) - 1 = n - 1 \geq (1-\varepsilon^2)n = (1-\varepsilon)(n + \varepsilon n) > (1-\varepsilon)(n + \kappa(H)).$$

Thus, we may now assume $\kappa(H) \geq \varepsilon n$, in particular, $\kappa(H) \geq \tilde{\varepsilon} n$. Using $n \geq N$ and Lemma 12 we now find that there exists an H-minor-free graph F whose vertex set is partitioned into two cliques A, B such that $|A| = \lfloor (1-2\tilde{\varepsilon})\kappa(H) \rfloor < \kappa(H)$ and $|B| = \lfloor (1-2\tilde{\varepsilon})n \rfloor$, and such that every vertex in B has at most $\tilde{\varepsilon} n$ non-neighbors in F. Let $d := \lfloor \tilde{\varepsilon} n \rfloor$ and $K := (|A| + |B| - 1)^{|A|}$. Let $F^{(K)}$ denote a K-fold pasting of F at the clique A. Since every vertex in B has at least $|A| - d$ neighbours in A, we can apply Lemma 15 to find that

$$\chi_\ell(F^{(K)}) \geq |A| + |B| - d \geq (1-2\tilde{\varepsilon})(\kappa(H) + n) - 2 - \tilde{\varepsilon} n$$

$$\geq (1-3\tilde{\varepsilon})(n + \kappa(H)) - 2 \geq (1-\varepsilon)(n + \kappa(H)),$$

using $n \geq n_0 \geq \frac{4}{\varepsilon^2}$ in the last step. In addition, since $|A| < \kappa(H)$, the graph $F^{(K)}$ is H-minor-free by repeated application of Lemma 13. We conclude that $f_\ell(H) \geq (1-\varepsilon)(\mathsf{v}(H) + \kappa(H))$, as desired. □

4. Proof of Theorem 7

In this section, we present the proof of Theorem 7. The theorem claims a lower bound on $f_\ell(H)$ for almost all graphs H on n vertices and $\lceil Cn \log n \rceil$ edges for some large constant $C > 0$. However, in fact the only condition on the graph H our lower bound proof relies upon is the following pseudo-random graph property, guaranteeing the existence of many edges between every pair of disjoint linear-size vertex subsets in H.

Definition 16 (Property Q, graph family $\mathcal{Q}_n$). *Let $\delta > 0$ and $D > 1$ be arbitrary. We say that a graph H with n vertices satisfies property $\mathsf{Q}(\delta, D)$ if for every two disjoint vertex sets $A, B \subseteq V(H)$ with $|A|, |B| \geq \delta n$, we have $\mathsf{e}_H(A, B) \geq Dn \log n$. Let $\mathcal{Q}_n(\delta, D)$ denote the family of n-vertex graphs H that satisfy property $\mathsf{Q}(\delta, D)$.*

Crucially, property $\mathsf{Q}(\delta, D)$ is satisfied for almost all graphs on n vertices with an average degree of $C \log n$ for a large enough constant C. The proof uses a standard probabilistic argument and is therefore omitted.

Lemma 17. *Let $\delta > 0$, $D > 1$ be arbitrary and let $m: \mathbb{N} \to \mathbb{N}$ be defined as $m(n) = \lceil \frac{D^2}{\delta^2} n \log n \rceil$. Then with high probability as $n \to \infty$, a random graph $H = G(n;m(n))$ drawn uniformly from all n-vertex graphs with $m(n)$ edges satisfies property $Q(\delta, D)$.*

In our next step towards proving Theorem 7, we establish the following statement somewhat analogous to Lemma 12, showing how to build small and close-to-complete H-minor-free graphs for a given graph $H \in \mathcal{Q}_n(\delta, D)$.

Lemma 18. *Let $\delta \in (0, 1)$, $D > 1$, $n \in \mathbb{N}$, and $H \in \mathcal{Q}_n(\delta, D)$ be arbitrary. Moreover, let G be a bipartite graph with bipartition $\{A, B\}$, $|A| = |B| = \lfloor (1-3\delta)n \rfloor$ satisfying property $P(H, \delta, s)$ for $s = \lceil Dn \log n \rceil$. Then its complement graph G^{C} does not contain $H[U]$ as a minor for any $U \subseteq V(H)$ with $|U| \geq (1-\delta)n$.*

Proof. Assume G^{C} contains $H[U]$ as a minor for some $U \subseteq V(H)$ with $|U| \geq (1-\delta)n$. Let $(Z_h)_{h \in U}$ be an $H[U]$-minor model in G^{C} and define $X_A := \{h \in U \mid Z_h \subseteq A\}$, $X_B := \{h \in U \mid Z_h \subseteq B\}$, and $X_{AB} := \{h \in U \mid Z_h \cap A \neq \emptyset \neq Z_h \cap B\}$. We have $|X_A| + |X_{AB}| \leq |A|$, $|X_B| + |X_{AB}| \leq$

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$|B|$, and $|X_A| + |X_B| + |X_{AB}| = |U| \geq (1-\delta)n$, which implies $|X_A|, |X_B| \geq (1-\delta)n - (1-3\delta)n = 2\delta n$.

Since the branch sets $(Z_h)_{h \in X_A}$ in A and the branch-sets $(Z_h)_{h \in X_B}$ in B are pairwise disjoint, at most $\delta(1-3\delta)n < \delta n$ branch sets in each of $(Z_h)_{h \in X_A}$ and $(Z_h)_{h \in X_B}$ can be larger than $\frac{1}{\delta}$. Thus, there are at least $2\delta n - \delta n = \delta n$ branch sets of size at most $\frac{1}{\delta}$ in $(Z_h)_{h \in X_A}$ as well as in $(Z_h)_{h \in X_B}$. Thus for $k := l := \lceil \delta n \rceil$, there exist distinct vertices $x_1, \ldots, x_k \in X_A$, $y_1, \ldots, y_l \in X_B$ such that $|Z_{x_i}|, |Z_{y_j}| \leq \frac{1}{\delta}$ for all $1 \leq i, j \leq k = l$. Since $H \in \mathcal{Q}_n(\delta, D)$, we have $\mathsf{e}_H(\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_l\}) \geq \lceil Dn \log n \rceil = s$. Next we use our assumption that G satisfies property $\mathsf{P}(H, \delta, s)$. It implies that there exists an edge $x_i y_j \in E(H)$ with $(i,j) \in [k] \times [l]$ such that G contains all the edges xy with $(x,y) \in Z_{x_i} \times Z_{y_j}$. Then, however, there is an edge between vertices x_i and y_j in H, but no edge between the corresponding branch sets Z_{x_i} and Z_{y_j} in $G^{\complement}$, a contradiction. □

The next auxiliary statement we need is Lemma 19 below, which establishes a weak analogue of Lemma 13 for graphs $H \in \mathcal{Q}_n(\delta, D)$. Note that as these graphs may have sublinear minimum degree and connectivity, Lemma 13 cannot be used to obtain the same statement.

Lemma 19. *Let $\delta > 0$, $D > 1$, $H \in \mathcal{Q}_n(\delta, D)$ and let F be a graph with a clique $W \subseteq V(F)$ of size $\lfloor (1-3\delta)n \rfloor$. Let $K \in \mathbb{N}$ and let $F^{(K)}$ be a K-fold pasting of F at W. If $F^{(K)}$ contains H as a minor, then there exists $U \subseteq V(H)$ with $|U| \geq (1-\delta)n$ such that F contains $H[U]$ as a minor.*

Proof. In the following, let $F_1, \ldots, F_K$ denote the copies of F such that $F^{(K)} = \bigcup_{i=1}^K F_i$.

Suppose $F^{(K)}$ has an H-minor and fix an H-minor model $(Z_h)_{h \in V(H)}$ in H. Let us denote $X_W := \{h \in V(H) \mid Z_h \cap W \neq \emptyset\}$ and $\xi_W := |X_W|$, and $X_i := \{h \in V(H) \mid Z_h \subseteq V(F_i) \setminus W\}$ and $\xi_i := |X_i|$ for every $i \in [K]$. Note that since every branch-set Z_h induces a connected subgraph of F, every vertex $h \in V(H)$ appears in exactly one of the sets $X_W, X_1, \ldots, X_K$, i.e., they form a partition of $V(H)$. In particular, we have $\xi_W + \sum_{i=1}^K \xi_i = \mathsf{v}(H) = n$.

We have $\xi_W \leq |W| \leq n - 3\delta n$ and thus $\sum_{i=1}^K \xi_i = n - \xi_W \geq 3\delta n$. In the following, let us w.l.o.g. assume $[K]$ is ordered such that $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_K$. We claim that $\xi_1 \geq (1-\delta)n - \xi_W$. Towards a contradiction, suppose in the following that $\xi_1 < (1-\delta)n - \xi_W$. We first note that using this assumption, we have that $\sum_{i=2}^K \xi_i = n - (\xi_W + \xi_1) > n - (1-\delta)n = \delta n$.

Now suppose for a first case that $\xi_1 \geq \delta n$. Then the two disjoint sets of vertices X_1 and $\bigcup_{i=2}^K X_i$ in H are both of size at least δn. By property $\mathsf{Q}(\delta, D)$ this implies that $\mathsf{e}_H(X_1, \bigcup_{i=2}^K X_i) \geq Dn \log n > 0$. In particular there exists $2 \leq i \leq K$ and an edge $uv \in E(H)$ for some $u \in X_1$ and $v \in X_i$. This implies that there must exist an edge in $F^{(K)}$ connecting a vertex in $Z_u \subseteq V(F_1) \setminus W$ to a vertex in $Z_v \subseteq V(F_i) \setminus W$. However, by construction of $F^{(K)}$ no such edges exist, and so we arrive at the desired contradiction in this first case.

For the second case, suppose that $\xi_1 < \delta n$ (and thus in particular $\xi_i < \delta n$ for all $i \in [K]$). Let $j \in [K]$ be the smallest index such that $\sum_{i=1}^j \xi_i > \delta n$ (this is well-defined, since $\sum_{i=1}^K \xi_i \geq 3\delta n$, see above). By the minimality of j, we have $\sum_{i=1}^j \xi_i = \xi_j + \sum_{i=1}^{j-1} \xi_j \leq \delta n + \delta n = 2\delta n$. This implies that $\sum_{i=j+1}^K \xi_i = \sum_{i=1}^K \xi_i - \sum_{i=1}^j \xi_i \geq 3\delta n - 2\delta n = \delta n$. In consequence, we find that the two disjoint vertex sets $\bigcup_{i=1}^j X_i, \bigcup_{i=j+1}^K X_i$ in H are both of size at least δn. Hence, using property $\mathsf{Q}(\delta, D)$ we have $\mathsf{e}_H(\bigcup_{i=1}^j X_i, \bigcup_{i=j+1}^K X_i) \geq Dn \log n > 0$. Similar as above, this implies the existence of two indices i, i' with $1 \leq i \leq j < i' \leq K$ such that there exists an edge between $V(F_i) \setminus W$ and $V(F_{i'}) \setminus W$ in $F^{(K)}$. As this is impossible by construction of $F^{(K)}$, a contradiction follows also in the second case. Thus our initial assumption $\xi_1 < (1-\delta)n - \xi_W$ was false.

We therefore have $|X_1 \cup X_W| = \xi_1 + \xi_W \geq (1-\delta)n$. Let $U := X_1 \cup X_W$. For every $h \in U$, let $Z_h' := Z_h$ if $h \in X_1$ and $Z_h' := Z_h \cap V(F_1)$ if $h \in X_W$. We now show that $(Z_h')_{h \in U}$ is an $H[U]$-minor model in F_1, which will then conclude the proof of the lemma.

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First of all, note that $F_1[Z_h{}']$ is a connected graph for every $h \in U$. If $h \in X_1$, then $F_1[Z_h{}'] = F^{(K)}[Z_h]$ is connected since $(Z_h)_{h \in V(H)}$ is an H-minor model. And if $h \in X_W$, then the connectivity of $F_1[Z_h{}'] = F^{(k)}[Z_h \cap V(F_1)]$ follows since (1) $F^{(K)}[Z_h]$ is connected and (2) every path connecting two vertices in $Z_h{}'$ that is contained in $F^{(K)}[Z_h]$ can be shortened to a path whose vertex set is completely contained in $V(F_1)$ by short-cutting every segment of the path that starts and ends in the clique W by the direct connection between its endpoints.

Let us now consider any edge $uv \in E(H[U])$. Then there must exist an edge $xy \in E(F^{(K)})$ with $x \in Z_u, y \in Z_v$. If we have $x, y \in V(F_1)$, then this witnesses the existence of an edge between $Z_u{}'$ and $Z_v{}'$ in F_1, as desired. If on the other hand at least one of x, y lies outside of $V(F_1)$, then we necessarily must have $Z_u \cap W \neq \emptyset \neq Z_v \cap W$, and thus there exists an edge in the clique induced by W (and thus also in F_1) that connects a vertex in $Z_u{}'$ to a vertex in $Z_v{}'$. All in all, this shows that F_1 contains $H[U]$ as a minor. Since $|U| \geq (1-\delta)n$, this concludes the proof. □

With the previous auxiliary results at hand, we can now deduce Theorem 7.

Proof of Theorem 7. Let a constant $\varepsilon \in (0, 1)$ be given. Let $\delta > 0$ be chosen small enough such that $7\delta < \varepsilon$, set $p := \frac{\delta}{2}$, let $D = D(\delta, p) > 1$ be the constant given by Lemma 11, and let $C := \frac{D^2}{\delta^2}$.

For every $n \in \mathbb{N}$, put $s = s(n) = \lceil Dn \log n \rceil$. By Lemma 17, a random graph $H = G(n; \lceil Cn \log n \rceil)$ chosen uniformly from all n-vertex graphs with $\lceil Cn \log n \rceil$ edges satisfies property $\mathsf{Q}(\delta, D)$ w.h.p. as $n \to \infty$. Now assume the graph H satisfies property $\mathsf{Q}(\delta, D)$. By Lemmas 9 and 11, w.h.p. as $n \to \infty$, the random bipartite graph $G = G(\lfloor (1-3\delta)n \rfloor, \lfloor (1-3\delta)n \rfloor; p)$ has maximum degree at most $2p\lfloor (1-3\delta)n \rfloor \leq \delta n$ and satisfies property $\mathsf{P}(H, \delta, s)$. Now fix n large enough and consider a graph G with bipartition $\{A, B\}$, $|A| = |B| = \lfloor (1-3\delta n) \rfloor$ satisfying these two properties. By Lemma 18, G^{C} does not contain any induced subgraph $H[U]$ as a minor for any $U \subseteq V(H)$ with $|U| \geq (1-\delta)n$. Let $K := (|A| + |B| - 1)^{|A|}$ and let$(G^{\mathsf{C}})^{(K)}$ be a K-fold pasting of G^{C} at A. Then by Lemma 19, $(G^{\mathsf{C}})^{(K)}$ does not contain H as a minor. Moreover, by Lemma 15, applied with $d = \lfloor \delta n \rfloor$, we find that $(G^{\mathsf{C}})^{(K)}$ has list chromatic number at least $|A| + |B| - d > 2(1-3\delta)n - \delta n - 2 > (2-\varepsilon)n$ for n large enough. This shows that w.h.p. the random graph $H = G(n; \lceil Cn \log n \rceil)$ satisfies $f_\ell(H) \geq (2-\varepsilon)n$, which concludes the proof. □

5. Proof of Theorem 3

In this section we give the proof of Theorem 3, which is self-contained and independent of the results in the previous sections. A basic tool from extremal graph theory used in the proof is Turán's theorem, in the following form:

Theorem 20 (Turán). *Let $k \in \mathbb{N}$, $k \geq 2$ and let G be a graph. If $\mathsf{e}(G) > (1 - \frac{1}{k-1})\frac{\mathsf{v}(G)^2}{2}$ then G contains a clique on k vertices.*

We also use the following classical result regarding the minimum degree of K_t-minor-free graphs, as independently proved by Kostochka [13] and Thomason [32].

Theorem 21 ([13, 32]). *For every integer $t \geq 1$ there exists an integer $d = d(t) = O(t\sqrt{\log t})$ such that every graph of minimum degree at least d contains K_t as a minor. In particular, for every graph F there exists $d = d(F) \in \mathbb{N}$ such that all graphs of minimum degree at least d contain F as a minor.*

Proof of Theorem 3. We start by fixing an integer $d \in \mathbb{N}$ as guaranteed by Theorem 21, i.e. such that every graph of minimum degree at least d contains F as a minor. We now define $k_0(F) := \min\{d+1, 9 \cdot \mathsf{v}(F)^3\}$. Let $k \geq k_0(F)$ be any given integer. Let H denote the graph obtained from F by adding k isolated vertices. We will now show that every H-minor-free graph is $(\mathsf{v}(H)-2)$-degenerate, which then easily implies $f_\ell(H) = \mathsf{v}(H) - 1$.

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Towards a contradiction, suppose that there exists an H-minor-free graph G which is not $(\mathsf{v}(H)-2)$-degenerate, and let G be chosen such that $\mathsf{v}(G)$ is minimised. Note that the minimality assumption on G immediately implies that $\delta(G) \geq \mathsf{v}(H)-1 = \mathsf{v}(F)+k-1$. Observe that since $\delta(G) \geq k > d$, the graph $G-x$ for some $x \in V(G)$ has minimum degree at least d and thus must contain F as a minor. Let $X \subseteq V(G)$ be chosen of minimum size subject to $G[X]$ containing F as a minor. Note that from the above it follows that $|X| \leq \mathsf{v}(G)-1$ and hence that $V(G) \setminus X \neq \emptyset$. Let $(Z_f)_{f \in V(F)}$ be an F-minor model in $G[X]$. By minimality of X, we have that $(Z_f)_{f \in V(F)}$ forms a partition of X. With the goal of bounding the number of edges in $G-X$, we present our next argument as a separate claim. We will later use this bound and Turán's theorem to show the existence of an F-subgraph in $G-X$.

Claim 22. *For every $v \in V(G) \setminus X$ and every $f \in V(F)$, we have $|N(v) \cap Z_f| < 9\mathsf{v}(F)$.*

Proof. Let $v \in V(G) \setminus X$ and $f \in V(F)$ be arbitrary. For $|Z_f| = 1$ the inequality $|N(v) \cap Z_f| \leq 1 < 9\mathsf{v}(F)$ trivially holds for every $v \in V(G) \setminus X$. We may therefore assume $|Z_f| \geq 2$. Let T_f denote a spanning tree of the connected graph $G[Z_f]$, and let $L_f \subseteq Z_f$ be the set of leaves in T_f.

We first show that T_f has at most $\mathsf{v}(F)-1$ leaves. Note that for every $l \in L_f$ the graph $G[Z_f \setminus \{l\}]$ is still connected. However, by minimality of X, $G[X \setminus \{l\}]$ does not contain F as a minor, and thus in particular the set system consisting of $Z_f \setminus \{l\}$ together with the remaining branch-sets $(Z_{f'})_{f' \in V(F), f' \neq f}$ cannot be an F-minor model in G. In consequence, there has to exist some $f' \in V(F) \setminus \{f\}$ such that among all vertices in Z_f, the vertex l is the only one that has a neighbour in $Z_{f'}$. Since the above argument applies to any choice of $l \in L_f$, and since the respective elements f' have to be distinct for different choices of l, it follows that $|L_f| \leq |V(F) \setminus \{f\}| = \mathsf{v}(F)-1$.

We next describe a decomposition of T_f into strictly less than $2\mathsf{v}(F)$ edge-disjoint and internal-vertex-disjoint paths. Let $T_f{}'$ be a tree without degree 2-vertices such that T_f is a subdivision of $T_f{}'$, i.e., every edge in $T_f{}'$ corresponds to one maximal path of T_f all whose internal vertices are of degree 2. Then, since $T_f{}'$ is a tree and thus has average degree strictly less than 2, it has more leaves than vertices of degree 3 or more. As the number of leaves in $T_f{}'$ is exactly $|L_f|$, we have $\mathsf{v}(T_f{}') \leq |L_f| + (|L_f|-1) \leq 2\mathsf{v}(F)-3$ and therefore $\mathsf{e}(T_f{}') = \mathsf{v}(T_f{}')-1 \leq 2\mathsf{v}(F)-4 < 2\mathsf{v}(F)$. This means that T_f can be expressed as the edge-disjoint union of a collection of paths $(P_i)_{i=1}^r$ where $r < 2\mathsf{v}(F)$ and the internal vertices of each path P_i are of degree 2 in T_f.

Now choose a vertex set $Y \subseteq Z_f$ of size at most $\mathsf{v}(F)-1$ as follows: For each edge $ff' \in E(F)$, pick some vertex $y_{f'} \in Z_f$ that has at least one neighbour in $Z_{f'}$ and add it to Y. Let $\mathcal{R}$ denote the collection of internally disjoint paths in T_f obtained from $(P_i)_{i=1}^r$ by splitting each path P_i into its maximal subpaths that do not contain internal vertices in Y. It is easy to see that $|\mathcal{R}| \leq r + |Y| < 2\mathsf{v}(F) + \mathsf{v}(F) = 3\mathsf{v}(F)$, and that T_f equals the union of the paths in $\mathcal{R}$.

We next claim that for every vertex $v \in V(G) \setminus X$ and every $R \in \mathcal{R}$, we have $|N(v) \cap V(R)| \leq 3$. Indeed, suppose that v has at least 4 distinct neighbours on R. Let x and y be the two neighbours of v on R that are closest to the endpoints of R. Define R' as the path obtained from R by replacing its subpath between x and y (which has to contain at least two internal vertices) by the path $x-v-y$ of length two. Let A be the set of vertices on R strictly between x and y and observe that $|A| \geq 2$ and $A \cap Y = \emptyset$. For $X' := (X \setminus A) \cup \{v\}$ we have $|X'| < |X|$ and we can find an F-minor model in $G[X']$, namely the branch-sets $(Z_f \setminus A) \cup \{x\}$ together with $(Z_{f'})_{f' \in V(F), f' \neq f}$. Notice that $G[(Z_f \setminus A) \cup \{x\}]$ is indeed connected as all the internal vertices of R are of degree 2 in T_f. Also, since $Y \subseteq Z_f \setminus A$, there still exists a connection from a vertex in $(Z_f \setminus A) \cup \{x\}$ (namely, $y_{f'}$) to a vertex in $Z_{f'}$ for every edge $ff' \in E(F)$. This contradicts our initial choice of X and proves that our assumption was wrong, so indeed every vertex $v \in V(G) \setminus X$ satisfies $|N(v) \cap V(R)| \leq 3$ for every $R \in \mathcal{R}$.

Therefore, we have $|N(v) \cap Z_f| \leq \sum_{R \in \mathcal{R}} |N(v) \cap V(R)| \leq 3|\mathcal{R}| < 9\mathsf{v}(F)$ for every $v \in V(G) \setminus X$, which concludes the proof of the claim. □

It follows immediately from Claim 22 that $|N(v) \cap X| \leq \sum_{f \in V(F)} |N(v) \cap Z_f| < 9\mathsf{v}(F)^2$ for every $v \in V(G) \setminus X$. Additionally recalling that $\delta(G) \geq \mathsf{v}(F) + k - 1 \geq k$, we find that for every $v \in V(G) \setminus X$, we have $\deg_{G-X}(v) = |N(v) \setminus X| = \deg(v) - |N(v) \cap X| > k - 9\mathsf{v}(F)^2$. Having established $V(G) \setminus X \neq \emptyset$ at the beginning of the proof, it now follows that $G - X$ is a graph of minimum degree greater than $k - 9\mathsf{v}(F)^2$. Also note that since $G[X]$ contains F as a minor, we are not able to find k distinct vertices in $V(G) \setminus X$ as these could be used to augment the F-minor in $G[X]$ to an H-minor in G, contradicting our assumptions. We thus have $\mathsf{v}(G - X) < k$. Using our choice of k_0 and $k \geq k_0$, it now follows that

$$\delta(G - X) > k - 9\mathsf{v}(F)^2 > \left(1 - \frac{1}{\mathsf{v}(F) - 1}\right) k > \left(1 - \frac{1}{\mathsf{v}(F) - 1}\right) \mathsf{v}(G - X).$$

Therefore, $G - X$ has more than $\left(1 - \frac{1}{\mathsf{v}(F)-1}\right)\frac{\mathsf{v}(G-X)^2}{2}$ edges and thus Theorem 20 implies the existence of a clique on $\mathsf{v}(F)$ vertices in $G - X$. In particular, $G - X$ and thus G contain a subgraph isomorphic to F. Let $K \subseteq V(G)$ be the vertex set of such a copy of F. Then, since $\mathsf{v}(G) \geq \delta(G) + 1 \geq \mathsf{v}(F) + k$, there are at least k vertices outside of K in G, which can be added to the copy of F on vertex set K to create a subgraph of G that is isomorphic to H. In particular, this means that G contains H as a minor, a contradiction. All in all, we find that our initial assumption, namely regarding the existence of a smallest counterexample G to our claim, was wrong. This concludes the proof that all H-minor-free graphs are $(\mathsf{v}(H) - 2)$-degenerate.

It is a well-known fact and easy to prove by induction that for every $a \in \mathbb{N}$ all a-degenerate graphs are $(a + 1)$-choosable. Thus what we have proved also implies that every H-minor-free graph is $(\mathsf{v}(H) - 1)$-choosable, as desired. All in all, it follows that $f_\ell(H) = \mathsf{v}(H) - 1$, concluding the proof of the theorem. □

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