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A RESTRICTION OF *EUCLID*

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Abstract

Euclid is a well-known two-player impartial combinatorial game. A position in *Euclid* is a pair of positive integers and the players move alternately by subtracting a positive integer multiple of one of the integers from the other integer without making the result negative. The player who makes the last move wins. There is a variation of *Euclid* due to Grossman in which the game stops when the two entries are equal. We examine a further variation which we called *M-Euclid* where the game stops when one of the entries is a positive integer multiple of the other. We solve the Sprague–Grundy function for *M-Euclid* and compare the Sprague–Grundy functions of the three games.

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1. Introduction

Euclid is a two-player impartial combinatorial game, introduced by Cole and Davie [5]. In *Euclid*, a position is a pair of positive integers. The players move alternately, and each move is to subtract a positive integer multiple of one of the entries from the other without making the result negative. The player who reduces one of the entries to zero wins. In the variation of *Euclid* due to Grossman [6], the game stops when the two entries are equal. Various aspects of *Euclid* and Grossman's game have been examined in the literature; see the references in [4].

In this note, we examine a variation, which we call *M-Euclid*, where the game stops when one of the entries is a positive integer multiple of the other. Recall that the *Sprague–Grundy function* of a game is defined recursively as follows: the terminal positions have value 0, and the value of a position p is the smallest nonnegative integer m such that there is no move from p to a position with value m, but for all $0 \le i < m$ there is a move from p to a position with value i. We denote the Sprague–Grundy functions of *Euclid*, Grossman's game and *M-Euclid* by $\mathcal{G}_E$, $\mathcal{G}_G$ and $\mathcal{G}_M$, respectively. We first recall the results for $\mathcal{G}_E$ and $\mathcal{G}_G$. The convention here is that we write continued fractions $[a_0, a_1, \dots, a_n]$ so that $a_n > 1$ if $n > 0$.

THEOREM 1.1 [4, 7]. *Let $0 < a < b$. Consider the continued fraction expansion $[a_0, a_1, \ldots, a_n]$ of b/a, and let $\mathcal{I}(a, b)$ be the largest nonnegative integer i such that $a_0 = \cdots = a_{i-1} \leq a_i$. Then*

$$\mathcal{G}_E(a, b) = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0 & \text{if } \mathcal{I}(a, b) \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore, for Grossman's game, $\mathcal{G}_G(a, b) = \mathcal{G}_E(a, b)$ except when $a_0 = a_1 = \cdots = a_n$, in which case

$$\mathcal{G}_G(a, b) = \mathcal{G}_E(a, b) - (-1)^{\mathcal{I}(a,b)}.$$

Typically, small variations in the terminal condition of a combinatorial game can produce wildly different Sprague–Grundy functions. Interestingly, the Sprague–Grundy functions of *Euclid*, Grossman's game and *M-Euclid* are closely related. We have the following theorem.

THEOREM 1.2. *Let $0 < a < b$ where b is not a multiple of a. Consider the continued fraction expansion $[a_0, a_1, \ldots, a_n]$ of b/a, and let $\mathcal{J}(a, b)$ be the largest nonnegative integer $j < n$ such that $a_0 = \cdots = a_{j-1} \leq a_j$. Then*

$$\mathcal{G}_M(a, b) = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0 & \text{if } \mathcal{J}(a, b) \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

REMARK 1.3. We draw the reader's attention to the subtle difference in the definitions of $\mathcal{I}(a, b)$ and $\mathcal{J}(a, b)$. For $\mathcal{J}(a, b)$ we have imposed $\mathcal{J}(a, b) < n$. So $\mathcal{J}(a, b) = \min\{\mathcal{I}(a, b), n - 1\}$.

COROLLARY 1.4. *With the notation of Theorems 1.1 and 1.2, $\mathcal{G}_M(a, b) = \mathcal{G}_E(a, b)$ except when $a_0 = a_1 = \cdots = a_{n-1} \leq a_n$, in which case*

$$\mathcal{G}_M(a, b) = \mathcal{G}_E(a, b) - (-1)^{\mathcal{I}(a,b)}.$$

Furthermore, $\mathcal{G}_M(a, b) = \mathcal{G}_G(a, b)$ except when $a_0 = a_1 = \cdots = a_{n-1} < a_n$, in which case

$$\mathcal{G}_M(a, b) = \mathcal{G}_G(a, b) - (-1)^{\mathcal{I}(a,b)}.$$

Having found the right formulation of Theorem 1.2, its proof is straightforward. We follow closely the proof of [4, Theorem 1].

This paper continues our investigations of variations of *Euclid* and related questions; see [1–4].

2. Proof of Theorem 1.2

For convenience we write $\mathcal{G}$ instead of $\mathcal{G}_M$ and, in an abuse of language, we write $\mathcal{J}(p)$ and $\mathcal{G}(p)$ for their values at a position $p = [a_0, a_1, \ldots, a_n]$. It suffices to establish the following two properties.

(1) For every move $p \mapsto q$, we have $\mathcal{G}(q) \neq \mathcal{G}(p)$.
(2) If $\mathcal{G}(p) > 0$, then for all integers k with $0 \le k < \mathcal{G}(p)$, there exists a move $p \mapsto q$ such that $\mathcal{G}(q) = k$.

We will make repeated use of the following fact: if $p = [a_0, a_1, \ldots, a_n]$ and $\mathcal{J}(p)$ is odd, then $a_0 \le a_1$ and $n > 1$; indeed, if $n = 1$ or $a_0 > a_1$, then we would have $\mathcal{J}(p) = 0$. Similarly, if $\mathcal{J}(p)$ is even then either $a_0 \ge a_1$ or $n = 1$.

First observe that Theorem 1.2 holds for $n = 1$. Indeed, clearly $\mathcal{G}(1, a_1) = 1$ for all a_1 and hence, by induction, $\mathcal{G}([a_0, a_1]) = a_0$ for all n. Since $a_0 = \lfloor b/a \rfloor$ and $\mathcal{J}(a, b) = 0$, the result follows. So we need only deal with positions p having $n > 1$.

To establish (1), suppose that we have a move $p \mapsto q$ with $\mathcal{G}(q) = \mathcal{G}(p)$. First suppose that $q = [a_0 - i, a_1, \ldots, a_n]$ for some $1 \le i < a_0$. From the definition of $\mathcal{G}$, it is clear that $i = 1$, $\mathcal{J}(p)$ is odd and $\mathcal{J}(q)$ is even. As $\mathcal{J}(p)$ is odd, $a_0 \le a_1$, and so as $\mathcal{J}(q)$ is even, $a_0 - 1 \ge a_1$. Hence $a_0 \le a_1 \le a_0 - 1$, which is impossible. So we may assume that $q = [a_1, \ldots, a_n]$. At first sight, as $\mathcal{G}(q) = \mathcal{G}(p)$, there are three possibilities:

(i) $a_0 = a_1 - 1$ and $\mathcal{J}(p)$ is even and $\mathcal{J}(q)$ is odd;
(ii) $a_0 = a_1 + 1$ and $\mathcal{J}(p)$ is odd and $\mathcal{J}(q)$ is even;
(iii) $a_0 = a_1$ and $\mathcal{J}(p)$ and $\mathcal{J}(q)$ have the same parity.

But case (i) is impossible, since $a_0 \ge a_1$ when $\mathcal{J}(p)$ is even, case (ii) is impossible since $a_0 \le a_1$ when $\mathcal{J}(p)$ is odd, and case (iii) contradicts the definition of $\mathcal{J}$.

To establish (2), suppose that $0 \le k < \mathcal{G}(p)$. First suppose that $\mathcal{J}(p)$ is odd, so $\mathcal{G}(p) = a_0 - 1$. Consider the position $q = [k + 1, a_1, \ldots, a_n]$. Since $\mathcal{J}(p)$ is odd, $a_0 \le a_1$. In particular, $k + 1 < a_1$ and thus $\mathcal{J}(q) = 1$. It follows that $\mathcal{G}(q) = k$, as required. So it remains to treat the case where $\mathcal{J}(p)$ is even. In this case, $\mathcal{G}(p) = a_0$ and $a_0 \ge a_1$.

We first treat the situation where $k = 0$. Assume for the moment that $a_0 > 1$. Consider $q = [1, a_1, \ldots, a_n]$. Notice that we may assume that $\mathcal{J}(q)$ is even, since otherwise $\mathcal{G}(q) = 0$, as required. In particular, we have $a_1 = 1$. Let $q' = [a_1, \ldots, a_n]$. But if $\mathcal{J}(q)$ is even, then $\mathcal{J}(q')$ is odd and hence $\mathcal{G}(q') = a_1 - 1 = 0$, as required. Similarly, if $a_0 = 1$, then as $\mathcal{J}(p)$ is even, we have $a_1 = 1$, and since $\mathcal{J}(p)$ is even, $\mathcal{J}(q')$ is odd and $\mathcal{G}(q') = 0$. This completes the case $k = 0$.

Now suppose that $0 < k < \mathcal{G}(p)$ and let $q = [k, a_1, \ldots, a_n]$. If $\mathcal{J}(q)$ is even, then $\mathcal{G}(q) = k$, as required. So we may assume that $\mathcal{J}(q)$ is odd and thus $k \le a_1$. In this case, we have $\mathcal{G}(q) = k - 1$. Let $q' = [k + 1, a_1, \ldots, a_n]$. If $\mathcal{J}(q')$ is odd, then $\mathcal{G}(q') = k$, as required, so we may assume that $\mathcal{J}(q')$ is even, and therefore $k + 1 \ge a_1$. Thus $k + 1 \ge a_1 \ge k$. Hence, either $k + 1 = a_1$ or $k = a_1$. Consider $q'' = [a_1, \ldots, a_n]$. If $k = a_1$, then as $\mathcal{J}(q)$ is odd, $\mathcal{J}(q'')$ is even, and hence $\mathcal{G}(q'') = a_1 = k$, as required. Finally, if $k + 1 = a_1$, then as $\mathcal{J}(q')$ is even, $\mathcal{J}(q'')$ is odd, and hence $\mathcal{G}(q'') = a_1 - 1 = k$, as required.

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