A STUDY OF TENSORS WHICH CHARACTERIZE A HYPERSURFACE OF A FINSLER SPACE

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1. Introduction. The literature on Finsler geometry contains more than one definition for the normal curvature vector of a hypersurface and for coefficients of the second fundamental form; see Berwald (**1**), Davies (**3**), and Rund (**5**). In the first case this situation has arisen from the basically different approach to the subject adopted by the authors; Davies, following the locally Euclidean school and Rund the locally Minkowskian theory. In both cases, a comparison of the definitions shows that they are linked by expressions in a vector M_α which was introduced in the paper by Rund (**7**). In the same paper, it was shown that the relationship between the induced and intrinsic curvature theories depends to a large extent upon a tensor $M_{\alpha\beta}$. We investigate both M_α and $M_{\alpha\beta}$ and find that the conditions $M_\alpha = 0$ and $M_{\alpha\beta} = 0$ lead to two classes of hypersurfaces whose properties are intermediate between those of Finsler and Riemannian hypersurfaces.

In § 1 it will be shown that M_α and $M_{\alpha\beta}$ arise naturally as coefficients in the decomposition of A_{ijk} into tangential and normal components relative to a hypersurface. § 2 is devoted to a study of M_α which is a gradient vector and is readily expressed in terms of the metric of F_n and the metric induced on the hypersurface. The significance of M_α is illustrated in § 3, where we compare two definitions for the coefficients of the second fundamental form, and for normal curvature vectors. The tensor $M_{\alpha\beta}$ is investigated in § 4 and its significance illustrated by a comparison of the induced and intrinsic connections in § 5. The final section contains a brief study of totally geodesic hypersurfaces.

We consider an n-dimensional Finsler space, F_n, with fundamental function $F(x^i, \dot{x}^i)$ which is positively homogeneous of degree one in $\dot{x}^i$ and satisfies the usual conditions for

$$g_{ij}(x, \dot{x}) = \frac{1}{2}\frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j} \tag{1.1}$$

to be a metric tensor for F_n; see Rund (**5**, Chapter I). (Throughout this paper, Latin indices take values $1, 2, \ldots, n$ and the summation convention is observed.)

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The vector $l^i = \dot{x}^i/F(x, \dot{x})$ is a unit vector in the direction of the line element $(x, \dot{x})$. The tensor defined by

$$A_{ijk}(x, \dot{x}) = \frac{F(x, \dot{x})}{2} \frac{\partial g_{ij}(x, \dot{x})}{\partial \dot{x}^k} \tag{1.2}$$

is completely symmetric and satisfies

$$A_{ijk}(x, \dot{x})l^i = A_{ijk}(x, \dot{x})l^j = A_{ijk}(x, \dot{x})l^k = 0. \tag{1.3}$$

Riemannian spaces are characterized by the vanishing of this tensor.

The quantities we deal with are functions of the line element $(x, \dot{x})$ and this will be omitted from equations unless confusion is likely to occur. With the exception of the fundamental function F, all functions may be assumed to be homogeneous of degree zero in $\dot{x}$. We employ the covariant derivative of Cartan (see Rund (**5**, p. 73)), so that for a vector X^i,

$$DX^i = dX^i + X^j\Gamma^i_{jk}\, dx^k + X^j A^{\,i}_{jk} Dl^k. \tag{1.4}$$

The connection coefficient Γ^i_{jk} is usually denoted by Γ^{*i}_{jk} in the literature, and may be expressed in terms of the metric tensor and its derivatives; see Rund (**5**, p. 71 (1.28)).

A hypersurface, F_{n-1}, may be represented parametrically by the equations

$$x^i = x^i(u^\alpha), \qquad \alpha = 1, 2, \ldots, n-1, \tag{1.5}$$

where u^α are Gaussian coordinates on F_{n-1}. (Greek indices take the values $1, 2, \ldots, n-1$.) We assume that the functions (1.5) are at least of class C^3, and the matrix whose elements are

$$B^i_\alpha(x) = \frac{\partial x^i}{\partial u^\alpha} \tag{1.6}$$

is of rank $(n-1)$. The following notation is employed

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \qquad B^{ij}_{\alpha\beta} = B^i_\alpha B^j_\beta.$$

The functions $B^i_\alpha(x)$ may be considered to be components of a set of $n-1$ linearly independent vectors tangent to F_{n-1} at a point $P(x^i)$, and so they form a basis for the tangent space of F_{n-1} at P. Any vector, X^i, tangent to F_{n-1} at P may be expressed uniquely in the form

$$X^i = B^i_\alpha X^\alpha, \tag{1.7}$$

where X^α are components of the vector relative to the u^α-coordinate system. In particular, we assume the line elements are tangential to F_{n-1} so that

$$\dot{x}^i = B^i_\alpha \dot{u}^\alpha. \tag{1.8}$$

The metric induced on F_{n-1} is

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x})B^{ij}_{\alpha\beta} \tag{1.9}$$

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and the corresponding contravariant tensor $g^{\alpha\beta}$ is used to define a set of $n-1$ covariant vectors

$$B_i^\alpha(x, \dot{x}) = g^{\alpha\beta}(u, \dot{u}) g_{ij}(x, \dot{x}) B_\beta^j(x) \tag{1.10}$$

which satisfy

$$B_\alpha^i B_i^\beta = \delta_\alpha^\beta. \tag{1.11}$$

At each point P of the hypersurface, a unit normal vector $N^i(x, \dot{x})$ is defined by

$$N_i B_\alpha^i = 0, \qquad g^{ij} N_i N_j = 1, \qquad N_i = g_{ij} N^j. \tag{1.12}$$

The n linearly independent vectors (N^i, B_α^i) form a basis for vectors tangent to F_n at P, while (N_i, B_i^α) span the corresponding dual tangent space. Covariant tensors may be expressed in terms of the latter set of vectors and, in particular,

$$g_{ij} = g_{\alpha\beta} B_{ij}^{\alpha\beta} + N_i N_j, \tag{1.13}$$

$$\begin{aligned} A_{ijk} = {} & A_{\alpha\beta\gamma} B_{ijk}^{\alpha\beta\gamma} + M_{\alpha\beta}(B_{ij}^{\alpha\beta} N_k + B_{jk}^{\alpha\beta} N_i + B_{ki}^{\alpha\beta} N_j) \\ & + M_\alpha (B_i^\alpha N_j N_k + B_j^\alpha N_i N_k + B_k^\alpha N_i N_j) + M N_i N_j N_k. \end{aligned} \tag{1.14}$$

This introduces four more quantities; $A_{\alpha\beta\gamma}$ which is the projection of A_{ijk} onto the hypersurface, M which is the normal component of A_{ijk}, and

$$M_{\alpha\beta} = A_{ijk} B_{\alpha\beta}^{ij} N^k, \tag{1.15}$$

$$M_\alpha = A_{ijk} B_\alpha^i N^j N^k. \tag{1.16}$$

We note that (1.15) and (1.16) differ from those in the paper by Rund (**7**, (1.15)) by a factor F. It is easily shown, using (1.2), (1.8), and (1.9), that

$$A_{\alpha\beta\gamma} = \frac{\bar{F}(u, \dot{u})}{2} \frac{\partial g_{\alpha\beta}}{\partial \dot{u}^\gamma}, \qquad \bar{F}(u, \dot{u}) = F(x, \dot{x}),$$

and thus the vanishing of $A_{\alpha\beta\gamma}$ indicates that the metric induced on the hypersurface is Riemannian.

2. The vector M_α. The principal theorem proves that M_α is a gradient function, and gives expression to it in terms of the metric tensor of F_n and the induced metric of F_{n-1}.

THEOREM 2.1. *If a hypersurface of a Finsler space is represented by equations* (1.5) *and $\dot{x}^i$ satisfies* (1.8), *then*

$$M_\alpha(u, \dot{u}) = \bar{F}(u, \dot{u}) \frac{\partial}{\partial \dot{u}^\alpha} \left(\log \sqrt{\frac{g(x, \dot{x})}{\bar{g}(u, \dot{u})}} \right), \tag{2.1}$$

where $g(x, \dot{x}) = \det|g_{ij}(x, \dot{x})|$ *and* $\bar{g}(u, \dot{u}) = \det|g_{\alpha\beta}(u, \dot{u})|$.

Proof. Since g^{ij} = cofactor of g_{ij}/g we have, from the rule for differentiation of determinants, that

$$\frac{\partial g}{\partial \dot{x}^k} = \frac{\partial g_{ij}}{\partial \dot{x}^k} g^{ij} g.$$

Thus

$$F \frac{\partial}{\partial \dot{x}^k} (\log \sqrt{g}) = A_{ijk} g^{ij} = A_k. \tag{2.2}$$

Similarly,

$$\bar{F} \frac{\partial}{\partial \dot{u}^\gamma} (\log \sqrt{\bar{g}}) = A_{\alpha\beta\gamma} g^{\alpha\beta} = A_\gamma. \tag{2.3}$$

Substituting for $N^j N^k$ from (1.13), (1.12) in (1.16) gives

$$M_\alpha = A_k B_\alpha^k - A_\alpha = F \frac{\partial}{\partial \dot{u}^\alpha} \left(\log \sqrt{\frac{g}{\bar{g}}} \right) \tag{2.4}$$

since from (1.8) $B_\alpha^i = \partial \dot{x}^i / \partial \dot{u}^\alpha$.

COROLLARY 1. *$M_\alpha = 0$ over a hypersurface if and only if there exists a function $k(u)$ which is independent of direction and such that*

$$g(x, \dot{x}) = k(u) \bar{g}(u, \dot{u}). \tag{2.5}$$

COROLLARY 2. *If the metric induced on a particular hypersurface is Riemannian and $M_\alpha = 0$, then g is independent of directions tangential to this hypersurface.*

From (2.2) it follows that A_i then vanishes on this hypersurface. It has been proved by Deicke (**4**) that $A_i = 0$ implies $A_{ijk} = 0$ and hence that the space is Riemannian. Thus, a necessary and sufficient condition for F_n to be Riemannian is that all its hypersurfaces should be Riemannian and that $M_\alpha = 0$ on each hypersurface.

We emphasize the fact that in view of (1.14) and the above remarks, the two conditions $A_\alpha = 0$ and $M_\alpha = 0$ imply that $M_{\alpha\beta} = 0$ and $M = 0$ on any hypersurface. The tensors $g_{\alpha\beta}$ and $A_{\alpha\beta\gamma}$ were defined as the projections of g_{ij} and A_{ijk}, respectively, onto F_{n-1}, and it has been shown that they may be derived from the fundamental function $\bar{F}(u, \dot{u})$ of F_{n-1} in the same way as g_{ij} and A_{ijk} are derived from $F(x, \dot{x})$. The tensor A_α was defined in F_{n-1} in the same way as A_i was defined in F_n ((2.2) and (2.3)), but it is evident from (2.4) that, in general, A_α is not the projection of A_i onto the hypersurface. In fact, a necessary and sufficient condition for A_α to be the projection of A_i on F_{n-1} is that $M_\alpha = 0$ on F_{n-1}.

Derivatives of the normal vector.

THEOREM 2.2. *A necessary and sufficient condition for the normal vector N_i to be independent of direction is that $M_\alpha = 0$ over the hypersurface.*

Proof. We differentiate equations (1.12) which define the normal vector. From the first equation, since B_α^i is independent of direction it follows that

$\partial N_i/\partial \dot{u}^\alpha$ is normal to F_{n-1}. Differentiation of $g_{ij}N^iN^j = 1$ and $N_iN^i = 1$ and use of (1.2) and (1.16) lead to

$$F\frac{\partial N_i}{\partial \dot{u}^\alpha} = M_\alpha N_i \tag{2.6}$$

and the result follows.

THEOREM 2.3. *The set of* $(n - 1)$ *vectors* $\partial N^i/\partial \dot{u}^\beta$ *of* F_n *are tangential to* F_{n-1} *if and only if* $M_\alpha = 0$ *over the hypersurface.*

Proof. Differentiate the final equation in (1.12) and use (1.2), (2.6) to get

$$F\frac{\partial N^i}{\partial \dot{u}^\alpha} = M_\alpha N^i - 2A^i{}_{jk}B^k_\alpha N^j. \tag{2.7}$$

The normal component is

$$F\frac{\partial N^i}{\partial \dot{u}^\alpha}N_i = -M_\alpha$$

and the result follows.

3. The second fundamental form of a hypersurface. We consider a curve, C, of the hypersurface, C being represented in F_n by the equations $x^i = x^i(s)$, where the parameter is arc length. Thus

$$\frac{ds}{dt} = F(x, \dot{x}) \quad \text{and} \quad l^i = \frac{dx^i}{ds} = B^i_\alpha l^\alpha.$$

Following Berwald, we define the normal curvature of C to be

$$N(u, \dot{u}) = \frac{Dl^i(x, \dot{x})}{ds}N_i(x, \dot{x}) \tag{3.1a}$$

and it can be shown from (1.4) that

$$N(u, \dot{u}) = N_i(x, \dot{x})(B^i_{\alpha\beta}l^\alpha l^\beta + \Gamma^i_{jk}l^jl^k). \tag{3.1}$$

The coefficients of the second fundamental form are then defined by

$$\Omega_{\alpha\beta}(u, \dot{u}) = \frac{1}{2}\frac{\partial^2(F^2N)}{\partial \dot{u}^\alpha \partial \dot{u}^\beta} \tag{3.2}$$

and, using homogeneity properties, we have

$$\Omega_{\alpha\beta}(u, \dot{u})l^\alpha l^\beta = N(u, \dot{u}). \tag{3.3}$$

Rund has defined coefficients, $\tilde{\Omega}_{\alpha\beta}$, of the second fundamental form in terms of a normal curvature vector (**6**, (1.17)), and it is easily shown that

$$\tilde{\Omega}_{\alpha\beta} = N_i(B^i_{\alpha\beta} + \Gamma^i_{jk}B^{jk}_{\alpha\beta}). \tag{3.4}$$

It is evident from (3.1) that

$$\tilde{\Omega}_{\alpha\beta}(u, \dot{u})\, l^\alpha l^\beta = N(u, \dot{u}) \tag{3.5}$$

therefore, Rund's normal curvature (**6**, (1.18)) coincides with that of Berwald. To obtain the relationship between $\Omega_{\alpha\beta}$ and $\tilde{\Omega}_{\alpha\beta}$, we multiply (3.1) by F^2 and differentiate with respect to direction. Using (2.6), (3.1), and (3.4), together with

$$\frac{\partial \Gamma^i_{hk}}{\partial \dot{x}^j}\dot{x}_h = A^{\ i}_{kj|h} l^h$$

(see **5**, p. 81), we get

$$\frac{\partial(F^2N)}{\partial \dot{u}^\gamma} = FNM_\gamma + 2\tilde{\Omega}_{\alpha\gamma}\dot{u}^\alpha. \tag{3.6}$$

We differentiate again, noting that

$$\frac{\partial(FN)}{\partial \dot{u}^\epsilon} = \frac{1}{F}\frac{\partial(F^2N)}{\partial \dot{u}^\epsilon} - Nl_\epsilon;$$

then using (3.6) in the right-hand side of this expression, we get

$$\frac{\partial^2(F^2N)}{\partial \dot{u}^\epsilon \partial \dot{u}^\gamma} = 2\Omega_{\epsilon\gamma} = (NM_\epsilon + 2\tilde{\Omega}_{\alpha\epsilon}l^\alpha - Nl_\epsilon)M_\gamma + FN\frac{\partial M_\gamma}{\partial \dot{u}^\epsilon} + 2\frac{\partial \tilde{\Omega}_{\alpha\gamma}}{\partial \dot{u}^\epsilon}\dot{u}^\alpha + 2\tilde{\Omega}_{\epsilon\gamma}. \tag{3.7}$$

From (3.4) and (2.6),

$$\frac{\partial \tilde{\Omega}_{\alpha\beta}}{\partial \dot{u}^\epsilon}\dot{u}^\alpha = \frac{1}{F}M_\epsilon\tilde{\Omega}_{\alpha\beta}\dot{u}^\alpha + N_iA^{\ i}_{kj|h}l^hB^{jk}_{\epsilon\beta}.$$

We therefore have from (3.7) that

$$\Omega_{\epsilon\gamma} = \tilde{\Omega}_{\epsilon\gamma} + \tfrac{1}{2}N\left(M_\epsilon M_\gamma - l_\epsilon M_\gamma + F\frac{\partial M_\gamma}{\partial \dot{u}^\epsilon}\right) + (M_\gamma\tilde{\Omega}_{\alpha\epsilon} + M_\epsilon\tilde{\Omega}_{\alpha\gamma})l^\alpha + N_iA^{\ i}_{kj|h}l^hB^{jk}_{\epsilon\gamma} \tag{3.8}$$

and it follows that

$$\Omega_{\epsilon\gamma}l^\epsilon = \tilde{\Omega}_{\epsilon\gamma}l^\epsilon + \tfrac{1}{2}NM_\gamma, \tag{3.9}$$

since M_γ is homogeneous of degree zero in $\dot{u}$ and $A^{\ i}_{kj|h}l^hl^j = 0$. (3.9) gives the following theorem.

THEOREM 3.1. *Assuming that $N \neq 0$, a necessary condition for $\Omega_{\alpha\beta}$ and $\tilde{\Omega}_{\alpha\beta}$ to coincide is that $M_\alpha = 0$.*

Finsler spaces for which Berwald's connection is independent of direction are said to be "affinely connected"; see Berwald (**2**, p. 47). It has been shown that $A_{ijk|h} = 0$ is a necessary and sufficient condition for a Finsler space to be affinely connected (**5**, p. 81). Thus, from (3.8), we have the following result.

THEOREM 3.2. *For affinely connected Finsler spaces, $M_\alpha = 0$ is a sufficient condition for $\Omega_{\alpha\beta}$ and $\tilde{\Omega}_{\alpha\beta}$ to coincide.*

Principal directions are usually defined to be the directions for which the normal curvature $N(u, \dot{u})$ assumes extreme values subject to the condition that $\dot{u}$ be a unit vector. Using the Lagrange multiplier rule it can be shown that $\dot{u}$ must satisfy

$$N(u, \dot{u})g_{\alpha\beta}(u, \dot{u})\dot{u}^{\beta} = \Omega_{\alpha\beta}(u, \dot{u})\dot{u}^{\beta}.$$

In (**6**, p. 235), Rund employed the term principal directions for directions which satisfy a similar equation, $\tilde{\Omega}_{\alpha\beta}$ replacing $\Omega_{\alpha\beta}$. From (3.9) we have the following result.

THEOREM 3.3. *Assuming that $N \neq 0$, $M_{\alpha} = 0$ is a necessary and sufficient condition for Rund's principal directions to coincide with the directions for which the normal curvature assumes extreme values.*

Normal curvature vectors. The normal curvature vectors provide an example of Finsler quantities which take different forms in the locally Euclidean and locally Minkowskian theories. In the Minkowskian theory, Rund considers a vector X^i which is tangential to F_{n-1}. The coefficient of $X^{\alpha}du^{\beta}$ in the normal component of δX^i is denoted by $I^i_{\alpha\beta}$ and is termed the normal curvature vector (**5**, p. 193 (7.18)). This vector may be written as

$$I^i_{\alpha\beta} = N^i\,\tilde{\Omega}_{\alpha\beta},$$

(see 7, (2.5)). In the Euclidean theory, Davies obtained a normal curvature vector $\mathring{H}^i_{\alpha\beta}$ which may be expressed in terms of $I^i_{\alpha\beta}$,

$$\mathring{H}^i_{\alpha\beta} = I^i_{\alpha\beta} + N^iN_jA^j{}_{hk}B^h_{\beta}\mathring{H}^k_{\alpha\lambda}l^{\lambda}$$

(see **5**, p. 193 and p. 165, (4.11a)). Multiplication by l^{β} and use of (1.3) lead to

$$\mathring{H}^i_{\alpha\beta}l^{\beta} = I^i_{\alpha\beta}l^{\beta} = N^i\tilde{\Omega}_{\alpha\beta}l^{\beta}.$$

This is substituted in the above equation and (1.16) is employed to give

$$\mathring{H}^i_{\alpha\beta} = N^i(\tilde{\Omega}_{\alpha\beta} + M_{\beta}\tilde{\Omega}_{\alpha\lambda}l^{\lambda}). \tag{3.10}$$

THEOREM 3.4. *Assuming $N \neq 0$, $M_{\alpha} = 0$ is a necessary and sufficient condition in order that the normal curvature vectors of Davies and Rund coincide.*

Proof. Clearly, the condition is sufficient. Suppose $\mathring{H}^i_{\alpha\beta} = I^i_{\alpha\beta}$, then, from (3.10),

$$M_{\beta}\tilde{\Omega}_{\alpha\lambda}l^{\lambda} = 0.$$

Either $M_{\beta} = 0$ or $\tilde{\Omega}_{\alpha\lambda}l^{\lambda} = 0$. If the latter result were true, then

$$\tilde{\Omega}_{\alpha\lambda}l^{\lambda}l^{\alpha} = N = 0$$

which is contrary to the hypothesis. Hence $M_{\beta} = 0$ is also a necessary condition.

4. The tensor $M_{\alpha\beta}$. $M_{\alpha\beta}$ does not appear to give rise to a simple metric expression corresponding to Theorem 2.1 for M_{α}. From the definition, it is clear that $M_{\alpha\beta}$ is a symmetric tensor, therefore we begin by investigating this property.

THEOREM 4.1. *$F^{-1}M_{\alpha\beta}$ is a second derivative with respect to $\dot{u}^\gamma$ if and only if*

$$M_{\alpha\epsilon}X^{\epsilon}_{\nu\beta} = M_{\beta\epsilon}X^{\epsilon}_{\nu\alpha}, \tag{4.1}$$

where

$$X^{\epsilon}_{\nu\beta} = 2A^{\epsilon}_{\nu\beta} - \delta^{\epsilon}_{\nu}M_{\beta}. \tag{4.2}$$

Proof. Equation (2.7) may be written in terms of tangential and normal components as

$$F\frac{\partial N^i}{\partial \dot{u}^\alpha} = -2g^{\beta\gamma}M_{\gamma\alpha}B^i_\beta - M_\alpha N^i. \tag{4.2a}$$

The tangential component of the second derivative of N^i is

$$\frac{\partial^2 N^i}{\partial \dot{u}^\alpha \partial \dot{u}^\beta} B^\mu_i = \frac{4}{F^2} A^{\mu\gamma}{}_\beta M_{\gamma\alpha} - 2g^{\gamma\mu}\frac{\partial}{\partial \dot{u}^\beta}\left(\frac{1}{F}M_{\alpha\gamma}\right) + \frac{2}{F^2}g^{\gamma\mu}M_\alpha M_{\gamma\beta}.$$

The skew-symmetric part in α, β is multiplied by $g_{\mu\nu}$ and rearranged to give

$$\frac{\partial(F^{-1}M_{\nu\alpha})}{\partial \dot{u}^\beta} - \frac{\partial(F^{-1}M_{\nu\beta})}{\partial \dot{u}^\alpha} = \frac{1}{F^2}(M_{\epsilon\alpha}X^{\epsilon}_{\nu\beta} - M_{\epsilon\beta}X^{\epsilon}_{\nu\alpha}),$$

where $X^{\epsilon}_{\nu\beta}$ is defined in (4.2). $F^{-1}M_{\nu\alpha}$ is a second derivative with respect to $\dot{u}^\beta$ if and only if the left-hand side vanishes, and the theorem is proved.

Clearly, $X^{\epsilon}_{\nu\beta} = 0$ satisfies the condition, but this implies that $M_{\alpha\beta} = 0$, for, from (4.2)

$$2A^{\epsilon}_{\nu\beta} = \delta^{\epsilon}_{\nu}M_\beta. \tag{4.3}$$

Multiplication by l^ν leads to $l^\epsilon M_\beta = 0$ and hence $M_\beta = 0$. But then $A^{\epsilon}_{\nu\beta} = 0$ from (4.3), and F_{n-1} is Riemannian, therefore, from Theorem 2.1, Corollary 2, $M_{\alpha\beta} = 0$ over the hypersurface.

A further situation in which (4.1) is satisfied is when M_α, $M_{\alpha\beta}$, and $A_{\alpha\beta\gamma}$ are of the form

$$M_\alpha = \lambda m_\alpha, \qquad M_{\alpha\beta} = \mu m_\alpha m_\beta, \qquad A_{\alpha\beta\gamma} = \nu m_\alpha m_\beta m_\gamma,$$

where we have taken account of the symmetry of the tensors. m_α must be orthogonal to the vector l^α since the contractions of the above tensors with l^α vanish. In the tangent space to F_{n-1} at a point $P(x^i)$ we can find $(n-2)$ linearly independent vectors orthogonal to l^α, and, in general, M_α, $M_{\alpha\beta}$, and $A_{\alpha\beta\gamma}$ will be linear combinations of all of these vectors. Thus, it is seen that the condition for $F^{-1}M_{\alpha\beta}$ to be a second derivative is quite a stringent one. Nevertheless, for the case when $n = 3$, there is only one vector tangent to F_{n-1} and orthogonal to l^α, and hence, for an F_2 embedded in F_3, $F^{-1}M_{\alpha\beta}$ is always a second derivative with respect to $\dot{u}^\gamma$. $M_{\alpha\beta}$ occurs in the derivative of N^i with respect to direction, and from (4.2a) we have the following result.

THEOREM 4.2. *The vector* $\partial N^i/\partial \dot{u}^\beta$ *of* F_n *is normal to the hypersurface if and only if* $M_{\alpha\beta} = 0$.

$M_{\alpha\beta}$ is also closely associated with the derivative of B_i^α. From the definition (1.6), it is clear that the coefficients B_α^i are independent of direction, but in general, this is not the case for B_i^α; see (1.10).

THEOREM 4.3. *A necessary and sufficient condition for* B_i^α *to be independent of direction is that* $M_{\alpha\beta} = 0$.

Proof. We differentiate equation (1.11) with respect to $\dot{u}^\gamma$,

$$B_\alpha^i \frac{\partial B_i^\beta}{\partial \dot{u}^\gamma} = 0.$$

Thus $\partial B_i^\beta/\partial \dot{u}^\gamma$ is normal to F_{n-1}. To obtain the normal component, we differentiate $N^i B_i^\alpha = 0$ and use (4.2a) giving

$$\frac{\partial B_i^\alpha}{\partial \dot{u}^\gamma} = 2g^{\alpha\nu}\frac{1}{F} M_{\nu\gamma} N_i.$$

The condition for B_i^α to be independent of $\dot{u}^\gamma$ is therefore that $M_{\alpha\beta} = 0$.

5. The induced and intrinsic connections on the hypersurface. It is a well-known property of Finsler geometry that the connection induced on a hypersurface is not, in general, equal to the connection formed "intrinsically" from the induced metric tensor. The two connection coefficients have been expressed in various forms (see Davies (**3**), Berwald (**1**), and Rund (**5**)), but it appears that the conditions for them to coincide have not been obtained. We employ a new and simple method of relating the connection coefficients. It is based on the Ricci Lemma, i.e., on the vanishing of the covariant derivative of the metric tensor, and is actually an extension of the method of obtaining the Christoffel connection in Riemannian geometry.

The induced differential is obtained by multiplying (1.4) by B_i^α, and using (1.7), (1.15), and (1.16). It takes the form

$$\bar{D}X^\alpha = dX^\alpha + (\Gamma_{\beta\gamma}^\alpha + \lambda_{\beta\gamma}^\alpha)X^\beta du^\gamma + A_{\beta\gamma}^\alpha X^\beta \bar{D}l^\gamma, \tag{5.1}$$

where

$$\Gamma_{\beta\gamma}^\alpha = B_i^\alpha(B_{\beta\gamma}^i + \Gamma_{jk}^i B_{\beta\gamma}^{jk}) \tag{5.2}$$

is the connection induced on the hypersurface, and

$$\lambda_{\beta\gamma}^\alpha = g^{\alpha\delta} M_{\beta\delta} \tilde{\Omega}_{\gamma\nu} l^\nu. \tag{5.3}$$

We note that $\Gamma_{\beta\gamma}^\alpha$ is symmetric in its lower indices, but $\lambda_{\beta\gamma}^\alpha$ is not, even though $\lambda_{\beta\delta\gamma} = g_{\delta\alpha}\lambda_{\beta\gamma}^\alpha$ is symmetric in $\beta\delta$.

The definition of an induced differential is extended to tensors in the usual way. Since $\bar{D}g_{\alpha\beta}$ is the projection of Dg_{ij} onto the hypersurface, and $Dg_{ij} = 0$, it follows that $\bar{D}g_{\alpha\beta} = 0$, i.e., the Ricci Lemma holds for the induced differential.

The intrinsic connection coefficient $\hat{\Gamma}^{\alpha}_{\beta\gamma}$ is constructed from $g_{\alpha\beta}$ in the same way as Γ^{i}_{jk} is constructed from g_{ij}, and the intrinsic differential $\hat{D}X^{\alpha}$ has the same form as (1.4). $\hat{\Gamma}^{\alpha}_{\beta\gamma}$ thus inherits the properties of Γ^{i}_{jk}, and, in particular, it is symmetric in β, γ and satisfies the Ricci Lemma, $\hat{D}g_{\alpha\beta} = 0$. We proceed to compare these two expressions of a Ricci Lemma on the hypersurface.

$$\hat{D}g_{\alpha\beta} = 0 = dg_{\alpha\beta} - (\hat{\Gamma}_{\alpha\beta\gamma} + \hat{\Gamma}_{\beta\alpha\gamma})du^{\gamma} - 2A_{\alpha\beta\gamma}\hat{D}l^{\gamma},$$

$$\bar{D}g_{\alpha\beta} = 0 = dg_{\alpha\beta} - (\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} + 2\lambda_{\alpha\beta\gamma})du^{\gamma} - 2A_{\alpha\beta\gamma}\bar{D}l^{\gamma}.$$

We eliminate $dg_{\alpha\beta}$ between these equations and write

$$\Lambda_{\alpha\beta\gamma} = \hat{\Gamma}_{\alpha\beta\gamma} - \Gamma_{\alpha\beta\gamma}$$

so that

$$\hat{D}l^{\gamma} - \bar{D}l^{\gamma} = l^{\nu}\Lambda^{\gamma}_{\nu\delta}du^{\delta}. \tag{5.4}$$

Thus we obtain

$$\Lambda_{\alpha\beta\gamma} + \Lambda_{\beta\alpha\gamma} = 2(\lambda_{\alpha\beta\gamma} - A_{\alpha\beta\delta}\Lambda^{\delta}_{\nu\gamma}l^{\nu}).$$

Two further equations are obtained by cyclic permutation of α, β, γ, and the final equation is subtracted from the sum of the first two. Since $\Lambda_{\alpha\beta\gamma}$ is symmetric in α, γ, this leads to

$$\Lambda_{\beta\alpha\gamma} = \lambda_{\alpha\beta\gamma} + \lambda_{\gamma\alpha\beta} - \lambda_{\beta\gamma\alpha} - (A_{\alpha\beta\nu}\Lambda^{\nu}_{\delta\gamma} + A_{\gamma\alpha\nu}\Lambda^{\nu}_{\delta\beta} - A_{\beta\gamma\nu}\Lambda^{\nu}_{\delta\alpha})l^{\delta}. \tag{5.5}$$

We contract this expression with l^{β}, noting from (5.3) that contractions of $\lambda_{\alpha\beta\gamma}$ with l^{α} or l^{β} vanish. Thus

$$\Lambda_{\beta\alpha\gamma}l^{\beta} = \lambda_{\gamma\alpha\beta}l^{\beta} - A_{\gamma\alpha\nu}\Lambda^{\nu}_{\delta\beta}l^{\delta}l^{\beta},$$

and the final term vanishes since a further contraction with l^{γ} gives

$$\Lambda_{\beta\alpha\gamma}l^{\beta}l^{\gamma} = 0. \tag{5.6}$$

Thus, from (5.3) and (3.5),

$$\Lambda_{\beta\alpha\gamma}l^{\beta} = M_{\alpha\gamma}N, \tag{5.7}$$

while from (5.5),

$$\Lambda_{\beta\alpha\gamma}l^{\alpha} = -M_{\beta\gamma}N.$$

A substitution in (5.5) now gives the desired relationship, namely,

$$\Lambda_{\beta\alpha\gamma} = \lambda_{\alpha\beta\gamma} + \lambda_{\gamma\alpha\beta} - \lambda_{\beta\gamma\alpha} - (A_{\alpha\beta\nu}M^{\nu}_{\gamma} + A_{\alpha\gamma\nu}M^{\nu}_{\beta} - A_{\beta\gamma\nu}M^{\nu}_{\alpha})N. \tag{5.8}$$

THEOREM 5.1. *Assuming that $N \neq 0$, a necessary and sufficient condition for the induced and intrinsic connections to coincide is that $M_{\alpha\beta} = 0$ over F_{n-1}.*

Proof. It is obvious from (5.8) and (5.3) that the condition is sufficient. If $\Lambda_{\beta\alpha\gamma} = 0$, then $\Lambda_{\beta\alpha\gamma}l^{\beta} = 0$ and from (5.7), $M_{\alpha\gamma}N = 0$, therefore the condition is also necessary. Thus, we have a class of hypersurfaces of a Finsler space which possess only one type of connection, and so have properties similar to Riemannian hypersurfaces.

It is possible to obtain an expression for $M_{\alpha\beta}$ in terms of derivatives of the metric tensor from (5.7).

$$FNM^{\alpha}_{\gamma} = \hat{\Gamma}^{\alpha}_{\beta\gamma}\dot{u}^{\beta} - \Gamma^{\alpha}_{\beta\gamma}\dot{u}^{\beta} = \frac{\partial G^{\alpha}}{\partial \dot{u}^{\gamma}} - B^{\alpha}_{i}\left(B^{i}_{\beta\gamma}\dot{u}^{\beta} + \frac{\partial G^{i}}{\partial \dot{u}^{\gamma}}\right), \tag{5.9}$$

where we have used (5.2). Since G^i and G^α are defined in terms of Christoffel symbols formed from g_{ij} and $g_{\alpha\beta}$, respectively (**5**, p. 71), (5.9) expresses M^{α}_{γ} in terms of the metrics, provided $N \neq 0$.

6. Totally geodesic hypersurfaces. Theorems 3.1, 3.4, and 5.1, which illustrate the significance of M_α and $M_{\alpha\beta}$, were all prefaced by the assumption that $N \not\equiv 0$ over the hypersurface. We now investigate the properties of hypersurfaces for which $N \equiv 0$.

THEOREM 6.1. *$N \equiv 0$ over a hypersurface if and only if*

$$G^j = G^\alpha B^j_\alpha - \tfrac{1}{2}B^j_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta. \tag{6.1}$$

Proof. Contract (5.9) with $\dot{u}^\gamma$ and use the fact that G^i and G^α are homogeneous of degree two in $\dot{x}$ and $\dot{u}$, respectively. This gives

$$2G^\alpha = B^\alpha_i(2G^i + B^i_{\beta\gamma}\dot{u}^\beta\dot{u}^\gamma).$$

Multiply by B^j_α, noting that $B^\alpha_i B^j_\alpha = \delta^j_i - N^j N_i$, and use (3.1) to get

$$2G^j + B^j_{\beta\gamma}\dot{u}^\beta\dot{u}^\gamma = 2G^\alpha B^j_\alpha + N^j N.$$

The theorem then follows.

THEOREM 6.2. *A necessary and sufficient condition for a geodesic of F_{n-1} to be geodesic of F_n is that $N(u, \dot{u}) = 0$ along the curve.*

Proof. Let $x^i = x^i(s)$ be a curve, C, on the hypersurface. Using (3.1a) we write Dl^i/ds in terms of its tangential and normal components:

$$\frac{Dl^i}{ds} = B^i_\alpha \frac{\bar{D}l^\alpha}{ds} + N^i N(u, \dot{u}).$$

If C is a geodesic of F_{n-1}, then $\hat{D}l^\alpha/ds = 0$ and, from (5.4) and (5.6), $\bar{D}l^\alpha/ds = 0$. Thus $Dl^i/ds = N^i N(u, \dot{u})$, and C is a geodesic of F_n if and only if $N(u, \dot{u}) = 0$ along C. If $N \equiv 0$ on the hypersurface, then every geodesic of F_{n-1} is also a geodesic of F_n, and the hypersurface is said to be totally geodesic.

In Riemannian geometry, the coefficients of the second fundamental form vanish on a totally geodesic hypersurface. It is evident from (3.2) that Berwald's coefficient $\Omega_{\alpha\beta}$ also has this property.

THEOREM 6.3. $\tilde{\Omega}_{\alpha\beta} = 0$ *on a totally geodesic hypersurface of an affinely connected Finsler space.*

Proof. Put $N = 0$ in (3.8) and (3.9).

THEOREM 6.4. *The normal curvature vectors defined by Rund and Davies coincide on totally geodesic hypersurfaces.*

Proof. This follows from (3.10) and (3.9).

THEOREM 6.5. *The induced and intrinsic connections coincide on totally geodesic hypersurfaces.*

Proof. From (5.3), $\lambda_{\alpha\beta\gamma} = 0$ on a totally geodesic hypersurface, and it follows from (5.8) that $\Lambda_{\beta\alpha\gamma} = 0$.

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