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Characterizations of Extremals for some Functionals on Convex Bodies

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Abstract. We investigate equality cases in inequalities for Sylvester-type functionals. Namely, it was proven by Campi, Colesanti, and Gronchi that the quantity

$$\int_{x_0\in K}\cdots\int_{x_n\in K}[V(\operatorname{conv}\{x_0,\dots,x_n\})]^p\,dx_0\cdots dx_n,\ n\geq d,\ p\geq 1$$

is maximized by triangles among all planar convex bodies K (parallelograms in the symmetric case). We show that these are the only maximizers, a fact proven by Giannopoulos for $p=1$. Moreover, if $h\colon\mathbb{R}_+\to\mathbb{R}_+$ is a strictly increasing function and W_j is the j-th quermassintegral in $\mathbb{R}^d$, we prove that the functional

$$\int_{x_0\in K_0}\cdots\int_{x_n\in K_n}h(W_j(\operatorname{conv}\{x_0,\dots,x_n\}))\,dx_0\cdots dx_n,\ n\geq d$$

is minimized among the $(n+1)$-tuples of convex bodies of fixed volumes if and only if $K_0,\dots,K_n$ are homothetic ellipsoids when $j=0$ (extending a result of Groemer) and Euclidean balls with the same center when $j>0$ (extending a result of Hartzoulaki and Paouris).

1 Introduction

In this paper $V_d(\cdot)$ will denote the volume functional (*i.e.,* Lebesgue measure) in a d-dimensional vector space. If there is no possibility of confusion, we may omit the index and simply write $V(\cdot)$ instead of $V_d(\cdot)$.

Let K be a convex body in $\mathbb{R}^d$, $h\colon\mathbb{R}_+\to\mathbb{R}_+$ a strictly increasing function, and $n\geq d$ an integer. We define

$$S_h(K,n;d):=\int_{x_0\in K}\cdots\int_{x_n\in K}h[V(\operatorname{conv}\{x_0,\dots,x_n\})]\,dx_0\cdots dx_n\,, \tag{1.1}$$

where $\operatorname{conv}\{x_0,\dots,x_n\}$ denotes the convex hull of the points $x_0,\dots,x_n$. In the case when $h(t)=t^p$, $p\geq 1$, and K has volume 1, this definition coincides with the Sylvester functional.

A classical problem is to determine the convex bodies of prescribed volume for which $S_h(K,n;d)$ attains its extremal values.

Blaschke [1] proved that if $d=n=2$ and h is the identity, $S_h(K,n;d)$ is minimal if and only if K is an ellipse. Groemer [14, 15] proved for all n and d that ellipsoids are still the only minimizers, when h is in addition convex. Schöpf [19] extended

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Groemer's result for all strictly increasing h, provided that $n = d$, and Giannopoulos and Tsolomitis [13] proved that ellipsoids are minimizers in the general case.

A functional very similar to Sylvester's is the one defined by Busemann [4] (actually a natural generalization):

$$B_h(K, n; d) := \int_{x_1 \in K} \cdots \int_{x_n \in K} h[V(\operatorname{conv}\{0, x_1, \ldots, x_n\})]dx_1 \cdots dx_n. \tag{1.2}$$

Again when h is convex, $B_h(K, n; d)$ is minimal among all convex bodies of volume 1 if and only if K is an ellipsoid centered at the origin.

In fact, Busemann [4] established an inequality that gives even more information. If $K_1, \ldots, K_d$ are convex bodies in $\mathbb{R}^d$, set

$$B(K_1, \ldots, K_d) := \int_{x_1 \in K_1} \cdots \int_{x_d \in K_d} V(\operatorname{conv}\{0, x_1, \ldots, x_d\})dx_1 \cdots dx_d .$$

Then

$$B(K_1, \ldots, K_d) \geq B(B_1, \ldots, B_d), \tag{1.3}$$

where B_i are balls centered at 0 having volumes $V(B_i) = V(K_i)$, $i = 1, \ldots, d$. Here, equality holds if and only if K_i are homothetic origin symmetric ellipsoids.

Bourgain, Milman, Meyer, and Pajor [3] introduced another variation of $B_h(K, n; d)$. If $K_1, \ldots, K_n$ are convex bodies in $\mathbb{R}^d$, define

$$I_h(K_1, \ldots, K_n; d) := \int_{x_1 \in K_1} \cdots \int_{x_n \in K_n} h\left[V\left(\sum_{i=1}^{n}[0, x_i]\right)\right] dx_1 \cdots dx_n, \tag{1.4}$$

where $\sum_{i=1}^{n}[0, x_i]$ denotes the Minkowski sum of the line segments $[0, x_i]$, $i = 1, \ldots, n$. Note that $V(\sum_{i=1}^{d}[0, x_i]) = d! V(\operatorname{conv}\{0, x_1, \ldots, x_d\})$.

An inequality similar to (1.3) holds, extending Busemann's result:

$$I_h(K_1, \ldots, K_n; d) \geq I_h(B_1, \ldots, B_n; d), \tag{1.5}$$

where B_i is as in (1.3).

Motivated by (1.3) and (1.4) we can define the multi-entry versions of (1.1) and (1.2):

$$S_h(K_0, \ldots, K_n; d) := \int_{x_0 \in K_0} \cdots \int_{x_n \in K_n} h[V(\operatorname{conv}\{x_0, \ldots, x_n\})]dx_0 \cdots dx_n \tag{1.6}$$

$$B_h(K_1, \ldots, K_n; d) := \int_{x_1 \in K_1} \cdots \int_{x_n \in K_n} h[V(\operatorname{conv}\{0, x_1, \ldots, x_n\})]dx_0 \cdots dx_n. \tag{1.7}$$

Following the argument in [3], it is not difficult to obtain inequalities analogous to (1.5). Our purpose is to investigate cases of equality in inequalities of this type.

We prove the following.

Theorem 1.1 *If $K_0, \dots, K_n$ are convex bodies in $\mathbb{R}^d$, then*

$$D_h(K_0, \dots, K_n; d) \geq D_h(B_0, \dots, B_n; d), \tag{1.8}$$

where $D = S$, B, or I and B_i are balls of volume $V(B_i) = V(K_i)$ centered at 0, $i = 0, \dots, n$. Moreover, when $D = S$ (resp. $D = B$ or I), equality holds in (1.8) if and only if $K_0, \dots, K_n$ are ellipsoids with the same center (resp. centered at the origin), homothetic to each other. Here we set by convention $K_0 = \{0\}$, in the case when $D = B$ or I.

It is easily verified that the functional defined by (1.6) (resp. (1.4) and (1.7)) is invariant under transformations of the form $(\Phi, \dots, \Phi)$, where $\Phi\colon \mathbb{R}^d \to \mathbb{R}^d$ is a volume-preserving affine (resp. linear) map. Thus, once (1.8) is proven, the fact that equality in (1.8) holds for homothetic ellipsoids of the same center is immediate.

The quermassintegrals of K, $W_j(K)$, $j = 0, 1, \dots, d-1$, are defined by the Steiner formula:

$$V(K + tB_1) = \sum_{j=1}^{d} t^j \binom{d}{j} W_j(K), t > 0,$$

where B_1 is the unit ball in $\mathbb{R}^d$. We refer to [11, Chapters 4–5] or [18, Appendix] for basic results on quermassintegrals and related concepts.

A functional generalizing $S_h(K, n; d)$ was introduced in [16] by Hartzoulaki and Paouris by substituting the volume of the random polytope with its j-th quermassintegral, $j = 0, 1, \dots, n-1$. In the same spirit as (1.4), (1.6), and (1.7) we define

$$S_h(K_0, \dots, K_n; d; j) := \int_{x_0 \in K_0} \cdots \int_{x_n \in K_n} h(W_j(\operatorname{conv}\{x_0, \dots, x_n\})) dx_0 \cdots dx_n,$$

$$B_h(K_1, \dots, K_n; d; j) := \int_{x_1 \in K_1} \cdots \int_{x_n \in K_n} h(W_j(\operatorname{conv}\{0, x_1, \dots, x_n\})) dx_1 \cdots dx_n \ ,$$

$$I_h(K_1, \dots, K_n; d; j) := \int_{x_1 \in K_1} \cdots \int_{x_n \in K_n} h\Big(W_j\Big(\sum_{i=1}^{n} [0, x_i] \Big) \Big) dx_1 \cdots dx_n \ .$$

It was proven that balls are minimizers of $S_h(K, \dots, K; d; j)$, while if h is convex, these are the only minimizers. Note that $W_0(\cdot) = V(\cdot)$, hence $D_h(K_0, \dots, K_n; d; 0)$ coincides with $D_h(K_0, \dots, K_n; d)$, $D = S, B$ or I.

We prove the following analogue of Theorem 1.1.

Theorem 1.2 *The following inequality holds:*

$$D_h(K_0, \dots, K_n; d; j) \geq D_h(B_0, \dots, B_n; d; j), j = 1, \dots, d-1, \tag{1.9}$$

where $D = S$, B, or I and B_i are balls of volume $V(B_i) = V(K_i)$ centered at the same point when $D = S$ (resp. centered at 0 when $D = B$ or I). If for some j equality holds in (1.9), then $K_i = B_i$, $i = 0, \dots, n$.

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The similarity of the functionals S, B, and I allows us to treat all three cases simultaneously. Inequalities for Theorems 1.1 and 1.2 will be proven in Section 4. The proof of the uniqueness results will be given in Sections 5 and 6 respectively.

It can be easily checked that the functionals defined by (1.4), (1.6) and (1.7) do not attain a maximal value. However, the "original" version $S_h(K, n; d)$ is invariant under affine volume preserving maps, and $B_h(K, n; d)$, $I_h(K, n; d) := I_h(K, \ldots, K; d)$ are invariant under volume preserving linear transformations. Therefore, and since for every sequence of convex bodies there exist affine images of the same volume, contained in a ball, a compactness argument ensures the existence of maximizers in the class of all convex bodies for $S_h(K, n; d)$ and in the class of all convex bodies containing the origin for $B_h(K, n; d)$ and $I_h(K, n; d)$.

For each one of these functionals, the problem of determining their maximal value remains open when $d \geq 3$, but it is solved in the plane. Namely, Dalla and Larman [9] proved that triangles are maximizers of $S_h(K, n; 2)$, when h is the identity. Giannopoulos [12] showed that these are the only maximizers. Campi, Colesanti, and Gronchi proved in [5, 6] that when $h(t) = t^p$, $p \geq 1$, $S_h(K, n; 2)$ is maximized by triangles (parallelograms in the centrally symmetric case) and $I_h(K, n; 2)$ is maximized by triangles with one vertex at the origin (resp. origin centered parallelograms). Moreover, these are the only polygons having these properties. The same approach can be used to treat various types of optimization problems (see *e.g.*, [7] or [8]).

The key to the proof is the strict convexity of the functionals mentioned above under a family of transformations of convex bodies, the so-called parallel chord movements. In Section 3 we give a characterization (Theorem 3.1) of triangles and parallelograms as maximizers with respect to these types of functionals.

2 Preliminaries

Given a bounded family of points $A = \{x_i : i \in J\}$, a shadow system along a direction $\nu \in S^{d-1}$ is a family of convex sets of the form

$$K_t = \operatorname{conv}\{x_i + \alpha_i t\nu : i \in J\}, t \in [t_0, t_1],$$

where J is any set of indices and the set $\{\alpha_i : i \in J\}$ is a bounded subset of $\mathbb{R}$. The real number α_i is called speed of the point x_i with respect to the shadow system K_t. Clearly, a shadow system is a continuous transformation with respect to the parameter t. It is also obvious that the projection of K_t on the hyperplane $\nu^\perp = \{x \in \mathbb{R}^d : \langle x, \nu \rangle = 0\}$ does not change with t.

Shadow systems were introduced by Rogers and Shephard in [17] where a fundamental result was proved. The volume of the shadow system K_t is a convex function of the parameter. The proof is based on the fact that the length of the intersection of a line parallel to ν with K_t is also a convex function of t. Later, Shephard observed in [20] that the same convexity property also holds for other functionals such as the diameter, the mean width, or the maximal brightness of a convex body.

It is straightforward from the definition that the projection of a shadow system along a direction ν onto an affine subspace H is still a shadow system in the direction

$\nu \,|\, H$, where $\cdot\,|\,\cdot$ denotes the orthogonal projection of a vector or a set onto an affine subspace of $\mathbb{R}^d$. In addition, it can be shown (see [6]) that Minkowski sums of shadow systems along the same direction are also shadow systems.

Let K be a convex body, and $\alpha\colon K \to \mathbb{R}$ be any function with the property of being constant on each chord of K, parallel to the direction $\nu \in S^{d-1}$. If there exists an interval $[t_0, t_1]$ such that the set $K_t = \{x + \alpha(x)t\nu : \; x \in K\}$ is a convex body, for all $t \in [t_0, t_1]$, we say that the family $\{K_t\}_{t\in[t_0,t_1]}$ is a parallel chord movement. The function $\alpha\colon K \to \mathbb{R}$ is called the speed function of the parallel chord movement.

Clearly, parallel chord movements are special cases of shadow systems. Usual examples of such movements are translations and Steiner symmetrization. The notion of parallel chord movements first appeared in [5]. An important fact about a parallel chord movement is that its volume is constant with respect to the parameter t.

The proofs of the maximizing properties of triangles and parallelograms are based on the following two facts (see *e.g.,* [5]):

(i) If K_t is a parallel chord movement, then $S_h(K_t, n; 2)$ (resp. $I_h(K_t, n; 2)$) is a convex function of t and cannot be constant unless its speed function is affine (resp. linear).

(ii) If P is a convex polygon that is not a triangle, there exists a parallel chord movement that can reduce P to a polygon with less vertices. A similar result holds in the centrally symmetric case. We briefly describe this procedure.

Let $v_1, \; v_2$, and v_3 be three consecutive vertices of the polygon P. We define the function $\alpha\colon P \to \mathbb{R}$ with the properties: $\alpha(v_2) = 1$, $\alpha(v_1) = 0 = \alpha(v_3)$, α is linear in the triangle spanned by $v_1, \; v_2, \; v_3$, and $\alpha = 0$ elsewhere. Let ν be the direction parallel to $v_1 - v_3$. We set $P_t = \{x + \alpha(x)t\nu : x \in P\}$. Suppose that $[t_0, t_1]$ is the largest interval such that P_t is convex for all t in $[t_0, t_1]$. It can be easily checked that $t_0 < 0 < t_1$ and the family $\{P_t\}_{t\in[t_0,\, t_1]}$ is a parallel chord movement. Clearly, P_{t_0}, P_{t_1} are polygons with less vertices than P. Now, by 1 and since $P_0 = P$ we see that

$$S_h(P, n; 2) \leq max\{S_h(P_{t_0}, n, 2), S_h(P_{t_1}, n, 2)\}.$$

The fact that triangles are maximizers follows immediately from the last inequality.

3 Uniqueness of Maximizers

We prove the following theorem, which ensures the uniqueness of maximizers of $I_h, \; S_h \; (h(t) = t^p, \; p \geq 1)$ mentioned in the previous section.

Theorem 3.1 *Let D be a continuous, invariant under non-singular affine (resp. linear) maps functional from the class of convex bodies of $\mathbb{R}^2$ into $\mathbb{R}_+$ having the following properties:*

(i) *If K_t is a parallel chord movement, $t_0 < 0 < t_1$, then $D(K_t)$ is a convex function of t.*

(ii) *If the speed function α of the movement is not affine (resp. linear), then $D(K_t)$ is not constant.*

Then, triangles (resp. with a vertex at the origin) are the only maximizers of D in the class of all convex bodies (resp. containing 0) in $\mathbb{R}^2$ and parallelograms (resp. origin *symmetric parallelograms) are the only maximizers in the class of all centrally symmetric convex bodies.*

Proof Let K be a planar body, which contains 0, maximizes D, and is not a triangle. We write

$$K = \{x + y\nu : x \in K|\nu^{\perp}, f_{\nu}(x) \leq y \leq g_{\nu}(x)\},$$

where $\nu \in S^1$ and $f_{\nu}, -g_{\nu} \colon K|\nu^{\perp} \to \mathbb{R}$ are convex functions. It suffices to find a direction $\nu \in S^1$ and some function $\alpha \colon K|\nu^{\perp} \to \mathbb{R}$ that is not linear such that $f_{\nu} + t\alpha$ is convex and $g_{\nu} + t\alpha$ is concave in $[t_0, t_1]$.

Indeed, if we have found such an α, for $t \in [t_0, t_1]$ we set:

$$\begin{aligned} K_t &= \{x + \alpha(x|\nu^{\perp})t\nu : x \in K\} \\ &= \{x + y\nu : \ x \in K|\nu^{\perp}, \ (f_{\nu} + t\alpha)(x) \leq y \leq (g_{\nu} + t\alpha)(x)\}. \end{aligned}$$

The function $\alpha(x|\nu^{\perp})$ is constant on each chord parallel to ν, while K_t is convex for all t in $[t_0, t_1]$. Therefore, K_t is a parallel chord movement and by assumptions (i) and (ii) we conclude that

$$D(K) = D(K_0) < max\{D(K_{t_0}), D(K_{t_1})\},$$

so K cannot be a maximizer.

Special case: There exist at least two non-regular points (*i.e.,* the supporting lines on these points are not unique) of the boundary of K, which do not lie in the same line segment of ∂K.

We may assume that the chord through these two points is parallel to the x_2−axis. We write $[b, c] = K|\nu^{\perp}$, $f = f_{\nu}$, $g = g_{\nu}$, where $\nu = e_2 = (0, 1)$.

By assumption, there exists $x_0 \in (b, c)$ such that f, g are not differentiable at x_0. We define $\alpha(x) = \gamma(x)$ in $[b, x_0]$ and $\alpha(x) = \delta(x)$ in $[x_0, c]$, where $\gamma(x_0) = \delta(x_0)$ and γ, δ are affine functions satisfying

$$|\gamma' - \delta'| \leq \min\{f'_+(x_0) - f'_-(x_0), g'_-(x_0) - g'_+(x_0)\}.$$

Then, α is as wanted.

To avoid 0 being outside K_t (in the case when D is not translation invariant), we can take α so that $\alpha(0) = 0$, by replacing α with $\alpha - \alpha(0)$ if necessary.

General case: Since K has at least four extreme points, we can choose regular points x, y from the boundary of K such that if G_1, G_2 are the open half-planes defined by the chord $[x, y]$, the following are satisfied:

(i) $[x, y]$ is not contained in the boundary of K.

(ii) The tangent lines e_x, e_y at x, y respectively are not parallel and the intersection point p of these lines lies in G_1, while $0 \notin G_1$.

Then, there exists a sequence $\{K_n\}$ of convex bodies such that $K_n \uparrow K$, $K_n \cap \overline{G_2} = K \cap \overline{G_2}$ and $K_n \cap \overline{G_1} = P_n$, where P_n are polygons that have $[x, y]$ as one of their edges.

We apply the parallel chord movement described in Section 2 on vertices of P_n that do not lie in e_x or e_y. If there are no such vertices, we simply do nothing. We move vertices of P so that D does not decrease and stop the movement either when a vertex of P_n "vanishes" or when the moving vertex "reaches" one of the lines $e_x,\ e_y$. Repeating the same process as many times as needed, we find for each n a convex body K_n' such that: $V(K_n') = V(K_n)$, $D(K_n') \geq D(K_n)$ and K_n' has exactly one or two extreme points inside G_1.

Since $K_n' \subset \operatorname{conv}(K \cup \{p\})$, there exists a subsequence K_{n_m}' that converges to a convex body K'. Clearly, $D(K_{n_m}') \geq D(K_{n_m}) \to D(K)$, which gives $D(K') = D(K)$.

Thus, we have found a convex body K' with $D(K') = D(K)$ the boundary of which contains at least one non-regular point that lies inside G_1.

If 0 is the unique extreme point of K' in G_2, we fall in the special case discussed before. If not, we can find a chord $[z, w]$ contained in G_2 so that the two open half-planes G_1' and G_2' defined by $[z, w]$ have analogous properties to G_1 and G_2 but also $x,\ y$ are contained in G_2'. Thus, the sets $\overline{G_1} \cap K$, $\overline{G_1'} \cap K$ are disjoint.

Working as above we get a convex body K'' with $K'' \cap G_1 = K' \cap G_1$ and $D(K'') = D(K)$, so that there exists at least one non-regular point q' of the boundary of K'' contained in G_1'. Notice that $q,\ q'$ belong to the convex angle defined by $e_x,\ e_y$ and containing $x,\ y$, but q' does not lie in any of these two lines. Since $e_x,\ e_y$ are still supporting lines of K'', q and q' cannot be contained in a boundary segment. Consequently, the boundary of K'' has at least two non-regular points not contained in the same line segment of $\partial K''$, while $D(K'') = D(K) = maxD$. This is impossible, so K must be a triangle. In particular, if D is not translation invariant and since translations are parallel chord movements with non-linear speed function, one of the vertices of K must be the origin. The proof for the centrally symmetric case is similar. ■

4 Proof of Lower Bound Inequalities

It is well known that any convex body is reduced to a ball of the same volume after applying the process of Steiner symmetrization along an appropriate sequence of directions S^{d-1}. Clearly the functionals D_h are continuous with respect to the Haussdorff metric, thus we only have to prove that $D_h(K_0, \ldots, K_n; d)$ does not increase under Steiner symmetrization of $K_0, \ldots, K_n$ simultaneously along the same direction ν in S^{d-1}.

For $X = (x_0, \ldots, x_n) \in (\mathbb{R}^{d-1})^{n+1}$ and $t = (t_0, \ldots, t_n) \in \mathbb{R}^{n+1}$ we set

$$\begin{aligned}\Phi_{S,X,j}(t) &:= W_j(\operatorname{conv}\{(x_0, t_0), \ldots, (x_n, t_n)\}) \\ &= \frac{\omega_d}{\omega_{d-j}} \int_{\mathcal{G}_{d,\ d-j}} V_{d-j}(\operatorname{conv}\{(x_0, t_0), \ldots, (x_n, t_n)\}|E) d\mu(E), \\ \Phi_{B,X,j}(t) &:= W_j(\operatorname{conv}\{0, (x_1, t_1), \ldots, (x_n, t_n)\}) \\ &= \frac{\omega_d}{\omega_{d-j}} \int_{\mathcal{G}_{d,\ d-j}} V_{d-j}(\operatorname{conv}\{0, (x_1, t_1), \ldots, (x_n, t_n)\}|E)\, d\mu(E),\end{aligned}$$

$$\Phi_{I,X,j}(t) := W_j\Big(\sum_{i=1}^{n}[0,(x_i,t_i)]\Big) = \frac{\omega_d}{\omega_{d-j}}\int_{\mathcal{G}_{d,\,d-j}} V_{d-j}\Big(\sum_{i=1}^{n}[0,(x_i,t_i)]|E\Big)\,d\mu(E).$$

The last part of each of the above equalities is Kubota's formula (see [18]), ω_d is the volume of the d-dimensional unit ball and μ is the Haar probability measure defined on the Grassmanian $\mathcal{G}_{d,\,d-j}$ of $(d-j)$-dimensional subspaces of $\mathbb{R}^d$. Note that $\mathcal{G}_{d,\,d} = \{\mathbb{R}^d\}$.

The key property of these functions is convexity. To see this, observe that the restrictions of the integrated functions (as functions of $t = (t_0,\dots,t_n)$) on any line segment are exactly the volumes of shadow systems in the direction of the projection of x_d-axis onto E. This follows immediately by the definition and properties mentioned in Section 2. For instance, if $J = [t, t']$, the shadow system used is

$$\operatorname{conv}\{(x_0, t_0' + s(t_0 - t_0')), \dots, (x_n, t_n' + s(t_n - t_n'))\}|E, s \in [-1, 1].$$

If ν is any direction in S^{d-1}, K_i can be written:

$$K_i = \{x + \theta\nu : x \in K_i|\nu^\perp, f_{i,\nu}(x) \le \theta \le g_{i,\nu}(x)\},\ i = 0,\dots,n,$$

where $f_{i,\nu}, -g_{i,\nu}$ are convex functions on $K_i|\nu^\perp$. The Steiner symmetral of K_i is defined by

$$S_\nu(K_i) = \{x + \theta\nu : x \in K_i|\nu^\perp, -k_{i,\nu}(x) \le \theta \le k_{i,\nu}(x)\},$$

where $k_{i,\nu} = \frac{g_{i,\nu}-f_{i,\nu}}{2}$, $i = 0,\dots,n$. Set $u_{i,\nu} = \frac{g_{i,\nu}+f_{i,\nu}}{2}$. Then, for $i = 0,\dots,n$ we have:

$$K_i = \{x + \theta\nu : x \in K_i|\nu^\perp, -k_{i,\nu}(x) + u_{i,\nu}(x) \le \theta \le k_{i,\nu}(x) + u_{i,\nu}(x)\}.$$

We may assume that $\nu = e_d = (0,\dots,0,1)$. Using Fubini's theorem we find

$$D_h(K_0,\dots,K_n; d; j) = \int_{x_0\in K_0|\nu^\perp}\cdots\int_{x_n\in K_n|\nu^\perp}\Big[\int_{-k_0+u_0}^{k_0+u_0}\cdots\int_{-k_n+u_n}^{k_n+u_n} h(\Phi_{D,X,j}(t))dt\Big]\,dX,$$

where $k_i = k_{i,\,\nu}(x_i)$, $u_i = u_{i,\,\nu}(x_i)$, $i = 0,\dots,n$, and $X = (x_0,\dots,x_n)$.We also set for simplicity

$$T = T_\nu(X) = [-k_{0,\,\nu}(x_0), k_{0,\,\nu}(x_0)] \times \cdots \times [-k_{n,\,\nu}(x_n), k_{n,\,\nu}(x_n)]$$

and

$$u = (u_0,\dots,u_n) = u_\nu(X) = (u_{0,\,\nu}(x_0),\dots,u_{n,\,\nu}(x_n))\,.$$

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Now,

$$D_h(K_0,\dots,K_n;d;j) = \int_{x_0\in K_0|\nu^\perp}\cdots\int_{x_n\in K_n|\nu^\perp}\int_0^\infty V(\,[T+u]\cap\{h(\Phi_{D,X,j})\geq s\})\,ds\,dX.$$

Working similarly for the Steiner symmetrals of $K_0,\dots,K_n$ we get

$$D_h(S_\nu(K_0),\dots,S_\nu(K_n);d;j) = \int_{x_0\in K_0|\nu^\perp}\cdots\int_{x_n\in K_n|\nu^\perp}\int_0^\infty V(\,T\cap\{h(\Phi_{D,X,j})\geq s\})\,ds\,dX.$$

Clearly,

$$V(\,(T+u)\cap\{h(\Phi_{D,X,j})\geq s\}) = V(T) - V((T+u)\cap\{\Phi_{D,X,j} < h^{-1}(s)\}),$$

so it suffices to prove that

$$V((T+y)\cap\{\Phi_{D,X,j}<\zeta\}) \leq V(T\cap\{\Phi_{D,X,j}<\zeta\}) \tag{4.1}$$

for all $y\in\mathbb{R}^m$ and all $\zeta>0$, where $m=n+1$ if $D=S$ and $m=n$ if $D=B$ or I. Define the function

$$\eta(y) = V\big((T+y)\cap\{\Phi_{D,X,j}<\zeta\}\big),\, y\in\mathbb{R}^m.$$

Note that T, $\{\Phi_{D,X,j}<\zeta\}$ are convex and centrally symmetric ($\{\Phi_{D,X,j}<\zeta\}$ is convex because of the convexity of $\Phi_{D,X,j}$). Thus η is an even function and also, by the Brunn–Minkowski theorem, it is log-concave. Consequently, η attains its maximum at 0 and (1.8) and (1.9) follow.

5 Characterizations of Products of Homothetic Ellipsoids

First we need some geometric lemmas. We preserve the notation of the previous section.

Lemma 5.1 *If $\nu=e_d$ and*

$$D_h(S_\nu(K_0),\dots,S_\nu(K_n);d;j) = D_h(K_0,\dots,K_n;d;j)\,, \tag{5.1}$$

then for any choice of $X\in K_0|\nu^\perp\times\cdots\times K_n|\nu^\perp$ there exists a vertex L of the parallelepiped $T_\nu(X)$ such that $\Phi_{D,X,j}(L+su_\nu(X))$ is constant in $[-1,1]$.

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Proof First note that it suffices to prove our claim for $(n+1)$-tuples $X \in R :=$ $\operatorname{int}(K_0|\nu^\perp) \cap \cdots \cap \operatorname{int}(K_n|\nu^\perp)$. The general case follows by the fact that $u_{i,\nu}$ is continuous up to the boundary on any chord of $K_i|\nu^\perp$ (since $f_{i,\nu}, g_{i,\nu}$ are) using an approximation argument.

Now, since $u_{i,\,\nu}$ is continuous in R, (5.1) together with standard (but somewhat tedious) compactness arguments imply that equality must hold in (4.1) for all choices of $X \in R$ and $\zeta > 0$. For any $X \in R$ we have

$$V\big((T+u) \cap \{\Phi_{D,X,j} < \zeta\}\big) = V\big(T \cap \{\Phi_{D,X,j} < \zeta\}\big) = V\big((T-u) \cap \{\Phi_{D,X,j} < \zeta\}\big), \tag{5.2}$$

where $T = T_\nu(X),\ u = u_\nu(X)$ (the right hand equality follows from the fact that the function η defined above is even). Note that $\{\Phi_{D,X,j} < \zeta\}$ contains T when ζ is large. By the choice of X it follows that $V(T) > 0$, so there exists a $\zeta_0 > 0$ such that

$$\zeta_0 = \inf\big\{\, \zeta > 0 : T \subseteq \{\Phi_{D,X,j} < \zeta\}\big\}.$$

Clearly, the boundary of $\{\Phi_{D,X,j} < \zeta_0\}$ touches the boundary of T at some vertex L of T. However, (5.2) holds, which cannot happen unless $L+u,\ L-u$ belong to $\{\Phi_{D,X,j} \le \zeta_0\}$, and since L is contained in $\partial\{\Phi_{D,X,j} \le \zeta_0\}$, we conclude that the segment $[L-u, L+u]$ lies on the boundary of $\{\Phi_{D,X,j} \le \zeta_0\}$. Consequently, $\Phi_{D,X,j}(L+su) = \zeta_0$ for all s in $[-1,1]$. ∎

Lemma 5.2 (i) *Assume that the convex bodies $K_0, \ldots, K_n$ do not all have the same centroid, $n \ge 1$. Then there exist directions $\nu_0, \ldots, \nu_n$ in S^{d-1} such that the convex bodies $S_{\nu_n} \circ \cdots \circ S_{\nu_0}(K_i)$, $i = 0, \ldots, n$ contain the origin in their interior and at the same time do not all have the same centroid.*

(ii) *If $K_1, \ldots, K_n$ do not all have their centroid at 0, there exist directions $\nu_0, \ldots, \nu_n$ such that 0 is not the centroid of all $S_{\nu_n} \circ \cdots \circ S_{\nu_0}(K_i)$, $i = 0, \ldots, n$, but they all contain the origin in their interior.*

Proof (i) An inductive argument reduces to the case $n = 1$. If a_i is the centroid of K_i, $i = 0,\ 1$, we can choose a direction ν with the property that there exists a point $x_0 \in int(K_0)$ such that the segment $[0, x_0]$ is parallel to ν but ν is not parallel to $[a_0, a_1]$. Then $a_0|\nu^\perp \ne a_1|\nu^\perp$.

Clearly, the centroid of $S_\nu(K_i)$ is $a_i|\nu^\perp$, $i = 0,\ 1$ and $0 \in int(S_\nu(K_0))$. Note that every Steiner symmetral of $S_\nu(K_0)$ contains the origin in its interior. So by the above discussion we can find a direction $u \in S^{d-1}$ so that the centroids of $S_u \circ S_\nu(K_0),\ S_u \circ S_\nu(K_1)$ are different but they contain the origin in their interior.

(ii) We just take K_0 to be a centrally symmetric body and apply (i) for the convex bodies $K_0, \ldots, K_n$. ∎

The previous lemma allows us to assume that $0 \in int(K_i)$, $i = 0, \ldots, n$. Suppose for instance that $D = S$, the intersection of the interiors of all K_i is empty, and equality holds in (1.8). Since S_h does not increase under Steiner symmetrization, by Lemma 5.2 we can find convex bodies that contain 0 in their interior, but not all have the same centroid. In particular, these bodies are not homothetic ellipsoids with the same center.

Lemma 5.3 *For almost every direction $\nu \in S^{d-1}$ we have*

$$k_{0,\nu}(x) = \cdots = k_{n,\nu}(x) = 0, \text{ for all } x \in \partial(K_i|\nu^\perp),\ i = 0,\ldots,n. \tag{5.3}$$

Proof Let y be a point in K_i and ν a direction in S^{d-1}. If $k_{i,\,\nu}(y|\nu^\perp) \neq 0$, clearly y is contained in a line segment in the boundary of K_i, parallel to ν. However, it follows by a classical result of Ewald, Larman, and Rogers [10] that the set of all such directions is of measure zero, completing the proof. ■

Proof of uniqueness in Theorem 1.1 Suppose that equality holds in (1.8). Then, (5.1) is valid for every $\nu \in S^{d-1}$. We choose a direction ν such that (5.3) holds. We may assume that $\nu = e_d$. Choose arbitrary $x_i \in K_i|\nu^\perp$, $i = 1,\ldots,d-1$, so that the points $0, x_1,\ldots,x_{d-1}$ are affinely independent.

As mentioned, we may assume that 0 is contained in the interior of $K_i|\nu^\perp$, $i = 0,\ldots,n$. So, any line of $\nu^\perp$ through the origin crosses the boundary of $K_i|\nu^\perp$ for all $i = 0,\ldots,n$. Thus, we can choose points $x_i \in \partial(K_i|\nu^\perp)$, $i = 0, d, d+1,\ldots,n$ that lie in the same line through 0. Moreover, the choice can be made so that x_0 is an endpoint of the line segment spanned by $x_0, x_d,\ldots,x_n$.

Let υ be the other endpoint of the line segment $\operatorname{conv}\{x_i : i = 0, d,\ldots,n\}$ when $D = S,\ B$ or the line segment $\sum_{i=d}^{n}[0, x_i]$ when $D = I$. Lemma 5.1 ensures that there exists a vertex $L = (l_0,\ldots,l_n)$ of $T = T_\nu(X)$ such that $\Phi_{D,X,0}(L + su_\nu(X))$ is constant in $[-1,1]$ (where $X = (x_0,\ldots,x_n)$). Note that $l_i = \pm k_i$, so $l_i = 0$ for $i = 0, d,\ldots,n$.

Set also β for the speed of the point $(\upsilon, 0)$ with respect to the shadow system

$$\operatorname{conv}\{(x_i, l_i + su_i) : i = 0,\ldots,n\} \quad (\text{when } D = S, B) \quad \text{or}$$

$$\sum_{i=1}^{n}\bigl[0,(x_i, l_i + su_i)\bigr] \quad (\text{when } D = I).$$

Now, for $D = S$ or B we have

$$V\Bigl(\operatorname{conv}\bigl(\{(x_i, l_i + su_i) : i = 0,1,\ldots,d-1\} \cup \{(\upsilon, s\beta)\}\bigr)\Bigr) \le V\Bigl(\operatorname{conv}\{(x_i, l_i + su_i) : i = 0,\ldots,n\}\Bigr) \equiv \text{const},$$

where equality holds for $s = 0$. The left hand side is the volume of a shadow system and therefore is a convex function of s. Thus, equality must hold everywhere in $[-1,1]$. A similar result holds in the case $D = I$.

So, we have shown that

$$V\Bigl(\operatorname{conv}\bigl(\{(x_i, l_i + su_i) : i = 0,1,\ldots,d-1\} \cup \{(\upsilon, s\beta)\}\bigr)\Bigr) \equiv \text{const}, \tag{5.4}$$

$$V\Bigl(\sum_{i=0}^{d-1}\bigl[0,(x_i, l_i + su_i)\bigr] + \bigl[0,(\upsilon, s\beta)\bigr]\Bigr) \equiv \text{const}. \tag{5.5}$$

Observe that $x_0 = 0$ and $u_0 = l_0 = 0$, when $D = B$ or I. Consequently, (5.4) and (5.5) give:

$$\Bigl|\det\bigl[(x_0, l_0 + su_0, 1),\ldots,(x_{d-1}, l_{d-1} + su_{d-1}, 1),(\upsilon, s\beta, 1)\bigr]\Bigr| \equiv \text{const},$$

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hence

$$\det\big[(x_0,u_0,1),\dots,(x_{d-1},u_{d-1},1),(\upsilon,\beta,1)\big]=0.$$

It follows that the points $(x_i,u_i),\ i=0,\dots,d-1,\ (\upsilon,\beta)$ lie in a common hyperplane. Since x_1,x_2 were chosen arbitrarily, the graph of the functions $u_{1,\,\nu}$ and $u_{2,\,\nu}$ is contained in this hyperplane, which shows that $u_{1,\,\nu}$ and $u_{2,\,\nu}$ are restrictions of the same affine function. Applying the same argument to all pairs of indices, it follows that $u_{0,\,\nu},\dots,u_{n,\,\nu}$ are all restrictions of the same affine function. By Lemma 5.3 this property holds for almost every direction in S^{d-1}. Consequently, for almost every direction ν the midpoints of all chords of $K_0,\dots,K_n$, parallel to ν, are contained in a common hyperplane H_ν. This is actually true for every direction in S^{d-1}. Thus, (see [2]) $K_0,\dots,K_n$ are ellipsoids. Moreover, it easy to check that a linear map that transforms K_1 into an origin centered ball also transforms the rest of the K_i's into origin centered balls. Thus, all K_i are homothetic with the same center. In particular, if $D=B$ or I, $K_0=\{0\}$, so the center of K_i is the origin. ■

6 Characterizations of Products of Euclidean Balls

Lemma 6.1 *Let $Q\subseteq\mathbb{R}^d$ be a polytope with vertices $\upsilon_1,\dots,\upsilon_n$. Suppose that the vertices $\upsilon_1,\dots,\upsilon_k$, $k\geq d$ span a supporting hyperplane H of Q and x is a point of the polytope spanned by υ_i, $i=1,\dots,k$. Define the shadow system*

$$Q_s=\operatorname{conv}\big(\{\upsilon_1+\beta_1 s\nu,\dots,\upsilon_n+\beta_n s\nu,x+\beta_0 s\nu\}\cup P_s\big),\ s\in[s_0,s_1]$$

for some $\beta_0,\dots,\beta_n\in\mathbb{R}$, $s_0<0<s_1$ and any shadow system $\{P_s\}_{s\in[s_0,\,s_1]}$ along a direction $\nu\in S^{d-1}$. If the volume of Q_s is an affine function of the parameter s and $Q_0=Q$, then for all s in $[s_0,s_1]$, $x+\beta_0 s\nu$, $\upsilon_1+\beta_1 s\nu,\dots,\upsilon_k+\beta_k s\nu$ are contained in the same supporting hyperplane of Q_s.

Proof If Q is of dimension less than d, the fact that Q_s is affine implies that $V(Q_s)=0$, $s\in[s_0,s_1]$. This means that for each s, Q_s is contained in a hyperplane and the result follows.

We are left with the case in which Q is full dimensional. Using Fubini's theorem we have

$$Q_s=\int_{z\in Q|\nu^\perp}V_1\big(Q_s\cap(z+\nu\mathbb{R})\big)\,dz,$$

where $V_1(\,\cdot\,)$ is the 1-dimensional Lebesgue measure. The integrated function is convex on s and the volume of Q_s is affine. A continuity argument similar to the one used in Lemma 5.1 implies that the length of $Q_s\cap(z+\nu\mathbb{R})$ is affine in $[s_0,s_1]$ for all z in $Q|\nu^\perp$.

The intersection of Q with the line through x, parallel to ν, is a line segment $[x,y]$ (possibly degenerate). We may assume that x is contained in the convex hull of $\upsilon_1,\dots,\upsilon_d$, where υ_i, $i=1,\dots,d$, are affinely independent and that ν is not parallel to H (in the opposite case the result is obvious). Then $x=\sum_{i=1}^d\lambda_i\upsilon_i$ for some $\lambda_i\geq 0$ with $\sum_{i=1}^d\lambda_i=1$ and $y=\sum_{j=1}^d\mu_j\upsilon_{i_j}$ for some $i_1,\dots,i_d\in\{1,\dots,n\}$ and $\mu_j\geq 0$ with $\sum_{i=1}^d\mu_i=1$. We set $\gamma:=\sum_{i=1}^d\lambda_i\beta_i$ and $\delta:=\sum_{j=1}^d\lambda_{i_j}\beta_{i_j}$. To prove that

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$x+\beta_0 s\nu, \upsilon_1+\beta_1 s\nu, \dots, \upsilon_d+\beta_d s\nu$ lie on the same hyperplane, it suffices to show that $\beta_0=\gamma$. Clearly, $V_1([x+\beta_0 s\nu, y+\delta s\nu]),\ V_1([x+\gamma s\nu, y+\delta s\nu]) \le V_1(Q_s\cap(x+\mathbb{R}\nu))$, with equality for $s=0$. Since the functions on the left are convex and the one on the right is affine it follows that equality must hold in both inequalities everywhere in $[s_0,s_1]$, thus $\beta_0=\gamma$.

Next we take any point z from the interior of the simplex spanned by $\upsilon_1,\dots,\upsilon_d$. Thus, $z=\sum_{i=1}^d \lambda_i\upsilon_i$ for some $\lambda_i>0$. We set $\gamma'=\sum_{i=1}^d\lambda_i\beta_i$. Since ν and H are not parallel it is clear that $z+\gamma' s\nu$ is an interior point of the simplex spanned by $\upsilon_1+\beta_1 s\nu,\dots,\upsilon_d+\beta_d s\nu$. However, $Q_s=\operatorname{conv}\{Q_s\cup\{z+\gamma' s\nu\}\}$ so the assumptions of the lemma hold. It follows by the above discussion that $z+\gamma' s\nu$ is a boundary point of Q_s that shows that $\upsilon_1+\beta_1 s\nu,\dots,\upsilon_d+\beta_d s\nu$ span a supporting hyperplane of Q_s. Finally, we apply the same argument to all d-tuples of affinely independent points from $\upsilon_1,\dots,\upsilon_k$ to obtain that all $\upsilon_i+\beta_i s\nu$, $i=1,\dots,k$ lie on the same supporting hyperplane of Q_s. ■

Before proving Theorem 1.2, we state an interesting reformulation of Lemma 6.1.

Corollary 6.2 *Let Q be a full dimensional polytope and*

$$Q_s=\operatorname{conv}\{x+\beta(x)s\nu : x\in Q\},\ s\in[-1,1]$$

be a shadow system along the direction ν. If the volume of Q_s is an affine function, then Q_s is also a polytope, combinatorially equivalent to Q, for all s in $(-1,1)$.

Proof It suffices to prove our claim in a small neighborhood of 0. Let $\upsilon_1,\dots,\upsilon_n$ be the vertices of Q. Set $\beta_i:=\beta(\upsilon_i),\ i=1,\dots,n$. Clearly

$$Q_s=\operatorname{conv}\{\upsilon_1+s\beta_1\nu,\dots,\upsilon_n+s\beta_n\nu\},$$

since the volume of the right-hand part is convex and dominated by the affine function $V(Q_s)$. A continuity argument ensures that the vertices of Q_s are exactly the points $\upsilon_i+s\beta_i\nu$ for s near 0. By Lemma 6.1, it follows that if $\upsilon_1,\dots,\upsilon_m$ are the vertices of a facet F of Q, then the vertices $\upsilon_1+s\beta_1\nu,\dots,\upsilon_m+s\beta_m\nu$ are contained in a common facet F_s of Q_s. Interchanging the role of Q and Q_s we see that these are exactly the vertices of the facet F_s. Similarly, the facets of Q_s are exactly of the form F_s, where F is a facet of Q. The result follows. ■

Proof of uniqueness in Theorem 1.2 First suppose that $D=S$ or B. Choose arbitrary $x_1\in K_1|\nu^\perp$. By Lemma 5.2, we may assume that 0 is contained in the interior of all K_i, hence we can choose $x_2=0$ and $x_i\in K_i|\nu^\perp$, $i=0,3,\dots,n$ so that the polytope spanned by the points $x_0,\dots,x_n$ is k-dimensional, where $k=d-j$. Assume again that $\nu=e_d$. By Lemma 5.1 we find a point L of T and a $\zeta_0\ge 0$ such that

$$\Phi_{D,X,j}(L+su)=\Phi_{D,X,j}(L)=\zeta_0,\forall s\in[-1,1]. \tag{6.1}$$

We show that, if $D=S$ or B, then $u_{1,\nu}(x_1)=u_{0,\nu}(x_0)$, which means that $u_{1,\nu}=u_{0,\nu}=const$. This will imply that for each ν in S^{d-1} all the midpoints of the chords

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of K_0, K_1 lie on the same hyperplane orthogonal to ν, thus (see [2]) K_0, K_1 are balls and, similarly, so are $K_2, \dots, K_n$. Also, as in the proof of Theorem 1.1, $K_0, \dots, K_n$ will all have the same center and if $D = B$, the center of K_i will be the origin.

By Kubota's formula and (6.1) we get

$$\int_{\mathcal{G}_{d,k}} V_k(\mathrm{conv}\{(x_0, l_0 + su_0), \dots, (x_n, l_n + su_n)\}|E)\, d\mu(E) = \int_{\mathcal{G}_{d,k}} V_k(\mathrm{conv}\{(x_0, l_0), \dots, (x_n, l_n)\}|E)\, d\mu(E), s \in [-1, 1].$$

The convexity on s implies that the integrated function must be affine with respect to s. Clearly, the polytope P spanned by the points (x_i, l_i), $i = 0, \dots, n$, has dimension k or $k + 1$. We may assume that the points (x_i, l_i), $i = 0, \dots, k$, span a k-dimensional supporting affine subspace H_0 of P, not parallel to ν.

Let G be a $(k+1)$-dimensional subspace of $\mathbb{R}^d$ containing P and E a k-dimensional subspace of G, perpendicular to H_0 (*i.e.,* E contains a vector orthogonal to H_0). We set,

$$Q = \mathrm{conv}\{(x_0, l_0), \dots, (x_n, l_n)\}|E.$$

Then Q is of dimension k or $k-1$. Moreover, the points $(x_i, l_i)|E, i = 0, \dots, k$, are contained in the same $(k-1)$-dimensional face of Q. In addition, the k-dimensional volume of the shadow system

$$Q_s := \mathrm{conv}\{(x_0, l_0 + su_0), \dots, (x_n, l_n + su_n)\}|E$$

is an affine function of the parameter s. Note also that $(0, l_2) \in P$, so $\nu = e_d \in G$ (l_2 cannot be 0 since we assumed that the origin is an interior point of K_i) thus the points $(x_i, l_i + su_i)$, $i = 0, \dots, n$, are contained in G for all s in $[-1, 1]$.

Now, Lemma 6.1 implies that the points $(x_i, l_i + su_i)|E,\ i = 0, \dots, k$ are contained in the same hyperplane of E. It follows that the affine subspace H_s spanned by the points $(x_i, l_i + su_i),\ i = 0, \dots, k$, is still perpendicular to E for all s in $[-1, 1]$.

In particular, we have shown that for each k-dimensional subspace E of G, perpendicular to the affine subspace H_0 of G, E is also perpendicular to $H_1 \subseteq G$. This case can occur only if H_0 and H_1 are parallel. By assumption, the vector $\nu = e_d$ is not parallel to H_0. Since both H_0 and H_1 have the same dimension, the linear spaces spanned by $(x_i - x_0, l_i - l_0)$ and $(x_i - x_0, l_i - l_0 + u_i - u_0)$, $i = 1, \dots, k$ respectively are identical, so $u_1 = u_0$.

The proof when $D = I$ is based on the same idea. We briefly describe the argument. The choice of $(x_1, \dots, x_n) \in K_1|\nu^\perp \times \cdots \times K_n|\nu^\perp$ can be made so that the zonotope $P = \sum_{i=1}^n [0, (x_i, l_i)]$ is a $(k + 1)$-dimensional parallelepiped (indeed, we can choose some of the x_i's to be equal to 0 if necessary). As before, equality in (1.9) forces the k-dimensional volume of the zonotope $\sum_{i=1}^n [0, (x_i, l_i + su_i)]|E$ to be affine for every k-dimensional subspace E of $\mathbb{R}^n$. Assuming without loss of generality that the segments $[0, (x_i, l_i)]$, $i = 1, \dots, k$ span a facet F of P, not parallel to $\nu = e_d$ and taking E to be perpendicular to F, we conclude (using Lemma 6.1) that the k-dimensional parallelepiped spanned by $(x_i, l_i + su_i)$, $i = 1, \dots, k$, is always identical to F. This shows that $u_1 = \cdots = u_n = 0$ and the result follows. ■

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References

[1] W. Blaschke, *Über affine Geometrie XI: Lösung des "Vierpunktproblems" von Sylvester aus der Theorie der geometrischen Wahrscheinlichkeiten.* Leipziger Berichte **69**(1917), 436–453.

[2] W. Blaschke, *Kreis und Kugel.* 2te Aufl., Walter de Gruyter, Berlin, 1956.

[3] J. Bourgain, M. Meyer, V. Milman, and A. Pajor, *On a geometric inequality.* In: Geometric aspects of functional analysis, Lecture Notes in Math., 1317, Springer, Berlin, 1988, pp. 271-282.

[4] H. Busemann, *Volume in terms of concurrent cross-sections.* Pacific J. Math. **3**(1953), 1–12.

[5] S. Campi, A. Colesanti, and P. Gronchi, *A note on Sylvester's problem for random polytopes in a convex body.* Rend. Istit. Mat. Univ. Trieste **31**(1999), no. 1–2, 79–94.

[6] S. Campi and P. Gronchi, *Extremal convex sets for Sylvester-Busemann type functionals.* Appl. Anal. **85**(2006), no. 1–3, 129–141. doi:10.1080/00036810500277579

[7] ———, *On the reverse L^p-Busemann-Petty centroid inequality.* Mathematika **49**(2002), no. 1–2, 1–11. doi:10.1112/S0025579300016004

[8] ———, *On volume product inequalities for convex sets.* Proc. Amer. Math. Soc. **134**(2006), no. 8, 2393–2402. doi:10.1090/S0002-9939-06-08241-4

[9] L. Dalla and D. G. Larman, *Volumes of a random polytope in a convex set.* In: Applied geometry and discrete mathematics, Ser. Discrete Math. Theoret. Comput. Sci., 4, American Mathematical Society, Providence, RI, 1991, pp. 175–180.

[10] G. Ewald, D. G. Larman, and C. A. Rogers, *The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space.* Mathematika **17**(1970), 1–20. doi:10.1112/S0025579300002655

[11] R. J. Gardner, *Geometric tomography.* Encyclopedia of Mathematics and its Applications, 58, Cambridge University Press, Cambridge, 1995.

[12] A. A. Giannopoulos, *On the mean value of the area of a random polygon in a plane convex body.* Mathematika **39**(1992), no. 2, 279–290. doi:10.1112/S0025579300015023

[13] A. A. Giannopoulos and A. Tsolomitis, *Volume radius of a random polytope in a convex body.* Math. Proc. Cambridge Philos. Soc. **134**(2003), no. 1, 13–21. doi:10.1017/S0305004102006254

[14] H. Groemer, *On some mean values associated with a randomly selected simplex in a convex set.* Pacific J. Math. **45**(1973), 525–533.

[15] ———, *On the mean value of the volume of a random polytope in a convex set.* Arch. Math. **25**(1974), 86–90.

[16] M. Hartzoulaki and G. Paouris, *Quermassintegrals of a random polytope in a convex body.* Arch. Math. **80**(2003), no. 4, 430–438.

[17] C. A. Rogers and G. C. Shephard, *Some extremal problems for convex bodies.* Mathematika **5**(1958), 93–102. doi:10.1112/S0025579300001418

[18] R. Schneider, *Convex bodies: the Brunn-Minkowski theory.* Encyclopedia of Mathematics and its Applications, 44, Cambridge University Press, Cambridge, 1993.

[19] P. Schöpf, *Gewichtete Volumsmittelwerte von Simplices, welche zufällig in einem konvexen Körper der $\mathbb{R}^n$ gewählt werden.* Monatsh. Math. **83**(1977), no. 4, 331–337. doi:10.1007/BF01387909

[20] G. C. Shephard, *Shadow systems of convex sets.* Israel J. Math. **2**(1964), 229–236. doi:10.1007/BF02759738

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