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On a new finite non-abelian simple group of Janko

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Two new simple groups have recently been discovered by Z. Janko. One of these groups has order 50,232,960. As a first step in showing that there is precisely one (up to isomorphism) simple group of order 50,232,960, the author proves in this paper the following result: If G is a non-abelian simple group of order 50,232,960, then the structure of the centralizer of an element of order two in G is uniquely determined.

In a note added on 21 April 1969 to this paper, the author announces that he has proved the uniqueness of the simple group of order 50,232,960.

Introduction

Recently, Z. Janko [6] announced his discovery of another two new simple groups of finite order. More precisely, he has proved the following result:

Let G be a finite non-abelian simple group with the following properties:

(a) The centre $Z(S)$ of a Sylow 2-subgroup S of G is cyclic.

(b) If z is the involution in $Z(S)$, then the centralizer H of z in G is an extension of a group E of order 2^5 by A_5 (the alternating group on 5 letters).

Then we have the following two possibilities: If G has only one class of involutions, then G is a new simple group of order 50,232,960 and has a uniquely determined character table. If G has more than one

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class of involutions, then G is a new simple group of order 604,800 and G itself is uniquely determined.

It should be of interest to know that in a non-abelian simple group of order 50,232,960, the centralizer of a central involution (an involution in the centre of a Sylow 2-subgroup) has a unique structure. Hence in this present paper, we shall prove the following result:

MAIN THEOREM. *Let G be a non-abelian simple group of order 50,232,960. Then G has the following two properties:*

(i) The centre $Z(S)$ of a Sylow 2-subgroup S of G is cyclic.

(ii) If z is the involution in $Z(S)$, then the centralizer H of z in G is an extension of a group E of order 2^5 by A_5.

In this work, the modular theory developed by R. Brauer and his school is extremely useful. In §1, we shall list some known results in modular theory which are used very often in this work. In §2, the structure of the normalizers of a Sylow 19-subgroup and a Sylow 17-subgroup of a simple group of order 50,232,960 will be determined. The Sylow 5-normalizer will be determined in §3; and finally, the proof of the Main Theorem will be completed in §4 by employing the exceptional character theory of Suzuki [8].

1. Notation and known results

Notation. Let G be a finite group. If p is a prime dividing the order of G, then G_p will denote a Sylow p-subgroup of G. An element y of G is called p-regular if the order of y is prime to p, otherwise y is p-singular. If S is a subset of G, then $|S|$ will denote the number of elements of S, $C_G(S)$ the centralizer of S in G and $N_G(S)$ its normalizer in G. We use the notation $H \leqq G$ to mean H is a subgroup of G and $[G : H]$ the index of H in G. The word "character" always refer to a character afforded by a representation in the field of all complex numbers; $\chi_1 = 1_G$ will denote the principal character of G and $B_1(p)$ will denote the principal p-block.

We shall now list some results obtained by R. Brauer [2], R.G. Stanton [7] and H.F. Tuan [3] concerning groups whose order contains a prime to the first power. Throughout the rest of this section, except in the last two

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theorems, G will be a finite group of order $g = pg'$, where p is a prime number and $(p, g') = 1$. For a definition of other terms used in this section, see [2] and [4].

Now let s be an element of order p in the Sylow p-subgroup G_p of G; then $G_p = \langle s \rangle$. The centralizer $C_G(G_p)$ of G_p in G has the form

$$C_G(G_p) = V \times G_p , \tag{1.1}$$

where V is a subgroup of $C_G(G_p)$. The normalizer $N_G(G_p)$ of G_p in G contains both G_p and V as normal subgroups. The factor group $N_G(G_p)/C_G(G_p)$ is a cyclic group. Its order q divides $p - 1$, say $q = \frac{p-1}{t}$. Then t is the number of conjugate classes of elements of order p in G. Let m be an element of $N_G(G_p)$ such that

$$N_G(G_p) = \langle m \rangle \cdot C_G(G_p) . \tag{1.2}$$

Then $m^{-1}sm = s^{\gamma^t}$, where γ is a primitive root $(\mathrm{mod}\ p)$. Let $\chi_1 = 1_G, \chi_2, \ldots$ be the ordinary irreducible characters of G, and denote by $X_\mu = \chi_\mu(1)$ the degree of χ_μ and r_μ the number of p-conjugate characters of χ_μ. The characters $\chi_1 = 1_G, \chi_2, \ldots$ are then distributed into p-blocks $B_1(p), B_2(p), \ldots$. A p-block $B_\lambda(p)$, is either of defect 0 (highest kind) or of defect 1 (lowest kind). A p-block $B_\lambda(p)$ of defect 0 consists of one character χ_μ only, and we have $X_\mu = \chi_\mu(1) \equiv 0 \ (\mathrm{mod}\ p)$, and $r_\mu = 1$; further, χ_μ is modular irreducible and it vanishes on all p-singular elements of G. All degrees of characters in a p-block of defect 1 are prime to p. We shall now give some results of R. Brauer [2], concerning the p-blocks of defect 1.

THEOREM 1A ([2], Theorem 1). *Let $v_1, v_2, \ldots, v_l$ be a maximal system of elements of the group V in (1.1), such that no two of them are conjugate in $N_G(G_p)$. Then the group G possesses l p-blocks $B_1(p)$, $B_2(p), \ldots, B_l(p)$ of defect 1. All the other p-blocks $B_{l+1}(p)$,*

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$B_{l+2}(p)$, ... *are of defect* 0 .

THEOREM 1B ([2], Theorem 2). *With each of the p-blocks* $B_\lambda(p)$ $(\lambda = 1, 2, \ldots, l)$ *of defect* 1 *, there corresponds a certain multiple* $t_\lambda = \tau_\lambda t > 0$ *of* t (t_λ *also divides* $p-1$). *The block* $B_\lambda(p)$ *then consists (a) of one family of* t_λ *p-conjugate characters, the "exceptional" family and (b)* $\frac{p-1}{t_\lambda}$ *further characters* χ_μ *which belong to the field of g'-th roots of unity, i.e. each such* χ_μ *is p-conjugate only to itself.*

REMARK 1. With each character χ_μ in a p-block $B_\lambda(p)$ $(\lambda = 1, 2, \ldots, l)$ of defect 1 , a sign $\delta_\mu = \pm 1$ is assigned. All the t_λ "exceptional" characters have the same sign assigned to them. For details (see [2], §5).

DEFINITION 1. The group V in (1.1) is a normal subgroup of $N_G(G_p)$. For any irreducible character $\theta(v)$ of V and any fixed element y of $N_G(G_p)$, the expression $\theta(y^{-1}vy)$ also represents an irreducible character of V , if v ranges over V . Two such characters $\theta(v)$ and $\theta(y^{-1}vy)$ are termed *associated characters* of V with regard to $N_G(G_p)$. All the irreducible characters of V are distributed into classes of associated characters. In order to obtain all the characters associated with $\theta(v)$, it is sufficient to take y as a power of the element m defined in (1.2).

THEOREM 1C ([2], Theorem 4). *To each of the p-blocks* $B_\lambda(p)$ $(\lambda = 1, 2, \ldots, l)$ *of defect* 1 *, there corresponds an irreducible character* θ_λ *of* V . *No two characters* $\theta_1, \theta_2, \ldots, \theta_l$ *of* V *are associated with regard to* $N_G(G_p)$ *, but every irreducible character of* V *is associated with one of the characters* θ_λ . *There are exactly* $\tau_\lambda = \frac{t_\lambda}{t}$ *characters associated with* θ_λ *; they are*

$$\theta_\lambda(m^{-\kappa} v m^{\kappa}), (\kappa = 0, 1, 2, \ldots, \tau_\lambda - 1) ,$$

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where v *ranges over* V *and we have*

$$\theta_\lambda(m^{-\tau_\lambda} v m^{\tau_\lambda}) = \theta_\lambda(v) \; , \quad v \text{ in } V \; .$$

(Note: We take $\theta_\lambda(m^{-\kappa} v m^{\kappa}) = \theta_\lambda(v)$ for $\kappa = 0$, v in $V)$. If χ_μ is a character belonging to $B_\lambda(p)$ which is p-conjugate only to itself, then

$$(1.3) \qquad \chi_\mu(s^\rho v) = \delta_\mu \sum_{\kappa=0}^{\tau_\lambda - 1} \theta_\lambda(m^{-\kappa} v m^{\kappa}) \; , \; (\tau_\lambda = \frac{t_\lambda}{t})$$

for $\rho \not\equiv 0 \pmod p$ and v in V . Here $\delta_\mu = +1$ or $\delta_\mu = -1$. If χ_μ is a character of the exceptional family of $B_\lambda(p)$, we have

$$(1.4) \qquad \chi_\mu(s^\rho v) = -\delta_\mu \sum_{\kappa=0}^{q-1} \varepsilon^{\rho\gamma^{\kappa t}} \theta_\gamma(m^{-\kappa} v m^{\kappa}) \; , \; (q = \frac{p-1}{t})$$

where $\rho \not\equiv 0 \pmod p$ and v is in V . Here ε is a suitable primitive p-th root of unity, γ is a primitive root $(\text{mod } p)$ and $\delta_\mu = +1$ or $\delta_\mu = -1$.

THEOREM 1D ([2], Theorem 10). *Let* f_λ *be the degree of the character* θ_λ *of* V . *The degree* X_μ *of a character* χ_μ *of the p-block* $B_\lambda(p)$ $(\lambda = 1 , 2 , \ldots , l)$ *of defect* 1 *satisfies the congruence*

$$(1.5) \qquad X_\mu \equiv \frac{\delta_\mu t_\lambda f_\lambda}{t} \pmod p$$

or

$$(1.6) \qquad X_\mu \equiv \frac{\delta_\mu f_\lambda}{t} \pmod p$$

according as we have the case (1.3) or (1.4).

REMARK 2. If the degrees X_μ and f_λ are known, the sign δ_μ assigned to the character χ_μ can be obtained from (1.5), (1.6); at least for an odd p .

THEOREM 1E ([2], Theorem 6). *Let* $B_\lambda(p)$ *be a p-block of defect* 1 . *For any p-regular element* y *of* G , *we have*

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(1.7) $$\sum_{\mu} \delta_{\mu} \chi_{\mu}(y) = 0$$

where χ_{μ} *ranges over a complete system of characters representing the different families of* $B_{\lambda}(p)$ *(i.e.* χ_{μ} *ranges over the* $\frac{p-1}{t_{\lambda}}$ *non-exceptional characters of* $B_{\lambda}(p)$ *and one of the* t_{λ} *exceptional characters of* $B_{\lambda}(p)$*)* . *In particular*

(1.8) $$\sum_{\mu} \delta_{\mu} X_{\mu} = 0$$

where X_{μ} *is the degree of* χ_{μ} *and* μ *ranges over the same values as in (1.7).*

THEOREM 1F ([2], Theorem 11). *If the p-block* $B_1(p)$ *of defect 1 contains the principal character* $\chi_1 = 1_G$ *of* G *, then* θ_1 *is the principal character of* V . *Further,* $t_1 = t$. *If* χ_{μ} *is a character belonging to* $B_1(p)$ *which is p-conjugate only to itself, then* X_{μ} *the degree of* χ_{μ} *satisfies the congruence*

(1.9) $$X_{\mu} \equiv \delta_{\mu} = \pm 1 \pmod{p} .$$

If χ_{μ} *is a character of the exceptional family of* $B_1(p)$ *, then we have*

(1.10) $$X_{\mu} \equiv \frac{-\delta_{\mu}(p-1)}{t} = \pm \frac{(p-1)}{t} \pmod{p} .$$

If $\chi_1 = 1_G$ *,* χ_2 *, ... ,* χ_{q+1} $(q = \frac{p-1}{t})$ *represent the different families of* $B_1(p)$ *, then*

(1.11) $$1 + \delta_2 X_2 + \dots + \delta_{q+1} X_{q+1} = 0 .$$

From Theorems 1C and 1F, we note

COROLLARY 1. *Let* G *be a group of order* $g = pq^b g^*$ *where* p *and* q *are distinct primes;* b *and* g^* *are positive integers with* $(g^*, pq) = 1$. *Suppose there is an element of order* pq *in* G *, then* q^b *cannot divide the degree of any irreducible character* χ_{μ} *in the principal p-block* $B_1(p)$.

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DEFINITION 2. We shall say a character χ of $B_1(p)$ *is of type 0 for the prime* p if $\chi(1) \equiv 1 \pmod p$ or if χ belongs to the exceptional family of $B_1(p)$ and $\chi(1) \equiv \frac{-(p-1)}{t} \pmod p$; and χ is of type 1 for the prime p, if $\chi(1) \equiv -1 \pmod p$ or if χ belongs to the exceptional family and $\chi(1) \equiv +\frac{(p-1)}{t} \pmod p$.

THEOREM 1G ("The block-intersection theorem")([7], Lemma 6). *Let the group G be of order g. Assume p and p' are distinct primes which divide g to the first power only and that there are no elements of order pp' in G. Let a_{ij} be the number* [1] *of characters in $B_1(p) \cap B_1(p')$ which are of type i for p and type j for p', the indices i and j being zero or one as defined in Definition 2. Then*

$$a_{00} + a_{11} = a_{01} + a_{10} .$$

THEOREM 1H ([7], Lemma 2). *Let G be a finite group of order $g = p^a q^b g^*$ where p and q are distinct primes and a, b and g^* are positive integers, $(g^*, pq) = 1$. Let $\chi_1 = 1_G, \chi_2, \ldots, \chi_n$ be all the ordinary irreducible characters of G. If G contains no elements of order pq, and if*

$$\sum_{\mu=1}^{n} \delta_\mu \chi_\mu(x) = 0$$

for all p-regular elements x of G, where $\delta_1, \delta_2, \ldots, \delta_n$ denote some algebraic integers independent of x, then

$$\sum_{\chi_\mu \in B(q)} \delta_\mu \chi_\mu(y) = 0$$

for all q-singular elements y of G and the summation is performed over

[1] If a character χ in $B_1(p) \cap B_1(p')$ is a member of the exceptional family of $B_1(p)$, then we only count one of the members of the exceptional family of $B_1(p)$ in the number a_{ij}. Similarly, if χ is exceptional for the prime p', we only count one of the members of the family in the number a_{ij}. The character χ in $B_1(p) \cap B_1(p')$ cannot be exceptional for both primes p and p'.

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all the irreducible characters χ_μ *which belong to a fixed q-block* $B(q)$. *Further, if* 1 *is the identity element in* G *, then*

$$\sum_{\chi_\mu \in B(q)} \delta_\mu \chi_\mu(1) \equiv 0 \pmod{q^b} .$$

THEOREM 11 ([3], Lemma 1). *Let* G *be a finite group which is identical with its commutator subgroup* G' *, and assume that the principal p-block* $B_1(p)$ *(i.e. the p-block which contains the principal irreducible character* $\chi_1 = 1_G$ *of* G*) contains an irreducible faithful character* χ *(i.e. a character afforded by a faithful irreducible representation) of degree* $z < 2p$. *Then the order of the centralizer* $C_G(G_p)$ *of a Sylow p-subgroup* G_p *of* G *, is a power of* p .

2. The Sylow 19- and 17- normalizers of G

Notation. Throughout the rest of the paper, G will be a non-abelian simple group of order $g = 50{,}232{,}960 = 2^7.3^5.5.17.19$.

Now, let G_{19} be a Sylow 19-subgroup of G and $N_G(G_{19})$, its normalizer in G . Let $n_{19} = [G : N_G(G_{19})]$, then n_{19} has one of the following possibilities:

(i) $2^7.3^5$,
(ii) $2^7.3.5$,
(iii) $2^5.3$,
(iv) $2^2.5$,
(v) $2^2.3^4$,
(vi) $2^4.3^4.5$,
(vii) $2^5.3^3.17$,
(viii) $3^2.17$,
(xi) $2^2.3^2.5.17$

and

(x) $2^7.3^3.5.17$.

Also the number t_{19} , of conjugate classes of elements of order 19 in the simple group G , can be only 1,2,3,6 or 9 . We shall now proceed to rule out the possibilities (i) to (ix) for n_{19} :

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The cases $t_{19} = 6$, and $t_{19} = 9$ can be easily ruled out by using Theorems 1E, 1F and 1I. Since $t_{19} = 1, 2,$ or 3, the case $n_{19} = 2^7.3^5$ is not possible. Using Sylow theorems, we find that cases (ii) and (iii) are not possible. The case $n_{19} = 20$ is not possible because the symmetric group S_{20} on 20 letters has no elements of order 17.19 .

Now suppose $n_{19} = 2^2.3^4$, then $|N_G(G_{19})| = 2^5.3.5.17.19$ and $|C_G(G_{19})| = 2^4.5.17.19$. By Theorems 1B and 1F, the principal 19-block $B_1(19)$ consists of 6 non-exceptional characters $1_G, \chi_2, \ldots, \chi_6$ say, and 3 exceptional characters $\chi^{(\tau)}$ $(\tau = 1,2,3)$. By Theorems 1F, 1I and Corollary 1, the only possibilities for $\chi_i(1)$, $i = 2, 3, \ldots, 6$ are:-

$$96 , 324 \text{ or } 1728 .$$

and the possibilities for $\chi^{(\tau)}(1)$, $\tau = 1, 2, 3$ are:-

$$576 , 1944 \text{ or } 108 .$$

Since none of the above degrees is prime to 3, we have a contradiction by (1.11) . Hence case (v) for n_{19} is not possible.

In a similar way, case (vi) for n_{19} can also be ruled out. In cases (vii), (viii) and (ix), there are no elements of order 17.19 in G , and so, we can use the block-intersection theorem (Theorem 1G) for the primes 19 and 17. We then have a contradiction by looking at the possible degrees of irreducible characters in $B_1(19) \cap B_1(17)$. Hence cases (vii), (viii) and (ix) for n_{19} are also not possible. So we have proved the following:-

LEMMA 2A. *The normalizer* $N_G(G_{19})$ *of a Sylow* 19-*subgroup* G_{19} *of* G *is a Frobenius group of order* 19.9.

By Theorems 1A, 1B, 1C and 1F, we have:

COROLLARY 2A. *The principal* 19-*block* $B_1(19)$ *of* G *is the only* 19-*block of defect* 1 *, and consists of* 9 *non-exceptional characters and* 2 *exceptional characters which are also complex conjugate.*

Since there are no elements of order 19.17, Theorem 1G gives:

LEMMA 2B. *There is an irreducible character in* $B_1(19) \cap B_1(17)$ *with degree* 1920 *or* 18 .

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Now let G_{17} be a Sylow 17-subgroup of G and $N_G(G_{17})$ its normalizer in G. Then by Lemma 2A, the index $n_{17} = [G : N_G(G_{17})]$ has one of the following possibilities:

$$\text{(i)}\quad 2^7.19\ ,$$
$$\text{(ii)}\quad 3^2.19\ ,$$
$$\text{(iii)}\quad 2.3^4.19\ ,$$
$$\text{(iv)}\quad 2^2.3.5.19\ ,$$
$$\text{(v)}\quad 2^3.3^3.5.19$$

and

$$\text{(vi)}\quad 2^4.3^5.5.19\ .$$

By a transfer theorem of Burnside, the case $n_{17} = 2^7.19$ is not possible. By Lemma 2B, Corollary 1 and Theorem 1I, cases (ii), (iii) and (iv) are not possible.

Now suppose $n_{17} = 2^3.3^3.5.19$, then $|N_G(G_{17})| = 2^4.3^2.17$, and by Lemma 2B and Corollary 1, $|C_G(G_{17})| = 3^2.17$. Let χ be an irreducible character in $B_1(17) \cap B_1(19)$ with $\chi(1) = 1920$. It is then an easy matter to show that χ is real by using Theorems 1B, 1D, 1E and 1H. Now let $C_G(G_{17}) = V \times G_{17}$, where V is a group of order 9. Since V is characteristic in $C_G(G_{17})$, we have $N_G(V) \geqq N_G(G_{17})$. Since $N_G(V)$ is divisible by 3^3 and $[N_G(V) : N_G(G_{17})] \equiv 1 \pmod{17}$ we only have the following 2 possibilities for $|N_G(V)|$:-

$$\text{(a)}\quad |N_G(V)| = 2^5.3^4.17$$

or

$$\text{(b)}\quad |N_G(V)| = 2^7.3^3.5.17\ .$$

First suppose we have case (a) for $|N_G(V)|$. Then $|N_G(V)| = 2^5.3^4.17$ and by a result of P. Hall ([5], Theorem 9.3.1), $N_G(V)$ is a non-soluble group. Since the automorphism group of a group of order 9 is soluble, $C_G(V)$ and $C_G(V)/V$ are also non-soluble groups. Hence $|C_G(V)| = 2^4.3^4.17$ or $2^5.3^4.17$. In any case, let S_2 be a subgroup of $C_G(V)$ such that $|S_2| = 2^4$, and let v be an element of order 3 in V. Now consider the group $H = \langle v \rangle \times S_2$. Since $|C_G(V) \cap N_G(G_{17})| = 2^3.3^2.17$ or $2^4.3^2.17$

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and $C_G(v) \geqq C_G(V)$, we have $|C_G(v) \cap N_G(G_{17})| = 2^3.3^2.17$ or $2^4.3^2.17$. By Sylow Theorems, $|C_G(v)| = 2^4.3^4.17$, $2^5.3^4.17$ or $2^7.3^5.5.17$. But by a Theorem of Burnside ([5], Theorem 16.8.7), $|C_G(v)|$ cannot be $2^7.3^5.5.17$. So $|C_G(v)| = 2^4.3^4.17$ or $2^5.3^4.17$. We shall now find the value of the real character χ on the element v of V . Since χ belongs to a 2-block of defect 0 , χ vanishes on all 2-singular elements. Summing χ over the group H , we have

$$1920 + 2\chi(v) \equiv 0 \pmod{48} . \tag{2.1}$$

Consider the group $K = \langle v \rangle \times G_{17}$. By Theorems 1C and 1F, $\chi(\omega) = -1$ for all 17-singular elements ω in K . Summing χ over K , we have

$$1920 + 2\chi(v) - 48 \equiv 0 \pmod{51} . \tag{2.2}$$

From (2.1) and (2.2), we have

$$1920 + 2\chi(v) - 48 \equiv 0 \pmod{816} . \tag{2.3}$$

From the order of $C_G(v)$ and (2.3), we have

$$\chi(v) = -120 . \tag{2.4}$$

Now, let θ be the non-trivial linear character of H/S_2 (which may be regarded as an irreducible character of H with S_2 in the kernel), such that $\theta(v) = \rho$, where ρ is a primitive cube root of unity. Let $\chi|_H$ be the restriction of χ to H . Since χ vanishes on 2-singular elements and $\langle \chi|_H , \theta \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h)\overline{\theta(h)}$, we have from (2.4),

$$\langle \chi|_H , \theta \rangle_H = \frac{1}{48} (1920 - 120\bar{\rho} - 120\rho) = \frac{85}{2} .$$

This is impossible, and hence case (a) for $|N_G(V)|$ is not possible. A similar argument can be used to rule out the case $|N_G(V)| = 2^7.3^3.5.17$. With this, the case $n_{17} = 2^3.3^3.5.19$ is ruled out, and together with Theorem 1I and Lemma 2B we have proved the following:

LEMMA 2C. *The normalizer* $N_G(G_{17})$, *of a Sylow* 17-*subgroup* G_{17} *of* G , *is a Frobenius group of order* 17.8.

Again, using Theorems 1A, 1B, 1C and 1F, we have:

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COROLLARY 2B. *The principal* 17-*block* $B_1(17)$ *of* G *is the only* 17-*block of defect* 1 . *There are 8 non-exceptional characters and 2 exceptional characters in* $B_1(17)$. *The 2 exceptional characters in* $B_1(17)$ *are real.*

It is not possible to give here the considerable amount of numerical work required to determine all the degrees of irreducible characters in each of $B_1(19)$ and $B_1(17)$. We shall only content ourselves here that they are all determined, and we give the results in the form of block relations, (asterisks denote the families of p-conjugate characters; also, we shall use the convention that the integer "m" shall mean "the irreducible character whose degree is m").

$$B_1(19): \quad 1 + 324 + 1920 + 1920' + 1920'' = 816 + 2754 + 1215 + 1215' + 85^* .$$

$$B_1(17): \quad 1 + 324 + 1140 + 2432 + 3078 = 1920 + 1920' + 1920'' + 1215^* .$$

By application of Theorem 1H to the block relations above, we have

$$\begin{aligned} B_1(17) \cap B_1(19) \cap B_1(5) &= \{1 , 324\} , \\ B_1(19) \cap B_2(5) &= \{816 , 2754\} , \\ B_1(17) \cap B_3(5) &= \{2432 , 3078\} , \end{aligned} \tag{2.5}$$

where $B_2(5)$ and $B_3(5)$ are distinct 5-blocks of defect 1 .

$$\begin{aligned} 2754(\omega) &= - 816(\omega) \\ 3078(\omega) &= - 2432(\omega) , \end{aligned} \tag{2.6}$$

where ω is any 5-singular element of G .

We also have one 3-block of defect 1 , say $B_2(3)$, where $B_2(3) = \{324 , 2754 , 3078\}$, and by a result of R. Brauer [*1*], we have

$$324(\tau) + 2754(\tau) = 3078(\tau) , \tag{2.7}$$

where τ is any 3-regular element of G , and

$$324(c) = 2754(c) = - 3078(c) , \tag{2.8}$$

where c is any 3-singular element of G .

3. The Sylow 5-normalizer of G

Notation. In this section, we shall use all the notations introduced in §1, with the prime $p = 5$. Throughout the rest of the paper, the irreducible characters $\chi_1 = 1_G , \chi_2 , \ldots , \chi_{14}$ of G will be so chosen

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that $\chi_2(1) = \chi_3(1) = \chi_4(1) = 1920$, $\chi_5(1) = 324$, $\chi_6(1) = 3078$, $\chi_7(1) = 2432$, $\chi_8(1) = 1140$, $\chi_9(1) = \chi_{10}(1) = 1215$, $\chi_{11}(1) = 2754$, $\chi_{12}(1) = 816$ and $\chi_{13}(1) = \chi_{14}(1) = 85$.

Now let $\omega = sv$, where $\langle s \rangle = G_5$ is a Sylow 5-subgroup of G and v is any 3-singular element of the group V in (1.1) (see §1). Since the element ω is 5-singular and the character χ_5 belongs to $B_1(5)$, we have by (2.6) and (2.8) the following results:-

$$\chi_5(sv) = \chi_7(sv) = \chi_{11}(sv) = -1 , \quad \chi_6(sv) = \chi_{12}(sv) = +1 , \tag{3.1}$$

where v is any 3-singular element in V .

Now let $\omega = su$, where s is as defined in §1 and u is any 2-singular element of V . Since the character χ_7 has degree $\chi_7(1) = 2^7.19$, χ_7 belongs to a 2-block of defect 0 and hence $\chi_7(su) = 0$. So by (2.6) and (2.7) we have

$$\chi_5(su) = \chi_{12}(su) = -1 , \quad \chi_{11}(su) = +1 , \quad \chi_6(su) = \chi_7(su) = 0 , \tag{3.2}$$

where u is any 2-singular element in V .

From (3.1) and (3.2), we have the following result:

COROLLARY 3A. *The group G has no elements of order* 30.

We shall now determine the order of the Sylow 5-normalizer $N_G(G_5)$. By Lemma 2A and 2C, $|N_G(G_5)|$ has one of the following possibilities:

(a) $2^7.3^2.5$,
(b) $2^6.3.5$,
(c) $2^5.5$,
(d) $2^4.3^3.5$,
(e) $2^6.3^5.5$,
(f) $2^2.3^5.5$,
(g) $2^5.3^4.5$,
(h) $2^3.3^2.5$,
(i) $2.3^4.5$,
(j) 2.5
and (k) $2^2.3.5$.

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First suppose the number t of conjugate classes of elements of order 5 in G is 1. By theorems 1B and 1E, let t_3 be the multiple of t corresponding to the 5-block $B_3(5)$ in (2.5); then $t_3 = 1$ or 2. Now suppose $t_3 = 1$. Then by Theorem 1C, there is only one irreducible character of V corresponding to $B_3(5)$, say θ_3. If $f_3 = \theta_3(1)$, then by Theorem 1D, we have $\chi_7(1) = 2432 \equiv \delta f_3 \pmod 5$, where $\delta = \pm 1$. Hence $f_3 \equiv \pm 2 \pmod 5$ and by Theorem 1C, $\chi_7(su) = \delta\theta_3(u)$ for all 2-singular elements u in V. But from (3.2), $\chi_7(su) = 0$ for any such element u in V, and hence $f_3 = \theta_3(1)$ must be divisible by the highest 2-power dividing $|V|$. We shall now go through all the possibilities (a) to (k) for $|N_G(G_5)|$ and show that none of the cases can occur.

Case (a). In this case, $|N_G(G_5)| = 2^7.3^2.5$, and since $t = 1$, $|C_G(G_5)| = 2^5.3^2.5$ and so we have $|V| = 2^5.3^2$. Then $f_3 = 2^5$; but $f_3^2 > |V|$, which is not possible. So this case for $|N_G(G_5)|$ is not possible. A similar argument can be used to rule out the cases (b), (c) and (d) for $|N_G(G_5)|$.

Case (e). In this case, $|N_G(G_5)| = 2^6.3^5.5$ and $|C_G(G_5)| = 2^4.3^5.5$. Hence $|V| = 2^4.3^5$ and $f_3 = \theta_3(1) = 48$. Then $\delta = -1$ and $\chi_7(s) = -48$; and by (2.6), $\chi_6(s) = 48$. In (2.7) put $\tau = s$, then $\chi_{11}(s) = 49$ because $\chi_5(s) = -1$. Let t_2 be the multiple of t corresponding to $B_2(5)$ in (2.5), then $t_2 = 1$ or 2. If $t_2 = 1$, let θ_2 be the irreducible character corresponding to $B_2(5)$. Then by Theorem 1C, $\chi_{11}(s) = \theta_2(1) = 49$, which is not possible. If $t_2 = 2$, let θ_2, θ_2' be the 2 irreducible characters of V corresponding to $B_2(5)$, where $\theta_2(1) = \theta_2'(1)$. Then by Theorem 1C, we have $\chi_{11}(s) = 2\theta_2(1) = 49$, which is not possible. With this, we have ruled out case (e) for $|N_G(G_5)|$.

A similar argument can be used to rule out case (f) for $|N_G(G_5)|$.

Case (g). In this case, $|N_G(G_5)| = 2^5.3^4.5$ and $|C_G(G\ )| = 2^3.3^4.5$. Hence $|V| = 2^3.3^4$ and $f_3 = \theta_3(1) = 8$. Now let v be any element of order 3 in V, then by (3.1) and Theorem 1C, we have $\chi_6(sv) = \theta_3(v) = +1$. But the degree of θ_3 is 8 and so this is not possible, and hence case (g) for $|N_G(G_5)|$ is not possible.

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Case (h). In this case, $|N_G(G_5)| = 2^3.3^2.5$ and $|C_G(G_5)| = 2.3^2.5$. So, $|V| = 2.3^2$ and $f_3 = \theta_3(1) = 2$. By Corollary 3A, there are altogether 6 conjugate classes of elements in V; 1 class of involutions with representative τ say, 4 classes of 3-elements, and the identity element 1 of V. By Theorems 1C, 1D and (3.1), $\chi_7(sv) = \theta_3(v) = -1$ for any 3-singular element v in V. Also, $\theta_3(\tau) = 0$ as $\theta_3(1) = 2$. We now compute the multiplicity $\langle 1_V, \theta_3\rangle_V$ of the principal character 1_V of V in θ_3. We have

$$\langle 1_V, \theta_3\rangle_V = \frac{1}{|V|}(\theta_3(1) + 9\theta_3(\tau) - 2 - 2 - 2 - 2)$$
$$= \frac{1}{18}(2 - 2 - 2 - 2 - 2) = -\frac{1}{3}.$$

This is absurd, and so case (h) for $|N_G(G_5)|$ is not possible. Cases (i), (j) and (k) are obviously not possible and hence we have proved that $t_3 \neq 1$. In a similar way, the case $t_3 = 2$ can also be ruled out. With this, we have proved that $t \neq 1$. Hence the number t of conjugate classes of elements of order 5 in G must be 2. Then by Theorems 1B and 1F, the 2 exceptional characters in $B_1(5)$ must have the common degree 323. Also, if t_2 and t_3 are the multiples of t corresponding to $B_2(5)$ and $B_3(5)$ respectively, then we have $t_2 = t_3 = 2$. Again by Theorems 1B and 1E, the 2 exceptional characters in $B_2(5)$ have the common degree 1938 and the 2 exceptional characters in $B_3(5)$ have the common degree 646. So we have the following block relations (we use the convention as introduced in §2)

$$\begin{aligned} B_1(5) &: \quad 1 + 323^* = 324 \\ B_2(5) &: \quad 816 + 1938^* = 2754 \\ B_3(5) &: \quad 2432 + 646^* = 3078 . \end{aligned}$$

REMARK 3. The sum of the squares of the degrees of the 20 irreducible characters of G found so far is 47,624,735 and so the sum of the squares of the remaining degrees of G must be $2{,}608{,}225 = 5^2.17^2.19^2$.

By Theorem 1C, let θ_2 and θ_3 be the irreducible characters of V corresponding to $B_2(5)$ and $B_3(5)$ respectively. By Theorem 1C and (3.2), we have

$$\chi_7(su) = \delta\theta_3(u) = 0 ,$$

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where $\delta = \pm 1$ and u is any 2-singular element of V. Hence $f_3 = \theta_3(1)$ must be divisible by the highest 2-power dividing the order of V. Then by arguments similar to those used to rule out the case $t = 1$, we find that case (k) is the only possibility for $|N_G(G_5)|$. Hence $|N_G(G_5)| = 60$ and $|C_G(G_5)| = 30$. By Corollary 3A, the group V must be a dihedral group of order 6 and has therefore 3 conjugate classes of elements. By Theorem 1A, $B_1(5)$, $B_2(5)$ and $B_3(5)$ must be the only 5-blocks of defect 1. So by Remark 3 above, the remaining irreducible character of G must have degree $1615 = 5.17.19$. By Corollary 3A, $N_G(G_5)$ must be the direct product of a dihedral group of order 6 by a dihedral group of order 10. Hence we have the following results:

LEMMA 3A. *The Sylow 5-normalizer* $N_G(G_5)$*, has order* 60 *and is the direct product of a dihedral group of order* 6 *by a dihedral group of order* 10.

LEMMA 3B. *The group* G *has* 21 *irreducible characters and hence has* 21 *conjugate classes of elements. The degrees of irreducible characters of* G *are:* 1, 1920, 1920, 1920, 324, 3078, 2432, 1140, 1215, 1215, 2754, 816, 85, 85, 323, 323, 1938, 1938, 646, 646 *and* 1615.

4. Completion of proof of Main Theorem

Let $G_5 = \langle s\rangle$ be a Sylow 5-subgroup of G and $N_3 = \langle v\rangle$, a Sylow 3-subgroup of $N_G(G_5)$. By Lemma 3A, let $N_G(G_5) = \langle t, v\rangle \times \langle \tau, s\rangle$, where t, τ are involutions and $\langle t, v\rangle$ is a dihedral group of order 6 and $\langle \tau, s\rangle$ a dihedral group of order 10. Any irreducible character of $N_G(G_5)$ has the form $\theta\lambda$, where θ is an irreducible character of $N_G(G_5)/\langle \tau, s\rangle$ and λ is an irreducible character of $N_G(G_5)/\langle t, v\rangle$. By an irreducible character of $N_G(G_5)/\langle \tau, s\rangle$, we mean an irreducible character of $N_G(G_5)$ with $\langle \tau, s\rangle$ in the kernel. Hence $N_G(G_5)$ has 12 irreducible characters and hence has 12 conjugate classes of elements with representatives:- 1, t, τ, $t\tau$, v, τv, s, s^2, ts, ts^2, vs and vs^2. Now let $H = N_G(G_5)$, 1_H the principal character of H, θ_1 the linear non-trivial character of $H/\langle \tau, s\rangle$, θ_2 the irreducible character

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of degree 2 of $H/\langle \tau , s \rangle$, λ_1 the non-trivial linear character of $H/\langle t , v \rangle$ and λ_2 , λ_3 two different irreducible characters of degree 2 of $H/\langle t , v \rangle$.

We shall now use the exceptional character theory of M. Suzuki [8]. We shall take the 6 conjugate classes of 5-singular elements in $H = N_G(G_5)$ as the collection D of "special classes" of H with respect to G . Hence we have 6 "special classes" and 6 non-special classes in H . The values of the irreducible characters 1_H , λ_1 and λ_2 of H on non-special classes are given below:-

Table 1

Element	Order of Element	No. of conjugates in H	1_H	λ_1	λ_2
1	1	1	1	1	2
t	2	3	1	1	2
τ	2	5	1	-1	0
$t\tau$	2	15	1	-1	0
v	3	2	1	1	2
τv	6	10	1	-1	0

We shall consider the following generalized character of H :

(4.1) $$\phi_1 = 1_H + \lambda_1 - \lambda_2 .$$

From Table 1, we see that ϕ_1 vanishes on all nonspecial classes of H .

Let ϕ_1^* be the induced generalized character of G corresponding to ϕ_1 . Since $\langle 1_H , \phi_1 \rangle_H = 1$, we have by the Frobenius reciprocity law $\langle 1_G , \phi_1^* \rangle_G = 1$. Also by a result of Suzuki [8], since $\langle \phi_1 , \phi_1 \rangle_H = 3$, we have $\langle \phi_1^* , \phi_1^* \rangle_G = 3$. Hence the generalized character ϕ_1^* of G has the following decomposition:

(4.2) $$\phi_1 = 1_G + \varepsilon_1 Y_1 + \varepsilon_2 Y_2 ,$$

where $\varepsilon_i = \pm 1$ $(i = 1, 2)$ and Y_1 , Y_2 are 2 distinct non-principal

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irreducible characters of G. By a result of Suzuki [8], $\phi_1^*(1) = 0$ and so we can assume $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$ in (4.2). By Lemma 3B, we must have $Y_1(1) = 323$ and $Y_2(1) = 324$, and so we have

$$\phi_1^* = 1_G + Y_1 - Y_2 , \tag{4.3}$$

where $Y_1(1) = 323$ and $Y_2(1) = 324$.

Now let z be an involution in the center of a Sylow 2-subgroup of G and let

$$c = |C_G(z)| , \quad Y_1(z) = \xi . \tag{4.4}$$

Then from (4.3) and a result of Suzuki [8], we have

$$Y_2(z) = 1 + \xi . \tag{4.5}$$

We are now in a position to apply Suzuki's order formula [8] to ϕ_1 and ϕ_1^*. Let I be the conjugate class of involutions of G with representative z, and $J = I \cap H$. Then by a result of Suzuki [8],

$$\frac{1}{|G|} \sum_{\sigma=0}^{2} \frac{(Y_\sigma(I))^2 b_\sigma}{Y_\sigma(1)} = \frac{1}{|H|} \sum_{\nu=0}^{2} \frac{(\lambda_\nu(J))^2 a_\nu}{\lambda_\nu(1)} , \tag{4.6}$$

where $Y_0 = 1_G$, $Y_\sigma(I) = \sum\limits_{x \in I} Y_\sigma(x)$, b_σ is the coefficient of Y_σ in (4.3); also, $\lambda_0 = 1_H$, $\lambda_\nu(J) = \sum\limits_{x \in J} \lambda_\nu(x)$ and a_ν is the coefficient of λ_ν in (4.1). If J is the empty set $\emptyset$, then we set $\lambda_\nu(\emptyset) = 0$ for $\nu = 0,1,2$. Since the order of G is $50,232,960 = 2^7.3^5.5.17.19$, we have from (4.3), (4.4), (4.5) and (4.6):

$$\frac{1}{|H|} \sum_{\nu=0}^{2} \frac{(\lambda_\nu(J))^2 a_\nu}{\lambda_\nu(1)} = \frac{2^5.3.5.(323-\xi)^2}{c^2} . \tag{4.7}$$

To evaluate the left-hand side of (4.7), we have to consider various cases for $J = I \cap H$:

Case (a). Suppose $J = \emptyset$, then the left hand side of (4.7) is 0, and we have $Y_1(z) = Y_1(1) = \xi = 323$, and so z belongs to the kernel of Y_1. This is not possible as G is simple, and so $J \neq \emptyset$.

Case (b). Suppose the involution t is in J and τ, $t\tau$ are not in J. Then from Table 1, J has 3 involutions which are conjugates of

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t in H . Hence $\lambda_1(J) = 3$ and $\lambda_2(J) = 6$. Again the left hand side of (4.7) is $\frac{1}{60}(3^2 + 3^2 - \frac{6^2}{2}) = 0$. So this case is also not possible.

Case (c). Suppose the involution τ is in J and t , $t\tau$ are not in J . Then the left-hand side of (4.7) is $\frac{1}{60}(5^2 + 5^2) = \frac{5}{6}$ and we have from (4.7) the following

$$c = 2^3.3.(323 - \xi) . \tag{4.8}$$

Since $\tau \in J$ and τ is centralized by an element of order 3, we have

$$c = |C_G(z)| = 2^7.3^i.5^j , \tag{4.9}$$

where $5 \geqq i \geqq 1$ and $1 \geqq j \geqq 0$.

$$\xi = 323 - 2^4.3^{i-1}.5^j , \quad 5 \geqq i \geqq 1 \text{ and } 1 \geqq j \geqq 0 . \tag{4.10}$$

From the orthogonality relations we find that ξ in (4.10) has no solutions and hence this case for J is also not possible. In a similar way, all other cases for J can be ruled out except the following:

Case (d). The 3 involutions t , τ and $t\tau$ are in J . Then from Table 1, $\lambda_1(J) = -17$, $\lambda_2(J) = 6$ and we have from (4.7)

$$\xi = 323 - \frac{c}{6} . \tag{4.11}$$

Since t and τ are in J , we must have

$$c = |C_G(z)| = 2^7.3^i.5 , \tag{4.12}$$

where $5 \geqq i \geqq 1$.

From (4.11) and (4.12), we have

$$\xi = 323 - 2^6.3^{i-1}.5 , \tag{4.13}$$

where $5 \geqq i \geqq 1$.

Since $|\xi| \leqq 323$, we have $i = 1$, $\xi = 3$ and $c = |C_G(z)| = 2^7.3.5$. Hence we have proved the following result:

LEMMA 4A. *Let z be an involution in the center of a Sylow 2-subgroup of the group G . Then the centralizer $C_G(z)$ of z in G has the order* 1920 .

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Now let z be an involution in the center of a Sylow 2-subgroup of G. Then by Lemma 4A, $|C_G(z)| = 2^7.3.5$. Let $C = C_G(z)$, then by Lemma 3A and a result of P. Hall ([5], Theorem 9.3.1), C cannot be soluble. Let $O_2(C)$ be the maximal normal 2-subgroup of C, then $C/O_2(C)$ must be a non-soluble group. Suppose $C/O_2(C)$ has a proper normal subgroup $\overline{H}$, and let H be the normal subgroup of C containing $O_2(C)$ such that $\overline{H} = H/O_2(C)$. Then $|H|$ must be divisible by 5 and H is non-soluble. By a Frattini lemma and Lemma 3A, we must have $[C : H] = 2$ or 4. If $[C : H] = 2$. then $|H| = 2^6.3.5$. Let G_5 be a Sylow 5-subgroup of H, then $|N_H(G_5)| = 2^2.3.5$, $2^5.5$ or 2.5. Hence by Lemma 3A, $|N_H(G_5)| = 2.5$ but z is in $N_H(G_5)$ and so by a transfer theorem of Burnside H is soluble, a contradition. If $[C : H] = 4$, then $|H| = 2^5.3.5$ and by Lemma 3A, $|N_H(G_5)| = 5$. Again H is soluble and so $[C : H] \neq 4$. With this, we have proved that $C/O_2(C)$ is a simple group and hence $C/O_2(C) \cong A_5$ (the alternating group in 5 letters). Let $E = O_2(C)$ and S a Sylow 2-subgroup of $C_G(z)$. Then S is also a Sylow 2-subgroup of G. Now let $Z(E)$ be the center of E. Then by Lemma 3A, $Z(E) = \langle z \rangle$ or $Z(E) = E$. If $Z(E) = E$, then E is an elementary abelian group of order 2^5, and since the factor group S/E is a Sylow 2-subgroup of A_5, S has no elements of order 8. This contradicts Lemma 2C and so $Z(E) = \langle z \rangle$. Let $Z(S)$ be the center of S. Since $C_G(z)/E \cong A_5$, we have $Z(S) \leq Z(E)$ and so $Z(S) = \langle z \rangle$. With this, the Main Theorem is proved.

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Note added on 21 April 1969. Since this paper was written, G. Higman and J. McKay have given a construction of a non-abelian simple group of order 50,232,960. For an announcement of their construction, see page 1036 of the *Notices of the American Mathematical Society* Volume 15, Number 7 (1968). Combining the Main Theorem of this paper with two further known results of J.G. Thompson and G. Higman the author has now shown (in his doctoral thesis) that there is precisely one (up to isomorphism) simple group of order 50,232,960.

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