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ON THE NUMBER OF REPRESENTATIONS OF INTEGERS BY CERTAIN QUADRATIC FORMS

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Abstract

Generating functions are used to derive formulas for the number of representations of a positive integer by each of the quadratic forms $x_1^2+x_2^2+x_3^2+2x_4^2$, $x_1^2+2x_2^2+2x_3^2+2x_4^2$, $x_1^2+x_2^2+2x_3^2+4x_4^2$ and $x_1^2+2x_2^2+4x_3^2+4x_4^2$. The formulas show that the number of representations by each form is always positive. Some of the analogous results involving sums of triangular numbers are also given.

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1. Introduction

Lagrange's four-squares theorem says that every positive integer may be expressed as a sum of four squares. Jacobi went further and used elliptic functions to prove that the number of solutions of

$$x_1^2+x_2^2+x_3^2+x_4^2=n$$

in integers is

$$8\sum_{\substack{d\mid n\\ d\not\equiv 0 \pmod 4}} d.$$

Since $d=1$ is a divisor of any positive integer n, the sum in Jacobi's theorem is positive and Lagrange's theorem follows immediately.

Fine [6, pp. 74–76] proved an analogue of Jacobi's theorem for the quadratic form $x_1^2+x_2^2+x_3^2+2x_4^2$, and showed that the number of representations of any positive integer by this form is always positive. The purpose of this paper is to present a short and simple proof of Fine's result, and to obtain analogous results for the quadratic forms $x_1^2+2x_2^2+2x_3^2+2x_4^2$, $x_1^2+x_2^2+2x_3^2+4x_4^2$ and $x_1^2+2x_2^2+4x_3^2+4x_4^2$. The proof depends on simple properties of theta functions and a well-known formula which expresses a difference of two Weierstrass elliptic functions as an infinite product.

2. Notation and lemmas

Let q be a complex number that satisfies $|q| < 1$. Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$
$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Some of their basic properties are summarized in the following three lemmas.

LEMMA 2.1.

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{2.1}$$
$$\varphi(q) - \varphi(-q) = 4q\psi(q^8), \tag{2.2}$$
$$\varphi(q)^2 + \varphi(-q)^2 = 2\varphi(q^2)^2, \tag{2.3}$$
$$\varphi(q)^2 - \varphi(-q)^2 = 8q\psi(q^4)^2. \tag{2.4}$$

PROOF. These can be proved using series manipulations. A complete proof is given in [3, pp. 40–41]. □

LEMMA 2.2.

$$\varphi(-q) = \prod_{j=1}^{\infty} \frac{(1-q^j)^2}{(1-q^{2j})}, \tag{2.5}$$
$$\psi(q) = \prod_{j=1}^{\infty} \frac{(1-q^{2j})^2}{(1-q^j)}. \tag{2.6}$$

PROOF. These are consequences of the Jacobi triple product identity. See [3, pp. 36–37] for proofs. □

LEMMA 2.3.

$$\varphi(-q^2)^2 = \varphi(q)\varphi(-q), \tag{2.7}$$
$$\psi(q)^2 = \varphi(q)\psi(q^2). \tag{2.8}$$

PROOF. These are consequences of Lemma 2.2. See [3, p. 40] for proofs. □

The final lemma in this section, and the main tool for this work, is a classical result that expresses a quotient of Weierstrass sigma functions as a difference of two Weierstrass elliptic functions.

LEMMA 2.4. *Let x, z and q be any complex numbers satisfying $|q| < |x|, |z| < |q|^{-1}$, and $x, z \neq 1$. Then*

$$\frac{(1-xz)(x-z)}{(1-x)^2(1-z)^2} \times \prod_{j=1}^{\infty} \frac{(1-xzq^j)(1-q^jx/z)(1-q^jz/x)(1-q^j/(xz))(1-q^j)^4}{(1-xq^j)^2(1-zq^j)^2(1-q^j/x)^2(1-q^j/z)^2}$$
$$= \frac{x}{(1-x)^2} - \frac{z}{(1-z)^2} + \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}(x^j + x^{-j} - z^j - z^{-j}).$$

PROOF. There are many proofs of this result in the literature. For a simple one, see [5]. For an equivalent statement in terms of the Weierstrass functions, see [7, p. 451, Exercise 1]. For generalizations, see [4]. □

We shall use the following special values of Jacobi's symbol:

$$\left(\frac{2}{n}\right) = \begin{cases} 1 & \text{if } n \equiv 1 \text{ or } 7 \pmod 8; \\ -1 & \text{if } n \equiv 3 \text{ or } 5 \pmod 8; \\ 0 & \text{otherwise;} \end{cases}$$

where n is a positive integer. The Jacobi symbol is multiplicative, that is,

$$\left(\frac{2}{mn}\right) = \left(\frac{2}{m}\right)\left(\frac{2}{n}\right)$$

for all positive integers m and n.

3. Theta function identities

The results in this section all hinge on the following result.

THEOREM 3.1.

$$\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 = 1 - 2\sum_{j=1}^{\infty}\left(\frac{2}{j}\right)\frac{jq^j}{1-q^j},$$
$$q\psi(q)^2\psi(q^2)\psi(q^4) = \sum_{j=1}^{\infty} \frac{j(q^j - q^{3j} - q^{5j} + q^{7j})}{1-q^{8j}}.$$

PROOF. Let $\omega = \exp(i\pi/4)$ and take $x = \omega^3$, $z = \omega$, in Lemma 2.4 to get

$$\frac{2(\omega^3-\omega)}{(1-\omega)^2(1-\omega^3)^2} \prod_{j=1}^{\infty} \frac{(1+q^j)^2(1+q^{2j})(1-q^j)^4}{(1-\omega q^j)^2(1-\omega^3 q^j)^2(1-\omega^5 q^j)^2(1-\omega^7 q^j)^2}$$
$$= \frac{\omega^3}{(1-\omega^3)^2} - \frac{\omega}{(1-\omega)^2} + \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}(\omega^{3j} + \omega^{-3j} - \omega^j - \omega^{-j}). \qquad (3.1)$$

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By straightforward calculations,

$$\frac{2(\omega^3-\omega)}{(1-\omega)^2(1-\omega^3)^2}=\frac{\omega^3}{(1-\omega^3)^2}-\frac{\omega}{(1-\omega)^2}=\sqrt{2}.$$

Next, using the algebraic rearrangements

$$1+x=\frac{1-x^2}{1-x}$$

and

$$(1-\omega x)(1-\omega^3 x)(1-\omega^5 x)(1-\omega^7 x)=\frac{1-x^8}{1-x^4},$$

the left-hand side of (3.1) simplifies to

$$\sqrt{2}\prod_{j=1}^{\infty}\frac{(1-q^j)^2(1-q^{2j})(1-q^{4j})^3}{(1-q^{8j})^2},$$

which, by (2.5), equals

$$\sqrt{2}\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2. \tag{3.2}$$

Furthermore, by checking the residue classes modulo 8, we find that

$$\omega^{3j}+\omega^{-3j}-\omega^j-\omega^{-j}=-2\sqrt{2}\left(\frac{2}{j}\right).$$

Therefore, the right-hand side of (3.1) simplifies to

$$\sqrt{2}-2\sqrt{2}\sum_{j=1}^{\infty}\left(\frac{2}{j}\right)\frac{jq^j}{1-q^j}. \tag{3.3}$$

Combining (3.2) and (3.3), we complete the proof of the first result.

Now let us prove the second result. In Lemma 2.4, replace (q, x, z) with (q^8, q, q^3) to get

$$\begin{aligned} q\prod_{j=1}^{\infty}&\frac{(1-q^{8j-4})^2(1-q^{8j-6})(1-q^{8j-2})(1-q^{8j})^4}{(1-q^{8j-7})^2(1-q^{8j-5})^2(1-q^{8j-3})^2(1-q^{8j-1})^2}\\ &=\frac{q}{(1-q)^2}-\frac{q^3}{1-q^3}+\sum_{j=1}^{\infty}\frac{jq^{8j}}{1-q^{8j}}(q^j+q^{-j}-q^{3j}-q^{-3j}). \end{aligned} \tag{3.4}$$

The left-hand side of (3.4) simplifies to

$$q\prod_{j=1}^{\infty}\frac{(1-q^{2j})^3(1-q^{4j})(1-q^{8j})^2}{(1-q^j)^2}$$

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which, by (2.6), equals

$$q\psi(q)^2\psi(q^2)\psi(q^4). \tag{3.5}$$

The right-hand side of (3.4) is

$$\begin{aligned}
&\sum_{j=1}^{\infty} j(q^j - q^{3j}) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j\{q^{(8k+1)j} + q^{(8k-1)j} - q^{(8k+3)j} - q^{(8k-3)j}\} \\
&\quad = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j\{q^{(8k-7)j} + q^{(8k-1)j} - q^{(8k-5)j} - q^{(8k-3)j}\} \\
&\quad = \sum_{j=1}^{\infty} \frac{j(q^j - q^{3j} - q^{5j} + q^{7j})}{1 - q^{8j}}.
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we complete the proof of the second result. □

The next theorem gives Lambert series expansions for four different products of theta functions.

THEOREM 3.2. *Let*

$$f_1(q) = 1 - 2\sum_{j=1}^{\infty}\left(\frac{2}{j}\right)\frac{jq^j}{1-q^j} \quad \textit{and} \quad f_2(q) = \sum_{j=1}^{\infty}\frac{j(q^j - q^{3j} - q^{5j} + q^{7j})}{1-q^{8j}}.$$

Then

$$\begin{aligned}
\varphi(q)^3\varphi(q^2) &= f_1(q) + 8f_2(q), \\
\varphi(q)\varphi(q^2)^3 &= f_1(q) + 4f_2(q), \\
\varphi(q)^2\varphi(q^2)\varphi(q^4) &= f_1(q^2) + 4f_2(q), \\
\varphi(q)\varphi(q^2)\varphi(q^4)^2 &= f_1(q^2) + 2f_2(q).
\end{aligned}$$

PROOF. By (2.4) followed by two applications of each of (2.7) and (2.8),

$$\begin{aligned}
\varphi(q)^3\varphi(q^2) &= \varphi(q)\varphi(q^2)\{\varphi(-q)^2 + 8q\psi(q^4)^2\} \\
&= \varphi(-q)\varphi(-q^2)^2\varphi(q^2) + 8q\varphi(q)\psi(q^2)^2\psi(q^4) \\
&= \varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 + 8q\psi(q)^2\psi(q^2)\psi(q^4).
\end{aligned} \tag{3.7}$$

Similarly, by (2.3) and (2.4), followed by two applications of each of (2.7) and (2.8),

$$\begin{aligned}
\varphi(q)\varphi(q^2)^3 &= \varphi(q)\varphi(q^2)\{\varphi(-q)^2 + 4q\psi(q^4)^2\} \\
&= \varphi(-q)\varphi(-q^2)^2\varphi(q^2) + 4q\varphi(q)\psi(q^2)^2\psi(q^4) \\
&= \varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 + 4q\psi(q)^2\psi(q^2)\psi(q^4).
\end{aligned} \tag{3.8}$$

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By (2.2) followed by three applications of each of (2.7) and (2.8),

$$\begin{aligned}\varphi(q)^2\varphi(q^2)\varphi(q^4) &= \varphi(q)\varphi(q^2)\varphi(q^4)\{\varphi(-q)+4q\psi(q^8)\}\\ &= \varphi(-q^2)^2\varphi(q^2)\varphi(q^4)+4q\varphi(q)\varphi(q^2)\psi(q^4)^2\\ &= \varphi(-q^2)\varphi(-q^4)^2\varphi(q^4)+4q\varphi(q)\psi(q^2)^2\psi(q^4)\\ &= \varphi(-q^2)\varphi(-q^4)\varphi(-q^8)^2+4q\psi(q)^2\psi(q^2)\psi(q^4). \quad (3.9)\end{aligned}$$

Similarly, by (2.1) and (2.2), followed by three applications of each of (2.7) and (2.8), we have (omitting some details which are similar to the above)

$$\begin{aligned}\varphi(q)\varphi(q^2)\varphi(q^4)^2 &= \varphi(q)\varphi(q^2)\varphi(q^4)\{\varphi(-q)+2q\psi(q^8)\}\\ &= \varphi(-q^2)\varphi(-q^4)\varphi(-q^8)^2+2q\psi(q)^2\psi(q^2)\psi(q^4). \quad (3.10)\end{aligned}$$

Substituting the results of Theorem 3.1 into the right-hand sides of (3.7)–(3.10), we complete the proof. □

Results involving the function $\psi(q)$ (and sometimes $\varphi(q)$ as well) can be obtained by dissecting one of the results in Theorem 3.1. These are given in the following result.

THEOREM 3.3.

$$q\psi(q^8)\varphi(q^8)^3 = \sum_{n=0}^{\infty}\bigg(\sum_{d|8n+1}\left(\frac{2}{d}\right)d\bigg)q^{8n+1},$$

$$q^3\psi(q^8)^3\varphi(q^8) = -\frac{1}{2}\sum_{n=0}^{\infty}\bigg(\sum_{d|8n+3}\left(\frac{2}{d}\right)d\bigg)q^{8n+3},$$

$$q^5\psi(q^8)^3\psi(q^{16}) = -\frac{1}{4}\sum_{n=0}^{\infty}\bigg(\sum_{d|8n+5}\left(\frac{2}{d}\right)d\bigg)q^{8n+5},$$

$$q^7\psi(q^8)\psi(q^{16})^3 = \frac{1}{8}\sum_{n=0}^{\infty}\bigg(\sum_{d|8n+7}\left(\frac{2}{d}\right)d\bigg)q^{8n+7}.$$

PROOF. If $f(q)=\sum_{n=0}^{\infty}c(n)q^n$, let us define $[q;a,b]f(q)$ by

$$[q;a,b]f(q)=\sum_{n=0}^{\infty}c(an+b)q^{an+b}.$$

By Lemma 2.1,

$$\begin{aligned}&\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2\\ &\quad=\{\varphi(q^4)-2q\psi(q^8)\}\{\varphi(q^8)-2q^2\psi(q^{16})\}\{\varphi(q^8)^2-4q^4\psi(q^{16})^2\}.\end{aligned}$$

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If we expand and extract terms of the form q^{8n+1}, q^{8n+3}, q^{8n+5} and q^{8n+7}, we obtain

$$[q;\,8,\,1]\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 = -2q\psi(q^8)\varphi(q^8)^3, \tag{3.11}$$

$$\begin{aligned}[q;\,8,\,3]\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 &= 4q^3\psi(q^8)\psi(q^{16})\varphi(q^8)^2\\ &= 4q^3\psi(q^8)^3\varphi(q^8) \quad \text{by (2.8)},\end{aligned} \tag{3.12}$$

$$\begin{aligned}[q;\,8,\,5]\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 &= 8q^5\psi(q^8)\varphi(q^8)\psi(q^{16})^2\\ &= 8q^5\psi(q^8)^3\psi(q^{16}) \quad \text{by (2.8)}\end{aligned} \tag{3.13}$$

and

$$[q;\,8,\,7]\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 = -16q^7\psi(q^8)\psi(q^{16})^3, \tag{3.14}$$

respectively. Next, by the first part of Theorem 3.1,

$$\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 = 1 - 2\sum_{n=1}^{\infty}\bigg(\sum_{d\mid n}\Big(\frac{2}{d}\Big)d\bigg)q^n. \tag{3.15}$$

Therefore, for $1 \le k \le 8$,

$$[q;\,8,\,k]\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2 = -2\sum_{n=0}^{\infty}\bigg(\sum_{d\mid 8n+k}\Big(\frac{2}{d}\Big)d\bigg)q^{8n+k}. \tag{3.16}$$

Using (3.16) in (3.11)–(3.14) we complete the proof. □

4. Combinatorial interpretations

In this section, we give combinatorial interpretations of the results in Theorems 3.1–3.3. We begin with a lemma.

LEMMA 4.1. *Let n be a positive integer and write $n = 2^\alpha m$, where m is odd and α is a nonnegative integer. Then:*

(1) *the coefficient of q^n in $\varphi(-q)\varphi(-q^2)\varphi(-q^4)^2$ is $-2\sum_{d\mid m}(2/d)d$;*
(2) *the coefficient of q^n in $q\psi(q)^2\psi(q^2)\psi(q^4)$ is $2^\alpha(2/m)\sum_{d\mid m}(2/d)d$.*

PROOF. The first result is immediate from (3.15), together with the fact that $(2/d) = 0$ if d is even. To prove the second result, observe that from Theorem 3.1,

$$q\psi(q)^2\psi(q^2)\psi(q^4) = \sum_{j=1}^{\infty}\frac{j(q^j - q^{3j} - q^{5j} + q^{7j})}{1 - q^{8j}} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big(\frac{2}{k}\Big)jq^{jk}.$$

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The coefficient of q^n is therefore

$$\begin{aligned}\sum_{jk=n}\left(\frac{2}{k}\right)j &= \sum_{\substack{jk=n\\ k \text{ odd}}}\left(\frac{2}{k}\right)j\\ &= \sum_{ik=m}\left(\frac{2}{k}\right)2^{\alpha} i\\ &= 2^{\alpha}\left(\frac{2}{m}\right)\sum_{ik=m}\left(\frac{2}{i}\right)i\\ &= 2^{\alpha}\left(\frac{2}{m}\right)\sum_{d\mid m}\left(\frac{2}{d}\right)d,\end{aligned}$$

where the multiplicative property of the Jacobi symbol was used in the penultimate step. □

Let a, b, c and d be positive integers. Let $r_{(a,b,c,d)}(n)$ denote the number of solutions of $ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 = n$ in integers (positive, negative and/or zero). The generating function is

$$\sum_{n=0}^{\infty} r_{(a,b,c,d)}(n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d).$$

THEOREM 4.2. *If n is a positive integer, write $n = 2^{\alpha}m$, where m is odd and α is a nonnegative integer. Then*

$$\begin{aligned} r_{(2,1,1,1)}(n) &= \left\{2^{\alpha+3}\left(\frac{2}{m}\right) - 2\right\}\sum_{d\mid m}\left(\frac{2}{d}\right)d,\\ r_{(2,2,2,1)}(n) &= \left\{2^{\alpha+2}\left(\frac{2}{m}\right) - 2\right\}\sum_{d\mid m}\left(\frac{2}{d}\right)d,\\ r_{(4,2,1,1)}(n) &= \left\{2^{\alpha+2}\left(\frac{2}{m}\right) - 1 - (-1)^n\right\}\sum_{d\mid m}\left(\frac{2}{d}\right)d,\\ r_{(4,4,2,1)}(n) &= \left\{2^{\alpha+1}\left(\frac{2}{m}\right) - 1 - (-1)^n\right\}\sum_{d\mid m}\left(\frac{2}{d}\right)d.\end{aligned}$$

PROOF. Use Lemma 4.1 to extract the coefficient of q^n from each of the results in Theorem 3.2. □

THEOREM 4.3. *Let o_j and e_j denote an odd integer and an even integer, respectively. For a positive integer n, let $A(n)$, $B(n)$, $C(n)$, $D(n)$ and $E(n)$ denote the number of*

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solutions of

$$\begin{aligned} o_1^2+2e_2^2+2e_3^2+2e_4^2&=n,\\ o_1^2+o_2^2+o_3^2+2e_4^2&=n,\\ o_1^2+o_2^2+o_3^2+2o_4^2&=n,\\ o_1^2+2o_2^2+2o_3^2+2o_4^2&=n \end{aligned}$$

and

$$o_1^2+o_2^2+2o_3^2+4o_4^2=n,$$

respectively. Then

$$\sum_{d|n}\left(\frac{2}{d}\right)d=\begin{cases} \frac{1}{2}A(n) & \text{if } n\equiv 1 \pmod 8,\\ -\frac{1}{4}B(n) & \text{if } n\equiv 3 \pmod 8,\\ -\frac{1}{4}C(n) & \text{if } n\equiv 5 \pmod 8,\\ \frac{1}{2}D(n) & \text{if } n\equiv 7 \pmod 8. \end{cases} \tag{4.1}$$

Since $A(n)$, $B(n)$, $C(n)$ and $D(n)$ are zero unless $n\equiv 1$, 3, 5 or 7, respectively, (4.1) completely determines the values of $A(n)$, $B(n)$, $C(n)$ and $D(n)$.

If we write $n=2^\alpha m$, where m is odd and α is a nonnegative integer, then

$$E(n)=2^{\alpha+1}\left(\frac{2}{m}\right)\sum_{d|n}\left(\frac{2}{d}\right)d \quad \text{if } n\equiv 0 \pmod 8, \tag{4.2}$$

and $E(n)=0$ otherwise.

PROOF. Observe that

$$\begin{aligned} \sum_{n=0}^{\infty}A(n)q^n&=\sum_{x_1,x_2,x_3,x_4=-\infty}^{\infty}q^{(2x_1+1)^2+2(2x_2)^2+2(2x_3)^2+2(2x_4)^2} \\ &=2q\psi(q^8)\varphi(q^8)^3. \end{aligned} \tag{4.3}$$

Similarly, we find that

$$\sum_{n=0}^{\infty}B(n)q^n=8q^3\psi(q^8)^3\varphi(q^8), \tag{4.4}$$

$$\sum_{n=0}^{\infty}C(n)q^n=16q^5\psi(q^8)^3\psi(q^{16}), \tag{4.5}$$

$$\sum_{n=0}^{\infty}D(n)q^n=16q^7\psi(q^8)\psi(q^{16})^3, \tag{4.6}$$

$$\sum_{n=0}^{\infty}E(n)q^n=16q^8\psi(q^8)^2\psi(q^{16})\psi(q^{32}). \tag{4.7}$$

The identity (4.1) follows from these by comparing coefficients using Theorem 3.3, while (4.2) follows from Lemma 4.1. □

The results for $C(n)$, $D(n)$ and $E(n)$ in Theorem 4.3 may be re-expressed in terms of sums of triangular numbers. Let a, b, c and d be positive integers, and let $t_{(a,b,c,d)}(n)$ denote the number of solutions in nonnegative integers of

$$\frac{ax_1(x_1+1)}{2}+\frac{bx_2(x_2+1)}{2}+\frac{cx_3(x_3+1)}{2}+\frac{dx_4(x_4+1)}{2}=n.$$

The generating function for $t_{(a,b,c,d)}(n)$ is

$$\sum_{n=0}^{\infty} t_{(a,b,c,d)}(n)q^n=\psi(q^a)\psi(q^b)\psi(q^c)\psi(q^d).$$

THEOREM 4.4. *Let n be a positive integer and write $n=2^{\alpha}m$, where m is odd and α is a nonnegative integer. Then*

$$\begin{aligned} t_{(2,1,1,1)}(n) &= -\frac{1}{4}\sum_{d\mid 8n+5}\left(\frac{2}{d}\right)d,\\ t_{(2,2,2,1)}(n) &= \frac{1}{8}\sum_{d\mid 8n+7}\left(\frac{2}{d}\right)d,\\ t_{(4,2,1,1)}(n-1) &= 2^{\alpha}\left(\frac{2}{m}\right)\sum_{d\mid n}\left(\frac{2}{d}\right)d. \end{aligned}$$

PROOF. From the generating function for $t_{(2,1,1,1)}(n)$ and (4.5),

$$\sum_{n=0}^{\infty} t_{(2,1,1,1)}(n)q^{8n+5}=q^5\psi(q^8)^3\psi(q^{16})=\frac{1}{16}\sum_{n=0}^{\infty}C(n)q^n.$$

Equating coefficients and applying Theorem 4.3, we find that

$$t_{(2,1,1,1)}(n)=\frac{1}{16}C(8n+5)=-\frac{1}{4}\sum_{d\mid 8n+5}\left(\frac{2}{d}\right)d.$$

The other two results may be proved similarly. □

Analogues of Lagrange's theorem can be deduced from Theorems 4.2 and 4.3.

THEOREM 4.5. *Let n be a nonnegative integer. Then $r_{(2,1,1,1)}(n)$, $r_{(2,2,2,1)}(n)$, $r_{(4,2,1,1)}(n)$ and $r_{(4,4,2,1)}(n)$ are all positive, as are $A(8n+1)$, $B(8n+3)$, $C(8n+5)$, $D(8n+7)$ and $E(8n+8)$.*

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PROOF. Let m be an odd integer, and let $m = \prod_p p^{\alpha_p}$ be its prime factorization. By geometric series,

$$\left(\frac{2}{m}\right) \sum_{d|m} \left(\frac{2}{d}\right) d = \prod_p \frac{p^{\alpha_p+1} - (2/p)^{\alpha_p+1}}{p - (2/p)},$$

which is clearly positive. The truth of the theorem follows from this, together with the results in Theorems 4.2 and 4.3. The trivial observation

$$r_{(2,1,1,1)}(0) = r_{(2,2,2,1)}(0) = r_{(4,2,1,1)}(0) = r_{(4,4,2,1)}(0) = 1$$

takes care of the case $n = 0$. □

REMARK 4.6. It is interesting to compare the results for $r_\lambda(n)$ with the corresponding results for $t_\lambda(n)$ in Theorem 4.4. We find that

$$r_{(2,1,1,1)}(8n+5) = -10 \sum_{d|8n+5} \left(\frac{2}{d}\right) d = 40 t_{(2,1,1,1)}(n)$$

and

$$r_{(2,2,2,1)}(8n+7) = 2 \sum_{d|8n+7} \left(\frac{2}{d}\right) d = 16 t_{(2,2,2,1)}(n).$$

These are instances of a theorem in [1], which says that $r_{(\lambda_1,\ldots,\lambda_k)}(8n + \lambda_1 + \cdots + \lambda_k)$ is a constant (independent of n) multiple of $t_{(\lambda_1,\ldots,\lambda_k)}(n)$, provided $\lambda_1 + \cdots + \lambda_k < 8$. Furthermore,

$$\begin{aligned} r_{(4,2,1,1)}(8n) - r_{(4,2,1,1)}(2n) &= (2^{\alpha+5} - 2^{\alpha+3}) \left(\frac{2}{m}\right) \sum_{d|m} \left(\frac{2}{d}\right) d \\ &= 24 \times 2^{\alpha} \left(\frac{2}{m}\right) \sum_{d|m} \left(\frac{2}{d}\right) d \\ &= 24 t_{(4,2,1,1)}(n-1), \end{aligned}$$

and this is an instance of a theorem in [2].

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